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A NULL CONTROLLABILITY PROBLEM WITH A FINITE NUMBER OF CONSTRAINTS ON THE NORMAL DERIVATIVE FOR THE SEMILINEAR HEAT EQUATION

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Abstract. We consider the semilinear heat equation in a bounded domain of $\mathbb{R}^m$. We prove the null controllability of the system with a finite number of constraints on the normal derivative, when the control acts on a bounded subset of the domain. First, we show that the problem can be transformed into a null controllability problem with constraint on the control, for a linear system. Then, we use an appropriate observability inequality to solve the linearized problem. Finally, we prove the main result by means of a fixed-point method.

1. Introduction

Let $m \in \mathbb{N}\backslash\{0\}$ and let $\Omega \subset \mathbb{R}^m$ be a bounded domain with boundary Γ of class C^2. Let also ω be a non empty subdomain of Ω and Γ_0 a non empty part of Γ. For a time $T > 0$, set $Q = \Omega \times (0, T)$, $\Sigma = \Gamma \times (0, T)$, $\Sigma_0 = \Gamma_0 \times (0, T)$ and $G = \omega \times (0, T)$. Consider the following system of semilinear heat equation:

$$(1)\qquad \begin{cases} \frac{\partial y}{\partial t} - \Delta y + f(y) & = & v\chi_\omega \quad \text{in } Q, \\ y|_\Sigma & = & 0, \\ y(0) & = & y^0 \quad \text{in } \Omega, \end{cases}$$

where f is a function of class C^1 on $\mathbb{R}$, $y^0 \in L^2(\Omega)$, $v \in L^2(G)$ represents the control function and χ_ω is the characteristic function of ω, the set where controls are supported. The function f is assumed to be globally Lipschitz all along the paper, i.e. there exists $K > 0$ such that

$$(2)\qquad |f(x) - f(z)| \leqslant K|x - z|, \forall x, z \in \mathbb{R},$$

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and assume for simplicity that

(3) $$f(0) = 0.$$

We denote y by $y(x,t,v)$ to mean that the solution y of (1) depends on the control v.
Null controllability problem with constraint on the control has been studied by O. Nakoulima in [1, 2], for the parabolic evolution equation. Indeed, he solved in [2] the following null controllability problem with constraint on the control: *Given a finite-dimensional subspace Y of $L^2(G)$ and $y^0 \in H_0^1(\Omega)$, find a control $v \in Y^\perp$, the orthogonal complement of Y in $L^2(G)$, such that the solution of*

(4) $$\begin{cases} \frac{\partial y}{\partial t} - \Delta y + a_0 y &= v\chi_\omega \quad \text{in } Q, \\ y &= 0 \quad \text{on } \Sigma, \\ y(0) &= y^0 \quad \text{in } \Omega, \end{cases}$$

satisfies $y(T) = 0$ in Ω.
The proof uses an observability inequality adapted to the constraint. The results obtained by O. Nakoulima allowed G. M. Mophou and O. Nakoulima to prove the existence of sentinels with given sensitivity in [3], and to solve a new type of controllability problem (see [4]): *Given e_i in $L^2(Q)$, $1 \leqslant i \leqslant M$, and $y^0 \in L^2(\Omega)$, find a control $v \in L^2(Q)$ such that the solution of* (4) *satisfies $y(T) = 0$ in Ω and*

(5) $$\int_0^T \int_\Omega y e_i \, dx dt = 0, 1 \leqslant i \leqslant M.$$

We also refer to [5] where a boundary null controllability with constraints on the state for a linear heat equation is solved. G. M. Mophou in [6] showed the null controllability with a finite number of constraints on the state, for a nonlinear heat equation involving gradient terms.

In this paper, we focus on a null controllability problem with constraint on the normal derivative that we describe now.
Let $\{e_1, \dots e_m\}$ be a family of vectors of

$$H_0^1(\Sigma) = \{\psi | \psi \in H^1(\Sigma), \psi(x,0) = 0, \psi(x,T) = 0 \text{ in } \Gamma\}$$

and let $\mathcal{E} = \text{Span}(\{e_1, \dots, e_m\})$ be the span of the family of vectors $\{e_1, \dots e_m\}$. Suppose that:

(6) the vectors $(e_j)_{j=1,\dots,m}$ are linearly independent on Σ_0.

The null controllability problem with constraint on the normal derivative for system (1) can be formulated as follows: *Given f a globally Lipschitz function of class C^1 on $\mathbb{R}$ satisfying (3), $y^0 \in L^2(\Omega)$ and $e_j \in H_0^1(\Sigma)$ $j = 1, \ldots, m$ satisfying (6), find $v \in L^2(G)$ such that if y is solution of (1), then*

$$\langle \frac{\partial y}{\partial \nu}, e_j \rangle_{H^{-1}(\Sigma_0), H_0^1(\Sigma_0)} = 0; j = 1, \ldots, m, \tag{7}$$

and

$$y(T) = 0 \text{ in } \Omega, \tag{8}$$

where ν is the unit exterior normal vector of Γ, $\frac{\partial y}{\partial \nu}$ is the normal derivative of y with respect to ν and $\langle ., . \rangle_{X,X'}$ denotes the duality bracket between the spaces X and X'.
The main result of this paper is as follows:

Theorem 1.1. *Let f be a globally Lipschitz function of class C^1 on $\mathbb{R}$ satisfying (3). Then for any $y^0 \in L^2(\Omega)$ and $e_j \in H_0^1(\Sigma)$ $j = 1, \ldots, m$ satisfying (6), there exists a unique control $\tilde{v}$ of minimal norm in $L^2(G)$, such that $(\tilde{v}, \tilde{y})$ satisfies the null controllability problem with constraint on the normal derivative (1), (7) and (8). Moreover there exists a positive constant $C = C(\Omega, \omega, K, T, \sum_{j=1}^{m} ||e_j||_{H_0^1(\Sigma)})$ such that*

$$||\tilde{v}||_{L^2(G)} \leqslant C||y^0||_{L^2(\Omega)}. \tag{9}$$

The proof of this theorem will be the subject of the last section. The rest of the paper is organized as follows. In Section 2, we show that problem (1), (7), (8) is equivalent to a null controllability problem with constraint on the control for a linearized system derived from (1). In Section 3, we prove an observability estimate for the linearized system. In Section 4, we use this estimate to prove the null controllability of the linearized system. Section 5 is devoted to proving Theorem 1.1.

2. Equivalence with null controllability problem with constraint on the control for linearized system

We introduce the notation

$$a_0(s) = \begin{cases} \frac{f(s)}{s} & \text{if } s \neq 0 \\ f'(0) & \text{if } s = 0. \end{cases}$$

In view of the globally Lipschitz assumption (2) on f, a_0 maps $L^2(Q)$ into a bounded set of $L^\infty(Q)$. Moreover

$$||a_0(y)||_{L^\infty(Q)} \leqslant K,\ \forall y \in L^2(Q), \tag{10}$$

K being the Lipschitz constant of f.
Thus, system (1) may be rewritten in the form

$$\begin{cases} \dfrac{\partial y}{\partial t} - \Delta y + a_0(y)y &= v\chi_\omega \quad \text{in } Q, \\ y|_\Sigma &= 0, \\ y(0) &= y^0 \qquad \text{in } \Omega. \end{cases} \tag{11}$$

Given $z \in L^2(Q)$, consider the linearized system

$$\begin{cases} \dfrac{\partial y}{\partial t} - \Delta y + a_0(z)y &= v\chi_\omega \quad \text{in } Q, \\ y|_\Sigma &= 0, \\ y(0) &= y^0 \qquad \text{in } \Omega, \end{cases} \tag{12}$$

Since $a_0(z) \in L^\infty(Q)$, $y^0 \in L^2(\Omega)$ and $v\chi_\omega \in L^2(Q)$, system (12) admits a unique solution y in

$$Ł^2(0,T; H_0^1(\Omega)) \cap C([0,T]; L^2(\Omega)).$$

Note that since $y \in L^2(0,T; H_0^1(\Omega))$ and $\Delta y \in H^{-1}(0,T; L^2(\Omega))$, we can define $\frac{\partial y}{\partial \nu}$ on Γ and $\frac{\partial y}{\partial \nu} \in H^{-1}(0,T; H^{-\frac{3}{2}}(\Gamma))$, which is a subset of $H^{-1}(\Sigma)$, the dual of $H_0^1(\Sigma)$.
Consequently our aim is: *For any $z \in L^2(Q)$, $a_0(z) \in L^\infty(Q)$, $y^0 \in L^2(\Omega)$ and $e_j \in H_0^1(\Sigma)$ $j = 1, \dots, m$, to find a control $v \in L^2(G)$ such that the solution y of (12) satisfies (7) and (8).*
As we said in the introduction, we show in the rest of this section that problem (12), (7), (8) is equivalent to a null controllability problem with constraint on the control.
For each e_j, $1 \leqslant j \leqslant m$, consider the adjoint of system (12):

$$\begin{cases} -\dfrac{\partial q_j}{\partial t} - \Delta q_j + a_0(z)q_j &= 0 \quad \text{in} \quad Q, \\ q_j &= e_j \quad \text{on} \quad \Sigma_0, \\ q_j &= 0 \quad \text{on} \quad \Sigma \setminus \Sigma_0, \\ q_j(T) &= 0 \quad \text{in} \quad \Omega. \end{cases} \tag{13}$$

The following lemma holds:

Lemma 2.1. *Under the hypothesis (6), the functions* $q_j\chi_\omega$, $1 \leqslant j \leqslant m$, *are linearly independent for any* $z \in L^2(Q)$.

Proof. Let $\gamma_j \in \mathbb{R}$, $1 \leqslant j \leqslant m$, be such that

$$\sum_{j=1}^{m} \gamma_j q_j = 0 \text{ in } G. \tag{14}$$

Since q_j is solution of (13) for each $j \in \{1, \dots, m\}$, then $\sum_{j=1}^{m} \gamma_j q_j := q$ satisfies:

$$\begin{cases} -\dfrac{\partial q}{\partial t} - \Delta q + a_0(z)q = 0 & \text{in } Q, \\ q = \sum_{j=1}^{m} \gamma_j e_j & \text{on } \Sigma_0. \end{cases} \tag{15}$$

Combining the first equation of (15) with (14), we deduce that, according to a unique continuation property for the evolution equation, $q = 0$ in Q. Therefore, we have in particular $q = 0$ on Σ_0. Since the second equation of (15) holds, the hypothesis (6) implies that $\gamma_j = 0$ for all $j \in \{1, \dots, m\}$ and the proof of Lemma (2.1) is complete. ■

If X is a closed vector subspace of $L^2(G)$, let us denote by $X^\perp$ the orthogonal of X in $L^2(G)$.

Proposition 2.2. *There exists a positive real function* θ *such that for any* $z \in L^2(Q)$, *there exist two finite dimensional vector subspaces* $\mathcal{U}, \mathcal{U}_\theta$ *of* $L^2(G)$, *and* $u_0(z) \in \mathcal{U}_\theta$ *such that the null controllability problem with constraint on the normal derivative (12), (7), (8) is equivalent to the following null controllability problem with constraint on the control: Given* $a_0(z) \in L^\infty(Q)$, $y^0 \in L^2(\Omega)$ *and* $u_0 \in \mathcal{U}_\theta$, *find*

$$u \in \mathcal{U}^\perp \tag{16}$$

such that if y *is solution of*

$$\begin{cases} \dfrac{\partial y}{\partial t} - \Delta y + a_0(z)y = (u_0 + u)\chi_\omega & \textit{in } Q, \\ y = 0 & \textit{on } \Sigma, \\ y(0) = y^0 & \textit{in } \Omega, \end{cases} \tag{17}$$

then

$$y(T) = 0 \textit{ in } \Omega. \tag{18}$$

Proof. Suppose that the null controllability problem with constraint on the normal derivative (12), (7), (8) holds.
Since $a_0(z) \in L^\infty(Q)$ and $e_j \in H_0^1(\Sigma)$, $j = 1, \dots, m$, for each j, system (13) admits a unique solution q_j in $L^2(0,T;H^2(\Omega)) \cap H^1(0,T;L^2(\Omega)) := H^{2,1}(Q)$.
Multiplying (12) by the solution q_j of (13), then integrating by parts over Q, we obtain

$$\int_\Omega y(T)q_j(T)dx - \int_\Omega y(0)q_j(0)dx - \langle \frac{\partial y}{\partial \nu}, q_j \rangle_{H^{-1}(\Sigma),H_0^1(\Sigma)} \\ + \int_\Sigma y \frac{\partial q_j}{\partial \nu} d\Gamma dt + \int_Q y(-\frac{\partial q_j}{\partial t} - \Delta q_j + a_0(z)q_j)dxdt = \int_Q v\chi_\omega q_j dxdt.$$

It follows that

$$-\int_\Omega y^0 q_j(0)dx - \langle \frac{\partial y}{\partial \nu}, e_j \rangle_{H^{-1}(\Sigma_0),H_0^1(\Sigma_0)} = \int_G vq_j dxdt.$$

In view of (7), we have

$$-\int_\Omega y^0 q_j(0)dx = \int_G vq_j dxdt. \tag{19}$$

Let $\mathcal{U} = \mathrm{Span}(\{q_1\chi_\omega, \dots, q_m\chi_\omega\})$ and let $\mathcal{U}_\theta = \frac{1}{\theta}\mathcal{U}$. Then there exists a unique $u_0 \in \mathcal{U}_\theta$ such that for any $j \in \{1, \dots, m\}$,

$$\int_G u_0 q_j dxdt = -\int_\Omega y^0 q_j(0)dx. \tag{20}$$

Thus according to (19), we have

$$\int_G u_0 q_j dxdt = \int_G vq_j dxdt, \text{ for any } j \in \{1, \dots, m\}.$$

Therefore, $v - u_0 \in \mathcal{U}^\perp$ and there exists $u \in \mathcal{U}^\perp$ such that $v = u_0 + u$. Now, replacing v by $u_0 + u$ in (12), we obtain (17).

Conversely, suppose that for any $z \in L^2(Q)$, $a_0(z) \in L^\infty(Q)$ and $y^0 \in L^2(\Omega)$ are given, and that the solution y of (17) satisfies $y(T) = 0$ in Ω. Let q_j, $j = 1, \dots, m$, be the solutions of (13), $u \in \mathcal{U}^\perp$ and u_0 satisfying (20). Multiplying (17) by q_j, then integrating by parts over Q, we have

$$-\int_\Omega y^0 q_j(0)dx - \langle \frac{\partial y}{\partial \nu}, e_j \rangle_{H^{-1}(\Sigma_0),H_0^1(\Sigma_0)} = \int_G u_0 q_j dxdt + \int_G uq_j dxdt.$$

In view of (20), we get

$$-\langle \frac{\partial y}{\partial \nu}, e_j\rangle_{H^{-1}(\Sigma_0),H_0^1(\Sigma_0)} = \int_G u q_j dxdt,$$

which ends the proof of Proposition 2.2, because $u \in \mathcal{U}^{\perp}$. ■

Remark 2.3. *The function u_0 is such that $\theta u_0 \in L^2(G)$.*
In the sequel, we will denote by P the orthogonal projection operator from $L^2(G)$ into $\mathcal{U}$.

3. Observability estimate

We prove in this section an observability estimate which is adapted to the constraint, deriving from a global Carleman inequality due to A. V. Fursikov and O. Yu. Imanuvilov [7].
Let $\psi \in C^2(\overline{\Omega})$ be such that

$$\begin{cases} \psi(x) & > & 0 & \forall x \in \Omega, \\ \psi(x) & = & 0 & \forall x \in \Gamma, \\ |\nabla\psi(x)| & \neq & 0 & \forall x \in \overline{\Omega - \omega}. \end{cases} \tag{21}$$

Then, for any $\lambda \in \mathbb{R}_+^*$, define

$$\varphi(x,t) = \frac{e^{\lambda(m||\psi||_{L^\infty(\Omega)}+\psi(x))}}{t(T-t)}, \tag{22}$$

$$\eta(x,t) = \frac{e^{2\lambda m||\psi||_{L^\infty(\Omega}} - e^{\lambda(m||\psi||_{L^\infty(\Omega}+\psi(x))}}{t(T-t)}, \tag{23}$$

for $(x,t) \in Q$ and $m > 1$.
We introduce the following notations

$$\begin{cases} \mathcal{V} & = & \{\rho \in C^\infty(\overline{Q}); \rho|_\Sigma = 0\}, \\ L\rho & = & \frac{\partial \rho}{\partial t} - \Delta\rho + a_0(z)\rho, \\ L^*\rho & = & -\frac{\partial \rho}{\partial t} - \Delta\rho + a_0(z)\rho, \end{cases} \tag{24}$$

where $a_0 \in L^\infty(Q)$.
Carleman's inequality can be formulated as follows:

Proposition 3.1 (Global Carleman's inequality [7, 8])**.** *Let ψ, φ and η the functions defined respectively by (21), (22), (23). Then there exist*

$\lambda_0 = \lambda_0(\Omega,\omega) > 1$, $s_0 = s_0(\Omega,\omega,T) > 1$ *and* $C = C(\Omega,\omega) > 0$ *such that, for* $\lambda \geqslant \lambda_0$, $s \geqslant s_0$, *and for any* $\rho \in \mathcal{V}$, *we have*

$$\begin{aligned} &\int_Q \frac{e^{-2s\eta}}{s\varphi}\left(\left|\frac{\partial\rho}{\partial t}\right|^2 + |\Delta\rho|^2\right)dxdt + \int_Q s\lambda^2\varphi e^{-2s\eta}|\nabla\rho|^2 dxdt \\ &\quad + \int_Q s^3\lambda^4\varphi^3 e^{-2s\eta}|\rho|^2 dxdt \\ &\leqslant C\left(\int_Q e^{-2s\eta}|L^*\rho|^2 dxdt + \int_G s^3\lambda^4\varphi^3 e^{-2s\eta}|\rho|^2 dxdt\right). \end{aligned} \tag{25}$$

Since φ does not vanish on Q, we set

$$\theta = \varphi^{-\frac{3}{2}} e^{s\eta}, \tag{26}$$

and from (25), we deduce the following corollary:

Corollary 3.2 ([4] p.546)**.** *There exist a positive real function* θ *(given by (26)) and a positive constant* $C = C(\Omega,\omega,K,T)$ *such that for any* $\rho \in \mathcal{V}$, *we have*

$$\int_\Omega |\rho(0)|^2 dx + \int_Q \frac{1}{\theta^2}|\rho|^2 dxdt \leqslant C\left(\int_Q |L^*\rho|^2 dxdt + \int_G |\rho|^2 dxdt\right). \tag{27}$$

Now, we are going to state the adapted observability inequality. The proof will require the two following lemmas.

Lemma 3.3. *Assume (6). Let* $\mu \in L^\infty(Q)$ *and let* ψ_j, $1 \leqslant j \leqslant m$, *be the solution of*

$$\begin{cases} -\dfrac{\partial\psi_j}{\partial t} - \Delta\psi_j + \mu\psi_j = 0 & \text{in } Q, \\ \psi_j = e_j & \text{on } \Sigma_0, \\ \psi_j = 0 & \text{on } \Sigma\setminus\Sigma_0, \\ \psi_j(T) = 0 & \text{in } \Omega. \end{cases} \tag{28}$$

Let ρ *be a function in* $\mathrm{Span}(\{\psi_1\chi_\omega, \ldots, \psi_m\chi_\omega\})$ *satisfying*

$$\begin{cases} -\dfrac{\partial\rho}{\partial t} - \Delta\rho + \mu\rho = 0 & \text{in } Q, \\ \rho = 0 & \text{on } \Sigma. \end{cases} \tag{29}$$

Then ρ *is identically null on* G.

Proof. Let ρ be a function in $\mathrm{Span}(\{\psi_1\chi_\omega,\dots,\psi_m\chi_\omega\})$ satisfying (29). There exist $\gamma_j \in \mathbb{R}$, $1 \leqslant j \leqslant m$, such that $\rho = \sum_{j=1}^{m} \gamma_j \psi_j \chi_\omega$. Set $\sigma = \sum_{j=1}^{m} \gamma_j \psi_j$. Then we have according to (28),

$$\begin{cases} -\dfrac{\partial \sigma}{\partial t} - \Delta\sigma + \mu\sigma & = 0 \quad \text{in} \quad Q, \\ \sigma & = \sum_{j=1}^{m} \gamma_j e_j \chi_{\Sigma_0} \quad \text{on} \quad \Sigma. \end{cases}$$

Since

$$\begin{cases} -\dfrac{\partial(\sigma-\rho)}{\partial t} - \Delta(\sigma-\rho) + \mu(\sigma-\rho) & = 0 \quad \text{in} \quad Q, \\ \sigma - \rho & = 0 \quad \text{in} \quad G, \end{cases}$$

we deduce that $\sigma = \rho$ in Q. In particular $\sigma|_\Sigma = 0$, which implies that $\sum_{j=1}^{m} \gamma_j e_j \chi_{\Sigma_0} = 0$. From (6), we deduce that $\gamma_j = 0$ for all $j \in \{1,\dots,m\}$, then $\rho = 0$ in G. ■

Lemma 3.4 ([4, 6]). *Let $(H, (.,.)_H)$ be a Hilbert space. For $n \in \mathbb{N}^*$, let $\{p_{n1},\dots,p_{nm}\}$ be a set of m linearly independent vectors of H and let h_n in the span of $\{p_{n1},\dots,p_{nm}\}$. We assume that there exists a set of linearly independent vectors $\{p_1,\dots,p_m\}$ of H, such that*

$$p_{ni} \to p_i \text{ strongly in } H \text{ for } 1 \leqslant i \leqslant m. \tag{30}$$

We also assume also that there exists a positive constant C such that

$$||h_n||_H \leqslant C, \tag{31}$$

where $||h_n||_H = (h_n, h_n)_H^{1/2}$. Then we can extract a subsequence such that

$$h_n \to h \in \mathrm{Span}(\{p_1,\dots,p_m\}) \text{ strongly in } H.$$

Proposition 3.5. *There exists a positive constant $C = C(\Omega, \omega, K, T)$ such that for all $z \in L^2(Q)$ and $\rho \in \mathcal{V}$,*

(32)

$$\int_\Omega |\rho(0)|^2 dx + \int_Q \frac{1}{\theta^2}|\rho|^2 dxdt \leqslant C\left(\int_Q |L^*\rho|^2 dxdt + \int_G |\rho - P\rho|^2 dxdt\right).$$

Proof. To prove (32), we argue by contradiction. If (32) does not hold, then for any $n \in \mathbb{N}^*$, there exist a sequence z_n of $L^2(Q)$ and a sequence σ_n of $\mathcal{V}$ such that:

$$\int_\Omega |\sigma_n(0)|^2 dx + \int_Q \frac{1}{\theta^2}|\sigma_n|^2 dxdt = 1, \tag{33}$$

$$\int_Q |L_n^*\sigma_n|^2 dxdt < \frac{1}{n}, \tag{34}$$

$$\int_G |\sigma_n - P_n\sigma_n|^2 dxdt < \frac{1}{n}, \tag{35}$$

where $L_n^*\sigma_n = -\frac{\partial \sigma_n}{\partial t} - \Delta\sigma_n + a_0(z_n)\sigma_n$ and P_n denotes the orthogonal projection operator from $L^2(G)$ into $\mathcal{U}(z_n) = \mathrm{Span}(\{q_1(z_n)\chi_\omega, \ldots, q_m(z_n)\chi_\omega\})$. For any $n \in \mathbb{N}^*$, we have:

$$\int_G \frac{1}{\theta^2}|P_n\sigma_n|^2 dxdt \leqslant 2\Big(\int_G \frac{1}{\theta^2}|\sigma_n|^2 dxdt + \int_G \frac{1}{\theta^2}|\sigma_n - P_n\sigma_n|^2 dxdt\Big).$$

The term $\displaystyle\int_G \frac{1}{\theta^2}|\sigma_n|^2 dxdt$ is bounded according to (33). Since $\dfrac{1}{\theta^2}$ is bounded, it follows from (35) that there exists a positive constant C such that:

$$\int_G \frac{1}{\theta^2}|P_n\sigma_n|^2 dxdt \leqslant C.$$

Since $P_n\sigma_n \in \mathcal{U}(z_n)$ and $\mathcal{U}(z_n)$ is a finite dimensional vector subspace of $L^2(G)$, we deduce that:

$$\int_G |P_n\sigma_n|^2 dxdt \leqslant C. \tag{36}$$

But we have:

$$\int_G |\sigma_n|^2 dxdt \leqslant 2\Big(\int_G |P_n\sigma_n|^2 dxdt + \int_G |\sigma_n - P_n\sigma_n|^2 dxdt\Big).$$

Using (35) and (36), we deduce that:

$$\int_G |\sigma_n|^2 dxdt \leqslant C. \tag{37}$$

Consequently, there exist a subsequence of $(\sigma_n)_n$ (still denoted by $(\sigma_n)_n$) and $\sigma \in L^2(G)$ such that

$$\sigma_n \rightharpoonup \sigma \text{ weakly in } L^2(G). \tag{38}$$

Now in view of (33) and the definition of $\frac{1}{\theta}$, we deduce that $(\sigma_n)_n$ is bounded in $L^2((\mu, T-\mu) \times \Omega), \forall \mu > 0$. Extracting subsequences, we can deduce that:

$$\sigma_n \rightharpoonup \sigma \text{ weakly in } L^2((\mu, T-\mu) \times \Omega), \forall \mu > 0.$$

Therefore,

$$\sigma_n \rightarrow \sigma \text{ in } \mathcal{D}'(Q). \tag{39}$$

Since for any $z \in L^2(Q)$, $q_j(z)$, $1 \leqslant j \leqslant m$ is solution of (13) and $e_j \in H_0^1(\Sigma)$, one can prove that $q_j(z) \in H^{2,1}(Q)$.
Moreover there exists a positive constant C such that

$$||q_j(z_n)||_{H^{2,1}(Q)} \leqslant C||e_j||_{H_0^1(\Sigma)}. \tag{40}$$

By extracting subsequences we may deduce that there exist $\psi_j \in H^{2,1}(Q)$ such that for $j \in \{1, \ldots, m\}$

$$q_j(z_n) \rightharpoonup \psi_j \text{ weakly in } H^{2,1}(Q).$$

As a consequence of the Aubin-Lions compactness Lemma, the injection from $H^{2,1}(Q)$ into $L^2(Q)$ is compact so that for $1 \leqslant j \leqslant m$

$$q_j(z_n) \rightarrow \psi_j \text{ strongly in } L^2(Q). \tag{41}$$

On the other hand, using (10), there exists a positive constant $C = C(T, \Omega)$ such that

$$||a_0(z_n)||_{L^2(Q)} \leqslant C||a_0(z_n)||_{L^\infty(Q)} \leqslant CK.$$

Consequently, there exist a subsequence of $a_0(z_n)$ (still denoted by $a_0(z_n)$) and $\mu \in L^2(Q)$ such that

$$a_0(z_n) \rightharpoonup \mu \text{ weakly in } L^2(Q). \tag{42}$$

Therefore in view of (40)-(42), ψ_j, $1 \leqslant j \leqslant m$ is solution of

$$
(43) \qquad \begin{cases} -\dfrac{\partial \psi_j}{\partial t} - \Delta\psi_j + \mu\psi_j &= 0 \quad \text{in } Q, \\ \psi_j|_{\Sigma_0} &= e_j, \\ \psi_j|_{\Sigma\setminus\Sigma_0} &= 0, \\ \psi_j(T) &= 0 \quad \text{in } \Omega. \end{cases}
$$

Since $P_n\sigma_n \in \mathcal{U}(z_n)$ and satisfies (36), we can apply Lemma 3.4 with $H = L^2(G)$, $p_{ni} = q_j(z_n)\chi_\omega$, $h_n = P_n\sigma_n$. There exists $g \in \text{Span}(\{\psi_1\chi_\omega, \dots, \psi_m\chi_\omega\})$ such that

$$P_n\sigma_n \to g \text{ strongly in } L^2(G).$$

On the other hand, it follows from (35) that

$$(44) \qquad \sigma_n - P_n\sigma_n \to 0 \text{ strongly in } L^2(G).$$

We can deduce that

$$\sigma_n \to g \text{ strongly in } L^2(G).$$

Hence from (38), we have:

$$(45) \qquad \sigma_n \to \sigma = g \text{ strongly in } L^2(G).$$

We conclude that $\sigma\chi_\omega \in \text{Span}(\{\psi_1\chi_\omega, \dots, \psi_m\chi_\omega\})$.
Since $L_n^* = \frac{\partial}{\partial t} - \Delta + a_0(z_n)I$ is weakly continuous in $\mathcal{D}'(Q)$, we have according to (39) and (42),

$$L_n^*\sigma_n \to -\frac{\partial\sigma}{\partial t} - \Delta\sigma + \mu\sigma \text{ in } \mathcal{D}'(Q).$$

But (34) implies that

$$(46) \qquad L_n^*\sigma_n \to 0 \text{ strongly in } L^2(Q),$$

we deduce that $-\dfrac{\partial\sigma}{\partial t} - \Delta\sigma + \mu\sigma = 0$ in Q. Since $\sigma_n \in \mathcal{V}$ satisfies (37) and (46), we can apply (25) to σ_n and deduce that σ_n is bounded in $L^2(]\mu, T-\mu[; H^2(\Omega)), \forall\mu > 0$. Then for any $\mu > 0$,

$$\sigma_n \rightharpoonup \sigma \text{ weakly in } L^2(]\mu, T-\mu[\times\Gamma).$$

Consequently,

$$\sigma_n \to \sigma \text{ in } \mathcal{D}'(\Sigma).$$

Hence from $\sigma_n|_\Sigma = 0$, we have

$$\sigma|_\Sigma = 0.$$

So σ satisfies $\sigma\chi_\omega \in \mathrm{Span}(\{\psi_1\chi_\omega, \dots, \psi_m\chi_\omega\})$ and

$$\left\{\begin{array}{rcll} -\dfrac{\partial \sigma}{\partial t} - \Delta\sigma + \mu\sigma &=& 0 & \text{in } Q, \\ \sigma &=& 0 & \text{on } \Sigma. \end{array}\right.$$

Using Lemma 3.3, we deduce that:

$$\sigma = 0 \text{ in } G,$$

and (45) can be rewritten in the form

$$\sigma_n \to 0 \text{ strongly in } L^2(G).$$

As $(\sigma_n)_n$ satisfies (27), then

$$\int_\Omega |\sigma_n(0)|^2 dx + \int_Q \left|\frac{1}{\theta}\sigma_n\right|^2 dxdt \to 0,$$

which is in contradiction with (33). ■

Let us now give a proposition that we will need to prove estimation (9). The proof requires the following two lemmas:

Lemma 3.6. *Assume (6). Let θ be the function given by Proposition 2.2. Let q_j, $1 \leqslant j \leqslant m$ and u_0 respectively defined by system (13) and (20). For any $z \in L^2(Q)$, set*

$$A_\theta(z) = \int_G \frac{1}{\theta} q_i(z) q_j(z) dxdt, \ 1 \leqslant i, j \leqslant m.$$

Then there exists $\delta > 0$ such that for any $z \in L^2(Q)$,

$$(A_\theta(z)X(z), X(z))_{\mathbb{R}^m} \geqslant \delta ||X(z)||^2_{\mathbb{R}^m}, \tag{47}$$

where $X(z) = (X_1(z), \dots, X_m(z)) \in \mathbb{R}^m$ and

$$(A_\theta(z)X(z), X(z))_{\mathbb{R}^m} = \int_G \frac{1}{\theta}\Big(\sum_{i=1}^m X_i(z) q_i(z)\Big)\Big(\sum_{j=1}^m X_j(z) q_j(z)\Big) dxdt.$$

Proof. To prove (47), we argue by contradiction. If (47) does not hold, then for any $n \in \mathbb{N}^*$, there exist a sequence $(z_n)_n$ of $L^2(Q)$ and a vector $X(z_n) = (X_1(z_n), \dots, X_m(z_n))$ of $\mathbb{R}^m$ such that

$$(A_\theta(z_n)X(z_n), X(z_n))_{\mathbb{R}^m} \leqslant \frac{1}{n}||X(z_n)||^2_{\mathbb{R}^m}.$$

Set $\tilde{X}(z_n) = \dfrac{X(z_n)}{||X(z_n)||_{\mathbb{R}^m}}$, then

(48) $$||\tilde{X}(z_n)||_{\mathbb{R}^m} = \Big(\sum_{j=1}^{m} |\tilde{X}_j(z_n)|^2\Big)^{1/2} = 1,$$

(49) $$(A_\theta(z_n)\tilde{X}(z_n), \tilde{X}(z_n))_{\mathbb{R}^m} \leqslant \frac{1}{n}.$$

Consequently, there exist subsequences of $\tilde{X}_j(z_n)$, $1 \leqslant j \leqslant m$ (still denoted by $\tilde{X}_j(z_n)$) and $\tilde{X}_j \in \mathbb{R}$ such that for $1 \leqslant j \leqslant m$,

(50) $$\tilde{X}_j(z_n) \to \tilde{X}_j \text{ in } \mathbb{R}.$$

Moreover,

(51) $$||\tilde{X}||_{\mathbb{R}^m} = \Big(\sum_{j=1}^{m} |\tilde{X}_j|^2\Big)^{\frac{1}{2}} = 1.$$

Now let $\tilde{\phi}_n = \sum_{j=1}^{m} \tilde{X}_j(z_n) q_j(z_n)$. Then from (40), (41), (50) and Lemma 3.4, it follows that

$$\tilde{\phi}_n \to \sum_{j=1}^{m} \tilde{X}_j \psi_j := \tilde{\phi} \text{ strongly in } L^2(Q).$$

But we deduce from (49) that

$$\int_G \frac{1}{\theta} |\tilde{\phi}_n|^2 dxdt = (A_\theta(z_n)\tilde{X}(z_n), \tilde{X}(z_n))_{\mathbb{R}^m} \leqslant \frac{1}{n},$$

so $\displaystyle\int_G \frac{1}{\theta}|\tilde{\phi}|^2 dxdt = 0$ and $\tilde{\phi} = 0$ in G.
Since ψ_j, $1 \leqslant j \leqslant m$ is solution of (43), $\tilde{\phi}$ satisfies:

$$\left\{\begin{array}{rcll} -\dfrac{\partial \tilde{\phi}}{\partial t} - \Delta \tilde{\phi} + \mu \tilde{\phi} & = & 0 & \text{in } Q, \\ \tilde{\phi}|_{\Sigma_0} & = & \sum_{j=1}^{m} \tilde{X}_j e_j. & \end{array}\right.$$

We deduce that $\tilde{\phi} = 0$ in Q, which implies that $\sum_{j=1}^{m} \tilde{X}_j e_j = 0$ on Σ_0. In view of assumption (6), $\tilde{X}_j = 0$ for all $j \in \{1, \ldots, m\}$, which is in contradiction with (51). ■

Proposition 3.7. *Let θ be the function given by Proposition 2.2, and let q_j, $1 \leqslant j \leqslant m$ and u_0 respectively defined by system (13) and (20).*

Then there exists a positive constant $C = C(\Omega, \omega, K, T, \sum_{j=1}^{m} ||e_j||_{H_0^1(\Sigma)})$ *such that for any* $z \in L^2(Q)$,

$$
(52) \qquad \begin{aligned} ||u_0(z)||_{L^2(G)} &\leqslant C||y^0||_{L^2(\Omega)}, \\ ||\theta u_0(z)||_{L^2(G)} &\leqslant C||y^0||_{L^2(\Omega)}. \end{aligned}
$$

Proof. In view of (20), we have for any $z \in L^2(Q)$,

$$
(53) \qquad \int_G u_0(z) q_j(z) dxdt = -\int_\Omega y^0 q_j(z)(0) dx, \; 1 \leqslant j \leqslant m.
$$

Since $u_0(z) \in \mathcal{U}_\theta(z)$, there exists $\alpha(z) = (\alpha_1(z), \ldots, \alpha_m(z)) \in \mathbb{R}^m$ such that

$$
(54) \qquad u_0(z) = \sum_{i=1}^{m} \alpha_i(z) \frac{1}{\theta} q_i(z) \chi_\omega.
$$

So (53) can be rewritten in the form

$$
\int_G \sum_{i=1}^{m} \alpha_i(z) \frac{1}{\theta} q_i(z) q_j(z) dxdt = -\int_\Omega y^0 q_j(z)(0) dx, \; 1 \leqslant j \leqslant m.
$$

Therefore,

$$
(55) \qquad \int_G \frac{1}{\theta} \Big(\sum_{i=1}^{m} \alpha_i(z) q_i(z) \Big) \Big(\sum_{j=1}^{m} \alpha_j(z) q_j(z) \Big) dxdt = -\int_\Omega y^0 \sum_{j=1}^{m} \alpha_j(z) q_j(z)(0) dx.
$$

Now applying Lemma 3.6 to the left-hand-side of (55), we get

$$
\delta ||\alpha(z)||^2_{\mathbb{R}^m} \leqslant -\int_\Omega y^0 \sum_{j=1}^{m} \alpha_j(z) q_j(z)(0) dx.
$$

Using the Cauchy-Schwarz inequality for the right-hand-member of the latter identity, it follows that

$$
(56) \qquad \delta ||\alpha(z)||^2_{\mathbb{R}^m} \leqslant ||y^0||_{L^2(\Omega)} \sum_{j=1}^{m} |\alpha_j(z)|.||q_j(z)(0)||_{L^2(\Omega)}.
$$

Since q_j is solution of (13) for $1 \leqslant j \leqslant m$, we have in addition to (40), the following energy inequality,

$$
||q_j(z)(0)||_{L^2(\Omega)} \leqslant C||e_j||_{H_0^1(\Sigma)}.
$$

Consequently, we obtain according to (56),

$$||\alpha(z)||^2_{\mathbb{R}^m} \leqslant \delta^{-1}C||y^0||_{L^2(\Omega)}||\alpha(z)||_{\mathbb{R}^m}\left(\sum_{j=1}^m ||e_j||^2_{H^1_0(\Sigma)}\right)^{\frac{1}{2}}.$$

Moreover, if follows from (54) that for any $z \in L^2(Q)$,

$$||u_0(z)||_{L^2(G)} \leqslant \left|\left|\frac{1}{\theta}\right|\right|_{L^\infty(Q)} ||\alpha(z)||_{\mathbb{R}^m}\left(\sum_{i=1}^m ||q_i(z)||^2_{L^2(G)}\right)^{\frac{1}{2}},$$

$$||\theta u_0(z)||_{L^2(G)} \leqslant ||\alpha(z)||_{\mathbb{R}^m}\left(\sum_{i=1}^m ||q_i(z)||^2_{L^2(G)}\right)^{\frac{1}{2}}.$$

Hence

$$||u_0(z)||_{L^2(G)} \leqslant \left|\left|\frac{1}{\theta}\right|\right|_{L^\infty(Q)} \delta^{-1}C||y^0||_{L^2(\Omega)}\left(\sum_{i=1}^m ||e_j||^2_{H^1_0(\Sigma)}\right)^{\frac{1}{2}},$$

$$||\theta u_0(z)||_{L^2(G)} \leqslant \delta^{-1}C||y^0||_{L^2(\Omega)}\left(\sum_{i=1}^m ||e_j||^2_{H^1_0(\Sigma)}\right)^{\frac{1}{2}},$$

which ends the proof of the Proposition. ■

4. Null controllability of the linearized system

We begin by proving the existence of a solution for problem (16), (17), (18).

We define on $\mathcal{V} \times \mathcal{V}$ the following symmetric bilinear form:

(57) $$b(\rho,\sigma) = \int_Q L^*\rho L^*\sigma dxdt + \int_G (\rho - P\rho)(\sigma - P\sigma)dxdt.$$

In view of Proposition 3.5, this bilinear form is an inner product on $\mathcal{V}$. Let $V = \overline{\mathcal{V}}$ be the completion of the pre-Hilbert space $\mathcal{V}$ with respect to the norm

(58) $$b(\rho,\rho) = \left(\int_Q |L^*\rho|^2 dxdt + \int_G |\rho - P\rho|^2 dxdt\right)^{\frac{1}{2}}.$$

The completion V of $\mathcal{V}$ is a Hilbert space.

Lemma 4.1. *For any $\rho \in V$, let*

$$\mathcal{L}(\rho) = \int_Q u_0\chi_\omega \rho dxdt + \int_\Omega y^0\rho(0)dx.$$

Then for any $z \in L^2(Q)$*, there exists a unique* $\rho_\theta = \rho_\theta(z) \in V$ *such that*

$$b(\rho_\theta, \sigma) = \mathcal{L}(\sigma) \quad \forall \sigma \in V,$$

in other words,

$$\int_Q L^*\rho_\theta L^*\sigma dxdt + \int_G (\rho_\theta - P\rho_\theta)(\sigma - P\sigma)dxdt = \int_Q u_0\chi_\omega \sigma dxdt + \int_\Omega y^0\sigma(0)dx, \forall \sigma \in V. \tag{59}$$

Proof. According to the Cauchy-Schwarz inequality, the bilinear form $b(.,.)$ is continuous on $V \times V$ and by definition, it is coercive on V. Moreover for every $\sigma \in V$, it follows from (32) that, the linear form $\mathcal{L}$ is continuous on V. Therefore in view of the Lax-Milgram Theorem, for any $z \in L^2(Q)$, there exists a unique $\rho_\theta \in V$ such that for all $\sigma \in V$, we have:

$$b(\rho_\theta, \sigma) = \mathcal{L}(\sigma),$$

and the proof of Lemma 4.1 is complete. ∎

Proposition 4.2. *For any* $y^0 \in L^2(\Omega)$ *and* $z \in L^2(Q)$*, let* ρ_θ *be the unique solution of (59). We set*

$$u_\theta = -(\rho_\theta - P\rho_\theta)\chi_\omega, \tag{60}$$

$$y_\theta = L^*\rho_\theta. \tag{61}$$

Then (u_θ, y_θ) *is solution of the controllability problem (16), (17), (18). Moreover there exists a positive constant* $C = C(\Omega, \omega, K, T, \sum_{j=1}^m ||e_j||_{H_0^1(\Sigma)})$ *such that:*

$$||\rho_\theta||_V \leqslant C||y^0||_{L^2(\Omega)}, \tag{62}$$

$$||u_\theta||_{L^2(G)} \leqslant C||y^0||_{L^2(\Omega)}, \tag{63}$$

$$||y_\theta||_{L^2(Q)} \leqslant C||y^0||_{L^2(\Omega)}. \tag{64}$$

Proof. On the one hand, since $\rho_\theta \in V$, we have $u_\theta \in L^2(G)$ and $y_\theta \in L^2(Q)$. On the other hand, since $P\rho_\theta \in \mathcal{U}$, $u_\theta \in \mathcal{U}^\perp$. Replacing $-(\rho_\theta - P\rho_\theta)\chi_\omega$ and $L^*\rho_\theta$ respectively by u_θ and y_θ in (59), we get:

(65)
$$\int_Q y_\theta L^*\sigma dxdt - \int_G u_\theta(\sigma - P\sigma)dxdt = \int_Q u_0\chi_\omega\sigma dxdt + \int_\Omega y^0\sigma(0)dx,$$

for any $\sigma \in V$. In particular for $\phi \in \mathscr{D}(Q)$, we obtain:

$$(66) \quad \langle y_\theta, L^*\phi\rangle_{\mathscr{D}'(Q),\mathscr{D}(Q)} - \langle u_\theta\chi_\omega, \phi\rangle_{\mathscr{D}'(Q),\mathscr{D}(Q)} = \langle u_0\chi_\omega, \phi\rangle_{\mathscr{D}'(Q),\mathscr{D}(Q)}.$$

We deduce that

$$(67) \qquad Ly_\theta = (u_0 + u_\theta)\chi_\omega \text{ in } Q.$$

As $y_\theta \in L^2(Q) = L^2(0,T;L^2(\Omega))$, we have on the one hand $\frac{\partial y_\theta}{\partial t} \in H^{-1}(0,T;L^2(\Omega))$, and from (67),

$$\Delta y_\theta = \frac{\partial y_\theta}{\partial t} + a_0 y_\theta - (u_0 + u_\theta)\chi_\omega \in H^{-1}(0,T;L^2(\Omega))$$

since $a_0 y_\theta - (u_0 + u_\theta)\chi_\omega \in L^2(Q)$. Therefore, $y_\theta|_\Sigma$ and $\left.\dfrac{\partial y_\theta}{\partial \nu}\right|_\Sigma$ exist and belong respectively to $H^{-1}(0,T;H^{-\frac{1}{2}}(\Gamma))$ and $H^{-1}(0,T;H^{-\frac{3}{2}}(\Gamma))$ (see [9]).

On the other hand, $\Delta y_\theta \in L^2(0,T;H^{-2}(\Omega))$ and from (67), we have:

$$\frac{\partial y_\theta}{\partial t} = \Delta y_\theta - a_0 y_\theta + (u_0 + u_\theta)\chi_\omega \in L^2(0,T;H^{-2}(\Omega)).$$

Consequently, $t \mapsto y_\theta(x,t)$ is continuous from $[0,T]$ into $H^{-1}(\Omega)$, which means that $y_\theta(T)$ and $y_\theta(0)$ are well defined in $H^{-1}(\Omega)$ (see [9]).
Multiplying (67) by $\phi \in C^\infty(\overline{Q})$ then integrating by parts over Q yield:

$$\begin{aligned}(68) \quad &\langle y_\theta(T), \phi(T)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} - \langle y_\theta(0), \phi(0)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \\ &-\langle \frac{\partial y_\theta}{\partial \nu}, \phi\rangle_{H^{-1}(0,T;H^{-\frac{3}{2}}(\Gamma)),H_0^1(0,T;H^{\frac{3}{2}}(\Gamma))} + \langle y_\theta, \frac{\partial \phi}{\partial \nu}\rangle_{H^{-1}(0,T;H^{-\frac{1}{2}}(\Gamma)),H_0^1(0,T;H^{\frac{1}{2}}(\Gamma))} \\ &+ \int_Q y_\theta L^*\phi dxdt = \int_Q u_0\chi_\omega\phi dxdt + \int_G u_\theta\phi dxdt.\end{aligned}$$

In particular for ϕ such that $\phi = 0$ on Σ, we have according to (65),

$$\begin{aligned}&\langle y_\theta(T), \phi(T)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} - \langle y_\theta(0), \phi(0)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \\ &+ \langle y_\theta, \frac{\partial \phi}{\partial \nu}\rangle_{H^{-1}(0,T;H^{-\frac{1}{2}}(\Gamma)),H_0^1(0,T;H^{\frac{1}{2}}(\Gamma))} + \int_\Omega y^0\phi(0)dx = 0,\end{aligned}$$

which is equivalent to

$$\langle y_\theta(T), \phi(T)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} + \langle y^0 - y_\theta(0), \phi(0)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} + \langle y_\theta, \frac{\partial\phi}{\partial\nu}\rangle_{H^{-1}(0,T;H^{-\frac{1}{2}}(\Gamma)),H_0^1(0,T;H^{\frac{1}{2}}(\Gamma))} = 0.$$

Choosing successively ϕ such that $\phi(T) = \phi(0) = 0$ in Ω, then $\phi(0) = 0$ in Ω, we conclude that:

$$\begin{cases} y_\theta &= 0 \quad \text{on } \Sigma, \\ y_\theta(T) &= 0 \quad \text{in } \Omega, \\ y_\theta(0) &= y^0 \quad \text{in } \Omega. \end{cases}$$

We deduce that (u_θ, y_θ) is solution of (16), (17), (18).

Now let us take $\sigma = \rho_\theta$ in (59), we have

$$(69) \qquad ||y_\theta||^2_{L^2(Q)} + ||u_\theta||^2_{L^2(G)} = \int_Q u_0\chi_\omega\rho_\theta dxdt + \int_\Omega y^0\rho_\theta(0)dx,$$

which according to the definition of the norm in V given by (58), is equivalent to

$$(70) \qquad ||\rho_\theta||^2_V = \int_Q u_0\chi_\omega\rho_\theta dxdt + \int_\Omega y^0\rho_\theta(0)dx.$$

Therefore, it follows from the Cauchy-Schwarz inequality and (32) that

$$(71) \qquad ||\rho_\theta||^2_V \leqslant C(||\theta u_0\chi_\omega||_{L^2(Q)} + ||y^0||_{L^2(\Omega)})||\rho_\theta||_V.$$

Applying Proposition 3.7, (71) can be reduced to (62). (63) and (64) follow from (69) and (70). ■

Proposition 4.3. *For any $z \in L^2(Q)$, there exists a unique control $\hat{u} = \hat{u}(z)$ such that*

$$||\hat{u}||_{L^2(G)} = \min\{||u||_{L^2(G)}, u \in \mathcal{F}\}$$

where $\mathcal{F} = \{u = u(z) \in L^2(G); (u, y)$ satisfies (16), (17), (18)$\}$. Moreover, there exists a positive constant $C = C(\Omega, \omega, K, T, \sum_{j=1}^m ||e_j||_{H_0^1(\Sigma)})$ such that

$$(72) \qquad ||\hat{u}||_{L^2(G)} \leqslant C||y^0||_{L^2(\Omega)}.$$

Proof. Proposition 4.2 guarantees that the set $\mathcal{F}$ is non empty. Since $\mathcal{F}$ is a closed convex subset of $L^2(G)$, we deduce the existence and the uniqueness of the optimal control $\hat{u}$. Therefore

$$||\hat{u}||_{L^2(G)} \leqslant ||u_\theta||_{L^2(G)} \leqslant C||y^0||_{L^2(\Omega)}.$$

■

We now arrive at the main result of this section.

Theorem 4.4. *Assume (6). Then for any $z \in L^2(Q)$, there exists a unique control $\tilde{u} = \tilde{u}(z)$ of minimal norm in $L^2(G)$ such that $(\tilde{u}, \tilde{y})$ is solution of the null controllability problem with constraint on the control (16), (17), (18). Furthermore, the control $\tilde{u}$ is given by*

$$\tilde{u} = \tilde{\rho}\chi_\omega - P\tilde{\rho}, \tag{73}$$

where $\tilde{\rho} = \tilde{\rho}(z)$ satisfies

$$\begin{cases} L^*\tilde{\rho} &= 0 \quad \text{in } Q, \\ \tilde{\rho}|_\Sigma &= 0. \end{cases} \tag{74}$$

Moreover, there exists a positive constant $C = C(\Omega, \omega, K, T, \sum_{j=1}^m ||e_j||_{H_0^1(\Sigma)})$ such that

$$||\tilde{\rho}||_V \leqslant C||y^0||_{L^2(\Omega)}, \tag{75}$$

$$||\tilde{\rho}||_{L^2(G)} \leqslant C||y^0||_{L^2(\Omega)}. \tag{76}$$

Proof. We divide the proof into three steps.
Step 1: Let $\varepsilon > 0$ and $z \in L^2(Q)$, and let $\mathcal{A}$ be given by

$$\mathcal{A} = \{(u, y); u = u(z) \in \mathcal{U}^\perp, y = y(z) \in L^2(Q), Ly \in L^2(Q), y|_\Sigma = 0, y(T) = 0 \text{ in } \Omega \text{ and } y(0) = y^0 \text{ in } \Omega\}.$$

For every pair (u, y) of $\mathcal{A}$, we define the functional

$$J_\varepsilon(u, y) = \frac{1}{2}||u||^2_{L^2(G)} + \frac{1}{2\varepsilon}||Ly - (u_0 + u)\chi_\omega||^2_{L^2(Q)}, \tag{77}$$

and we consider the optimal control problem:

$$\inf\{J_\varepsilon(u, y)|(u, y) \in \mathcal{A}\}. \tag{78}$$

We show that for every $\varepsilon > 0$, problem (78) has a unique solution. Indeed, since $(u_\theta, y_\theta) \in \mathcal{A}$, $\mathcal{A} \neq \emptyset$ and J_ε is bounded from below (by

0), we deduce that $\inf\{J_\varepsilon(u,y);(u,y)\in\mathcal{A}\} := I_\varepsilon$ exists. Let $(u_n,y_n) = (u(z_n),y(z_n))$ be a minimising sequence of $\mathcal{A}$, so $\exists n_0\in\mathbb{N}$, $\forall n\geqslant n_0$,

$$I_\varepsilon \leqslant J_\varepsilon(u_n,y_n) < I_\varepsilon + \frac{1}{n}. \tag{79}$$

But we have

$$I_\varepsilon \leqslant J_\varepsilon(u_\theta,y_\theta) = \frac{1}{2}||u_\theta||^2_{L^2(G)}. \tag{80}$$

Consequently in view of (79), (80) and (63), there exists a positive constant $C = C(\Omega,\omega,K,T,\sum_{j=1}^m ||e_j||_{H_0^1(\Sigma_0)})$ such that $J_\varepsilon(u_n,y_n) < C^2||y^0||^2_{L^2(\Omega)}$. Due to (77), we have

$$||u_n||_{L^2(G)} \leqslant C||y^0||_{L^2(\Omega)}, \tag{81}$$

$$||L_ny_n - (u_0+u_n)\chi_\omega||_{L^2(Q)} \leqslant C\sqrt{\varepsilon}||y^0||_{L^2(\Omega)}, \tag{82}$$

with $L_n = \frac{\partial}{\partial t} - \Delta + a_0(z_n)I$. Combining (81) and (82), we have according to (52),

$$||L_ny_n||_{L^2(Q)} \leqslant C\sqrt{\varepsilon}||y^0||_{L^2(\Omega)}. \tag{83}$$

It follows from (81) and (83) that there exist a subsequence of (u_n) (still denoted by (u_n)), a subsequence of (y_n) (still denoted by (y_n)), $u_\varepsilon = u_\varepsilon(z)\in L^2(G)$ and $\xi_\varepsilon\in L^2(Q)$ such that

$$u_n \rightharpoonup u_\varepsilon \text{ weakly in } L^2(G), \tag{84}$$

$$L_ny_n \rightharpoonup \xi_\varepsilon \text{ weakly in } L^2(Q), \tag{85}$$

and we have $u_\varepsilon\in\mathcal{U}^\perp$ which is a closed vector subspace of $L^2(G)$. Let $W(0,T)$ be defined by

$$W(0,T) = \{\phi\in L^2(0,T;H_0^1(\Omega)), \frac{\partial\phi}{\partial t}\in L^2(0,T;H^{-1}(\Omega))\}.$$

Since $(u_n,y_n)\in\mathcal{A}$, we have $y_n\in W(0,T)$ and due to (83) and the regularizing effect of the heat equation, we can write:

$$||y_n||_{W(0,T)} \leqslant C\sqrt{\varepsilon}||y^0||_{L^2(\Omega)}. \tag{86}$$

So there exist a subsequence of (y_n) (still denoted by (y_n)) and $y_\varepsilon = y_\varepsilon(z)\in W(0,T)$ such that

$$y_n \rightharpoonup y_\varepsilon \text{ weakly in } W(0,T). \tag{87}$$

But for any $\phi\in\mathscr{D}(Q)$, we have:

$$\langle L_ny_n,\phi\rangle_{\mathscr{D}'(Q),\mathscr{D}(Q)} = \langle y_n, L_n^*\phi\rangle_{\mathscr{D}'(Q),\mathscr{D}(Q)},$$

with $L_n^* = -\frac{\partial}{\partial t} - \Delta + a_0(z_n)I$. Passing to the limit $n \to +\infty$ in the latter equality, we get using (85), (87) and (42):
(88)
$$\langle \xi_\varepsilon, \phi \rangle_{\mathscr{D}'(Q),\mathscr{D}(Q)} = \langle y_\varepsilon, -\frac{\partial \phi}{\partial t} - \Delta\phi + \mu\phi \rangle_{\mathscr{D}'(Q),\mathscr{D}(Q)} = \langle \frac{\partial y_\varepsilon}{\partial t} - \Delta y_\varepsilon + \mu y_\varepsilon, \phi \rangle_{\mathscr{D}'(Q),\mathscr{D}(Q)}.$$

Thus $\frac{\partial y_\varepsilon}{\partial t} - \Delta y_\varepsilon + \mu y_\varepsilon = \xi_\varepsilon$ and (85) can be rewritten in the form:

$$L_n y_n \rightharpoonup \frac{\partial y_\varepsilon}{\partial t} - \Delta y_\varepsilon + \mu y_\varepsilon \text{ weakly in } L^2(Q). \tag{89}$$

Since $\frac{\partial y_\varepsilon}{\partial t} - \Delta y_\varepsilon + \mu y_\varepsilon \in L^2(Q)$ and $y_\varepsilon \in L^2(0,T;H_0^1(\Omega))$, we can define as in page 18, $y_\varepsilon|_\Sigma$ in $H^{-1}(0,T,H^{-\frac{1}{2}}(\Gamma))$, $\frac{\partial y_\varepsilon}{\partial \nu}\Big|_\Sigma$ in $H^{-1}(0,T,H^{-\frac{3}{2}}(\Gamma))$, $y_\varepsilon(0)$ and $y_\varepsilon(T)$ in $H^{-1}(\Omega)$.
Now let $\phi \in C^\infty(\overline{Q})$ be such that $\phi|_\Sigma = 0$. Using Green's Formula, we have

$$\int_Q (L_n y_n)\phi dxdt = -\int_\Omega y^0\phi(0)dx + \int_Q y_n(L_n^*\phi)dxdt.$$

In view of (89), (87) and (42), we can pass to the limit $n \to +\infty$ in the previous relation:

$$\begin{aligned}\int_Q \Big(\frac{\partial y_\varepsilon}{\partial t} - \Delta y_\varepsilon + \mu y_\varepsilon\Big)\phi dxdt &= -\int_\Omega y^0\phi(0)dx + \int_Q y_\varepsilon\Big(-\frac{\partial \phi}{\partial t} - \Delta\phi + \mu\phi\Big)dxdt \\ &= -\int_\Omega y^0\phi(0)dx - \langle y_\varepsilon(T), \phi(T)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \\ &\quad + \langle y_\varepsilon(0), \phi(0)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \\ &\quad - \langle y_\varepsilon, \frac{\partial \phi}{\partial \nu}\rangle_{H^{-1}(0,T,H^{-1/2}(\Gamma)),H_0^1(0,T,H^{1/2}(\Gamma))} \\ &\quad + \int_Q \Big(\frac{\partial y_\varepsilon}{\partial t} - \Delta y_\varepsilon + \mu y_\varepsilon\Big)\phi dxdt, \\ &\quad \forall \phi \in C^\infty(\overline{Q}) \text{ such that } \phi|_\Sigma = 0.\end{aligned}$$

Hence,

$$\begin{aligned}&\langle y_\varepsilon(0) - y^0, \phi(0)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} - \langle y_\varepsilon(T)\phi(T)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \\ &\quad - \langle y_\varepsilon, \frac{\partial \phi}{\partial \nu}\rangle_{H^{-1}(0,T,H^{-1/2}(\Gamma)),H_0^1(0,T,H^{1/2}(\Gamma))} = 0,\ \forall \phi \in C^\infty(\overline{Q}) \\ &\qquad \text{such that } \phi|_\Sigma = 0.\end{aligned}$$

Choosing successively ϕ such that $\phi(0) = \phi(T) = 0$ in Ω, then $\phi(T) = 0$ in Ω, we find that y_ε satisfies:

$$\left\{\begin{array}{rcll} y_\varepsilon|_\Sigma &=& 0, & \\ y_\varepsilon(0) &=& y^0 & \text{in } \Omega, \\ y_\varepsilon(T) &=& 0 & \text{in } \Omega. \end{array}\right. \tag{90}$$

Consequently $(u_\varepsilon, y_\varepsilon) \in \mathcal{A}$. J being lower semicontinuous, we have

$$J_\varepsilon(u_\varepsilon, y_\varepsilon) \leqslant \varliminf_{n\to+\infty} J_\varepsilon(u_n, y_n) = I_\varepsilon.$$

Therefore $J_\varepsilon(u_\varepsilon, y_\varepsilon) = I_\varepsilon$; the uniqueness is the consequence of the strict convexity of J_ε.

Step 2: We give the optimality system which characterizes the optimal solution of problem (78). The Euler-Lagrange optimality conditions which characterize $(u_\varepsilon, y_\varepsilon)$ are given by:

$$\begin{gathered} \frac{d}{d\mu} J_\varepsilon(u_\varepsilon + \mu u, y_\varepsilon)|_{\mu=0} = 0, \ \forall u \in \mathcal{U}^\perp, \\ \frac{d}{d\mu} J_\varepsilon(u_\varepsilon, y_\varepsilon + \mu\phi)|_{\mu=0} = 0, \ \forall \phi \in C^\infty(\overline{Q}) \\ \text{such that } \phi|_\Sigma = 0, \phi(0) = \phi(T) = 0 \text{ in } \Omega. \end{gathered}$$

After some calculations, we have

$$\int_G u_\varepsilon u\,dxdt - \frac{1}{\varepsilon}\int_Q \Big(Ly_\varepsilon - (u_0 + u_\varepsilon)\chi_\omega\Big) u\chi_\omega\,dxdt = 0, \forall u \in \mathcal{U}^\perp, \tag{91}$$

$$\frac{1}{\varepsilon}\int_Q \Big(Ly_\varepsilon - (u_0 + u_\varepsilon)\chi_\omega\Big) L\phi\,dxdt = 0, \forall \phi \in C^\infty(\overline{Q}) \tag{92}$$

$$\text{such that } \phi|_\Sigma = 0, \phi(0) = \phi(T) = 0 \text{ in } \Omega.$$

Set $\rho_\varepsilon = \frac{1}{\varepsilon}\Big(Ly_\varepsilon - (u_0 + u_\varepsilon)\chi_\omega\Big)$. Then $\rho_\varepsilon = \rho_\varepsilon(z) \in L^2(Q)$ and we have

$$Ly_\varepsilon = (u_0 + u_\varepsilon)\chi_\omega + \varepsilon\rho_\varepsilon \text{ in } Q,$$

which in addition to (90), gives:

$$\left\{\begin{array}{rcll} Ly_\varepsilon &=& (u_0 + u_\varepsilon)\chi_\omega + \varepsilon\rho_\varepsilon & \text{in } Q, \\ y_\varepsilon|_\Sigma &=& 0, & \\ y_\varepsilon(0) &=& y^0 & \text{in } \Omega, \\ y_\varepsilon(T) &=& 0 & \text{in } \Omega. \end{array}\right. \tag{93}$$

Replacing $\frac{1}{\varepsilon}\Big(Ly_\varepsilon - (u_0+u_\varepsilon)\chi_\omega\Big)$ by ρ_ε in (91) and (92) yields,

$$\int_G u_\varepsilon u dxdt - \int_Q \rho_\varepsilon u\chi_\omega dxdt = 0,\ \forall u \in \mathcal{U}^\perp, \tag{94}$$

$$\int_Q \rho_\varepsilon L\phi dxdt = 0,\ \forall \phi \in C^\infty(\overline{Q}) \tag{95}$$
$$\text{such that } \phi|_\Sigma = 0, \phi(0) = \phi(T) = 0 \text{ in } \Omega.$$

Relation (94) is equivalent to $\displaystyle\int_G (u_\varepsilon - \rho_\varepsilon)udxdt = 0,\ \forall u \in \mathcal{U}^\perp$, hence $u_\varepsilon - \rho_\varepsilon\chi_\omega \in \mathcal{U}$. We deduce that $u_\varepsilon - \rho_\varepsilon\chi_\omega = P(u_\varepsilon - \rho_\varepsilon\chi_\omega)$ and since $u_\varepsilon \in \mathcal{U}^\perp$, we can write $u_\varepsilon = \rho_\varepsilon\chi_\omega - P\rho_\varepsilon$.
Relation (95) holds in particular for $\phi \in \mathscr{D}(Q)$,

$$\langle \rho_\varepsilon, L\phi\rangle_{\mathscr{D}'(Q),\mathscr{D}(Q)} = \langle L^*\rho_\varepsilon, \phi\rangle_{\mathscr{D}'(Q),\mathscr{D}(Q)} = 0.$$

Consequently,

$$L^*\rho_\varepsilon = 0 \text{ in } Q. \tag{96}$$

Since $L^*\rho_\varepsilon \in L^2(Q)$ and $\rho_\varepsilon \in L^2(Q)$, we can define as in page 18, $\rho_\varepsilon|_\Sigma$ in
$H^{-1}(0,T,H^{-\frac{1}{2}}(\Gamma))$, $\dfrac{\partial\rho_\varepsilon}{\partial\nu}\Big|_\Sigma$ in $H^{-1}(0,T,H^{-\frac{3}{2}}(\Gamma))$, $\rho_\varepsilon(0)$ and $\rho_\varepsilon(T)$ in $H^{-1}(\Omega)$.
Multiplying (96) by $\phi \in C^\infty(\overline{Q})$, then integrating by parts over Q, we have:

$$\begin{aligned} & - \langle \rho_\varepsilon(T), \phi(T)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} + \langle \rho_\varepsilon(0), \phi(0)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \\ & -\langle \frac{\partial\rho_\varepsilon}{\partial\nu}, \phi\rangle_{H^{-1}(0,T;H^{-\frac{3}{2}}(\Gamma)),H_0^1(0,T;H^{\frac{3}{2}}(\Gamma))} + \langle \rho_\varepsilon, \frac{\partial\phi}{\partial\nu}\rangle_{H^{-1}(0,T,H^{-\frac{1}{2}}(\Gamma)),H_0^1(0,T,H^{\frac{1}{2}}(\Gamma))} \\ & + \int_Q \rho_\varepsilon L\phi dxdt = 0. \end{aligned} \tag{97}$$

Choosing ϕ such that $\phi|_\Sigma = 0, \phi(0) = \phi(T) = 0$ in Ω, and using (95), relation (97) can be rewritten in the form:

$$\langle \rho_\varepsilon, \frac{\partial\phi}{\partial\nu}\rangle_{H^{-1}(0,T,H^{-\frac{1}{2}}(\Gamma)),H_0^1(0,T,H^{\frac{1}{2}}(\Gamma))} = 0, \text{ for any function } \phi \in C^\infty(\overline{Q}) \text{ such that } \phi|_\Sigma = 0, \phi(0) = \phi(T) = 0 \text{ in } \Omega,$$

and we conclude that $\rho_\varepsilon|_\Sigma = 0$.

In summary, we have proved that $(u_\varepsilon, y_\varepsilon)$ is the optimal solution of (78) if and only if there exists a function ρ_ε such that the triplet $(u_\varepsilon, y_\varepsilon, \rho_\varepsilon)$ satisfies the following optimality system:

$$u_\varepsilon = \rho_\varepsilon \chi_\omega - P\rho_\varepsilon \tag{98}$$

$$\begin{cases} Ly_\varepsilon &= (u_0 + u_\varepsilon)\chi_\omega + \varepsilon\rho_\varepsilon & \text{in } Q, \\ y_\varepsilon|_\Sigma &= 0, & \\ y_\varepsilon(0) &= y^0 & \text{in } \Omega, \\ y_\varepsilon(T) &= 0 & \text{in } \Omega, \end{cases} \tag{99}$$

$$\begin{cases} L^*\rho_\varepsilon &= 0 & \text{in } Q, \\ \rho_\varepsilon|_\Sigma &= 0. \end{cases} \tag{100}$$

Step 3: We establish some useful estimates and we end the proof of the main theorem.

From (81), (82), (84) and (89), there exists a positive constant $C = C(\Omega, \omega, K, T, \sum_{j=1}^{m} ||e_j||_{H_0^1(\Sigma)})$ such that:

$$||u_\varepsilon||_{L^2(G)} \leqslant C||y^0||_{L^2(\Omega)}, \tag{101}$$

$$||Ly_\varepsilon - (u_0 + u_\varepsilon)\chi_\omega||_{L^2(Q)} \leqslant C\sqrt{\varepsilon}||y^0||_{L^2(\Omega)}. \tag{102}$$

Relation (102) and the fact that y_ε is solution of (99) imply:

$$||y_\varepsilon||_{W(0,T)} \leqslant C||y^0||_{L^2(\Omega)}. \tag{103}$$

In view of (98), it follows from (101) that:

$$||\rho_\varepsilon \chi_\omega - P\rho_\varepsilon||_{L^2(G)} \leqslant C||y^0||_{L^2(\Omega)}, \tag{104}$$

and since ρ_ε satisfies (100),

$$||\rho_\varepsilon||_V \leqslant C||y^0||_{L^2(\Omega)}. \tag{105}$$

Now applying inequality (32) to ρ_ε yields,

$$\Big|\Big|\frac{1}{\theta}\rho_\varepsilon\Big|\Big|_{L^2(Q)} \leqslant C||y^0||_{L^2(\Omega)}.$$

We have $\Big|\Big|\frac{1}{\theta}P\rho_\varepsilon\Big|\Big|_{L^2(G)} \leqslant \Big|\Big|\frac{1}{\theta}(\rho_\varepsilon\chi_\omega - P\rho_\varepsilon)\Big|\Big|_{L^2(G)} + \Big|\Big|\frac{1}{\theta}\rho_\varepsilon\chi_\omega\Big|\Big|_{L^2(G)}$ and since $\frac{1}{\theta} \in L^\infty(Q)$, then

$$\Big|\Big|\frac{1}{\theta}P\rho_\varepsilon\Big|\Big|_{L^2(G)} \leqslant C||y^0||_{L^2(\Omega)}.$$

Since $P\rho_\varepsilon \in \mathcal{U}$ and $\mathcal{U}$ is a finite dimensional vector subspace of $L^2(G)$, we deduce that:

$$||P\rho_\varepsilon||_{L^2(G)} \leqslant C||y^0||_{L^2(\Omega)}. \tag{106}$$

Using again (104), we obtain

$$||\rho_\varepsilon||_{L^2(G)} \leqslant C||y^0||_{L^2(\Omega)}. \tag{107}$$

By extracting subsequences, we have according to (101), (103), (105), (106) and (107),

$$u_\varepsilon \rightharpoonup \tilde{u} = \tilde{u}(z) \text{ weakly in } L^2(G), \tag{108}$$

$$y_\varepsilon \rightharpoonup \tilde{y} = \tilde{y}(z) \text{ weakly in } W(0,T), \tag{109}$$

$$\rho_\varepsilon \rightharpoonup \tilde{\rho} = \tilde{\rho}(z) \text{ weakly in } V, \tag{110}$$

$$P\rho_\varepsilon \rightharpoonup \tilde{\nu} = \tilde{\nu}(z) \text{ weakly in } L^2(G), \tag{111}$$

$$\rho_\varepsilon \rightharpoonup \tilde{\rho} \text{ weakly in } L^2(G), \tag{112}$$

and so $\tilde{u} \in \mathcal{U}^\perp$, $\tilde{\nu} \in \mathcal{U}$.
Since the injection from $W(0,T)$ into $L^2(Q)$ is compact, $(\tilde{u}, \tilde{y})$ is solution of the null controllability problem with constraint on the control (16), (17), (18).
We know, based on Proposition 4.3, that there exists a unique control $\hat{u}$ of minimal norm in $L^2(G)$, such that problem (16), (17), (18) holds. So, we have

$$\frac{1}{2}||\hat{u}||^2_{L^2(G)} \leqslant \frac{1}{2}||\tilde{u}||^2_{L^2(G)}.$$

Now, let $\hat{y}$ be the solution of (17) corresponding to $\hat{u}$; then we have

$$\frac{\partial \hat{y}}{\partial t} - \Delta\hat{y} + a_0(z)\hat{y} = (u_0 + \hat{u})\chi_\omega \text{ in } Q.$$

$(u_\varepsilon, y_\varepsilon)$ being the optimal solution of (78), we have:

$$\frac{1}{2}||u_\varepsilon||^2_{L^2(G)} \leqslant J_\varepsilon(u_\varepsilon, y_\varepsilon) \leqslant J_\varepsilon(\hat{u}, \hat{y}) = \frac{1}{2}||\hat{u}||^2_{L^2(G)}. \tag{113}$$

But because of (108), we can also write:

$$\frac{1}{2}||\tilde{u}||^2_{L^2(G)} \leqslant \liminf_{\varepsilon\to 0} \frac{1}{2}||u_\varepsilon||^2_{L^2(G)}, \tag{114}$$

and we deduce that $\tilde{u} = \hat{u}$.
In view of (114), (113) and (72), the following estimate holds:

$$||\tilde{u}||_{L^2(G)} \leqslant C||y^0||_{L^2(\Omega)}, \tag{115}$$

where $C = C(\Omega, \omega, K, T, \sum_{j=1}^{m} ||e_j||_{H_0^1(\Sigma)})$ is a positive constant.
On the other hand, ρ_ε satisfying (100), it follows from (110) that,

$$\begin{cases} L^*\tilde{\rho} = 0 & \text{in } Q, \\ \tilde{\rho}|_\Sigma = 0. \end{cases}$$

Furthemore, (104) implies that

$$\rho_\varepsilon \chi_\omega - P\rho_\varepsilon \rightharpoonup \tilde{\varsigma} \text{ weakly in } L^2(G),$$

so $\tilde{\varsigma} = \tilde{\varsigma}(z) \in \mathcal{U}^\perp$ and in view of (111) and (112), $\tilde{\rho} = \tilde{\nu} + \tilde{\varsigma}$. We conclude that $\tilde{\nu} = P\tilde{\rho}$ and

$$\rho_\varepsilon \chi_\omega - P\rho_\varepsilon \rightharpoonup \tilde{\rho} - P\tilde{\rho} \text{ weakly in } L^2(G).$$

(75) and (76) are consequences of (105), (107), (110) and (112), which establishes Theorem 4.4. ■

5. Proof of Theorem 1.1

For any $z \in L^2(Q)$, we showed that there exists a unique control $\tilde{u} = \tilde{u}(z)$ such that $(\tilde{u}, \tilde{y}(\tilde{u}))$ satisfies (16), (17), (18). Therefore in view of Proposition 2.2, there exists a unique control $\tilde{v} = \tilde{v}(z)$ satisfying

$$\tilde{v} = (u_0 + \tilde{u})\chi_\omega, \tag{116}$$

solution of the null controllability problem with constraint on the normal derivative (12), (7), (8). As a consequence of (116), (115) and (52), we have

$$||\tilde{v}||_{L^2(G)} \leqslant C||y^0||_{L^2(\Omega)}. \tag{117}$$

Thus, we have built a non-linear mapping

$$\begin{array}{rccc} \mathcal{S}: & L^2(Q) & \to & L^2(Q) \\ & z & \mapsto & S(z) = \tilde{y}(\tilde{v}) \end{array}$$

where $\tilde{y}(\tilde{v})$ is the solution of (12) corresponding to the control $\tilde{v} = (u_0 + \tilde{u})\chi_\omega$, with $u_0 \in \mathcal{U}_\theta$ and $\tilde{u} \in \mathcal{U}^\perp$ is defined by (73) and (74). The problem is then reduced to finding a fixed point of $\mathcal{S}$. Indeed, if $z \in L^2(Q)$ is such that $\mathcal{S}(z) = \tilde{y}(\tilde{v}) = z$, the solution $\tilde{y}$ of (12) is actually solution of (11). Then, the control $\tilde{v}$ is the one we were looking for, since by construction, $\tilde{y}(\tilde{v})$ satisfies (7) and (8).
In order to conclude the existence of a fixed point of $\mathcal{S}$, we can use

Schauder's fixed point Theorem. So it is sufficient to check the following three properties:

$$\mathcal{S} \text{ is continuous,}$$

$$\mathcal{S} \text{ is compact,}$$

$$\text{the range of } \mathcal{S} \text{ is bounded, i.e. } \exists R > 0; ||\mathcal{S}(z)||_{L^2(Q)} \leqslant R, \forall z \in L^2(Q).$$

5.1. **Continuity of $\mathcal{S}$.** We divide the proof into five steps.

Step 1: Let $(z_n)_n$ be a sequence of $L^2(Q)$ and assume that $z_n \to z$ strongly in $L^2(Q)$. Then there exists a subsequence $(z_{n_k})_k$ such that $z_{n_k}(x) \to z(x)$ almost everywhere in Q. f being a function of class C^1, the function a_0 is continuous and is such that

$$a_0(z_{n_k}(x)) \to a_0(z(x)) \text{ almost everywhere in } Q.$$

In view of (10), we have $|a_0(z_{n_k}(x))| \leqslant K$ and as a consequence of Lebesgue's Theorem,

$$a_0(z_{n_k}) \to a_0(z) \text{ strongly in } L^2(Q). \tag{118}$$

Step 2: The control $\tilde{v}_{n_k} = \tilde{v}(z_{n_k})$ is such that the solution $\tilde{y}_{n_k} = \tilde{y}(\tilde{v}_{n_k})$ of

$$\left\{\begin{array}{rcll} \frac{\partial \tilde{y}_{n_k}}{\partial t} - \Delta \tilde{y}_{n_k} + a_0(z_{n_k})\tilde{y}_{n_k} &=& \tilde{v}_{n_k}\chi_\omega & \text{in } Q, \\ \tilde{y}_{n_k}|_\Sigma &=& 0, & \\ \tilde{y}_{n_k}(0) &=& y^0 & \text{in } \Omega, \end{array}\right. \tag{119}$$

satisfies

$$\langle \frac{\partial \tilde{y}_{n_k}}{\partial \nu}, e_j \rangle_{H^{-1}(\Sigma_0), H_0^1(\Sigma_0)} = 0; j = 1, \ldots, m, \tag{120}$$

and

$$\tilde{y}_{n_k}(T) = 0 \text{ in } \Omega. \tag{121}$$

Moreover, $\tilde{v}_{n_k}$ is given by

$$\tilde{v}_{n_k} = (u_0(z_{n_k}) + \tilde{u}_{n_k})\chi_\omega, \tag{122}$$

where on the one hand, $u_0(z_{n_k}) \in \mathcal{U}_\theta(z_{n_k}) = \mathrm{Span}(\{\frac{1}{\theta}q_1(z_{n_k})\chi_\omega, \ldots, \frac{1}{\theta}q_m(z_{n_k})\chi_\omega\})$ satisfies in view of (20),

$$\int_G u_0(z_{n_k})q_j(z_{n_k})dxdt = -\int_\Omega y^0 q_j(z_{n_k})(0)dx, \tag{123}$$

with $q_j(z_{n_k})$ solution of

(124)
$$\begin{cases} -\dfrac{\partial q_j(z_{n_k})}{\partial t} - \Delta q_j(z_{n_k}) + a_0(z_{n_k}) q_j(z_{n_k}) & = 0 \quad \text{in} \quad Q, \\ q_j(z_{n_k}) & = e_j \quad \text{on} \quad \Sigma_0, \\ q_j(z_{n_k}) & = 0 \quad \text{on} \quad \Sigma \setminus \Sigma_0, \\ q_j(z_{n_k})(T) & = 0 \quad \text{in} \quad \Omega. \end{cases}$$

On the other hand, $\tilde{u}_{n_k} = \tilde{u}(z_{n_k})$ is given by

$$\tilde{u}_{n_k} = \tilde{\rho}(z_{n_k})\chi_\omega - P_{n_k}\tilde{\rho}(z_{n_k}), \tag{125}$$

where $\tilde{\rho}(z_{n_k}) \in V$ solves

$$\begin{cases} \frac{\partial \tilde{\rho}(z_{n_k})}{\partial t} - \Delta \tilde{\rho}(z_{n_k}) + a_0(z_{n_k})\tilde{\rho}(z_{n_k}) & = & 0 \quad \text{in } Q, \\ \tilde{\rho}(z_{n_k})|_\Sigma & = & 0, \end{cases} \tag{126}$$

and P_{n_k} denotes the orthogonal projection operator from $L^2(G)$ into $\mathcal{U}(z_{n_k}) = \text{Span}(\{q_1(z_{n_k})\chi_\omega, \dots q_m(z_{n_k})\chi_\omega\})$. Furthermore, in view of (75), (76), (52), (115) and (117), there exists a positive constant $C = C(\Omega, \omega, K, T, \sum_{j=1}^m ||e_j||_{H_0^1(\Sigma)})$ such that

$$||\tilde{\rho}(z_{n_k})||_V \leqslant C||y^0||_{L^2(\Omega)}, \tag{127}$$

$$||\tilde{\rho}(z_{n_k})||_{L^2(G)} \leqslant C||y^0||_{L^2(\Omega)}, \tag{128}$$

$$||u_0(z_{n_k})||_{L^2(G)} \leqslant C||y^0||_{L^2(\Omega)}, \tag{129}$$

$$||\theta u_0(z_{n_k})||_{L^2(G)} \leqslant C||y^0||_{L^2(\Omega)}, \tag{130}$$

$$||\tilde{u}_{n_k}||_{L^2(G)} \leqslant C||y^0||_{L^2(\Omega)}, \tag{131}$$

$$||\tilde{v}_{n_k}||_{L^2(G)} \leqslant C||y^0||_{L^2(\Omega)}. \tag{132}$$

By extracting subsequences, we may deduce that

$$\tilde{\rho}(z_{n_k}) \rightharpoonup \rho \text{ weakly in } V, \tag{133}$$

$$\tilde{\rho}(z_{n_k}) \rightharpoonup \rho \text{ weakly in } L^2(G), \tag{134}$$

$$u_0(z_{n_k}) \rightharpoonup x \text{ weakly in } L^2(G), \tag{135}$$

$$\theta u_0(z_{n_k}) \rightharpoonup x_1 \text{ weakly in } L^2(G), \tag{136}$$

$$\tilde{u}_{n_k} \rightharpoonup u \text{ weakly in } L^2(G), \tag{137}$$

and so $u \in \mathcal{U}$. Hence from (122), we have

$$\tilde{v}_{n_k} \rightharpoonup (x+u)\chi_\omega = v\chi_\omega \text{ weakly in } L^2(Q). \tag{138}$$

Step 3: Since $\tilde{y}_{n_k}$ solves (119), we have according to (132),

$$||\tilde{y}_{n_k}||_{W(0,T)} \leqslant C||y^0||_{L^2(\Omega)}, \tag{139}$$

where $C = C(\Omega, \omega, K, T, \sum_{j=1}^{m} ||e_j||_{H_0^1(\Sigma)})$. On the one hand, we deduce that

$$\left|\left|\frac{\partial \tilde{y}_{n_k}}{\partial \nu}\right|\right|_{H^{-1}(\Sigma)} \leqslant C||y^0||_{L^2(\Omega)},$$

on the other hand, by Aubin-Lions compactness Lemma, it follows that

$$\tilde{y}_{n_k} \to y \text{ strongly in } L^2(Q). \tag{140}$$

Therefore, using (118), (138), (139) and (140), we can pass to the limit $k \to +\infty$ in (119), (120) and (121) and we obtain that $(v, y = y(v))$ satisfies

$$\begin{cases} \frac{\partial y}{\partial t} - \Delta y + a_0(z)y &= v\chi_\omega \quad \text{in } Q, \\ y|_\Sigma &= 0, \\ y(0) &= y^0 \quad \text{in } \Omega, \end{cases}$$

$$\langle \frac{\partial y}{\partial \nu}, e_j \rangle_{H^{-1}(\Sigma_0), H_0^1(\Sigma_0)} = 0; j = 1, \dots, m,$$

and

$$y(T) = 0 \text{ in } \Omega.$$

Step 4: $q_j(z_{n_k})$ being solution of (124), we have in view of (40),

$$||q_j(z_{n_k})||_{H^{2,1}(Q)} \leqslant C||e_j||_{H_0^1(\Sigma)}, \tag{141}$$

and once again, by Aubin-Lions compactness Lemma, it follows that for $j \in \{1, \dots, m\}$,

$$q_j(z_{n_k}) \to \psi_j \text{ strongly in } L^2(Q). \tag{142}$$

Moreover, we also have the following energy inequality

$$||q_j(z_{n_k})(0)||_{L^2(\Omega)} \leqslant C||e_j||_{H_0^1(\Sigma)}. \tag{143}$$

Passing to the limit $k \to +\infty$ on (124), we obtain according to (118), (141) and (142),

$$\begin{cases} -\frac{\partial \psi_j}{\partial t} - \Delta \psi_j + a_0(z)\psi_j &= 0 \quad \text{in } Q, \\ \psi_j|_{\Sigma_0} &= e_j, \\ \psi_j|_{\Sigma \setminus \Sigma_0} &= 0, \\ \psi_j(T) &= 0 \quad \text{in } \Omega. \end{cases}$$

Thus for each e_j, $1 \leqslant j \leqslant m$, ψ_j solves (13). From (143), we get for $j \in \{1, \dots, m\}$

$$q_j(z_{n_k})(0) \rightharpoonup \psi_j(0) \text{ weakly in } L^2(\Omega). \tag{144}$$

The uniqueness of the solution of (13) implies that forall $j \in \{1, \dots, m\}$,

$$\psi_j(z) = q_j(z). \tag{145}$$

Step 5: Since $\theta u_0(z_{n_k}) \in \mathcal{U}(z_{n_k})$ and (130), (142), (145) and (136) hold, we can apply Lemma 3.4 with $H = L^2(G)$, $h_n = \theta u_0(z_{n_k})$, $p_{ni} = q_j(z_{n_k})$, $p_i = q_j$, we deduce that there exist $\alpha_j \in \mathbb{R}$, $1 \leqslant j \leqslant m$ such that

$$\theta u_0(z_{n_k}) \to x_1 = \sum_{j=1}^{m} \alpha_j q_j \text{ strongly in } L^2(G).$$

Then, using (135) and the fact that $\frac{1}{\theta}$ is bounded in $L^\infty(Q)$,

$$u_0(z_{n_k}) = \frac{1}{\theta}\theta u_0(z_{n_k}) \to x = \frac{1}{\theta}\sum_{j=1}^{m} \alpha_j q_j \text{ strongly in } L^2(G). \tag{146}$$

In view of (146), (142), (144) and (145) , we can pass to the limit $k \to +\infty$ in (123),

$$\int_G x q_j(z) dx dt = -\int_\Omega y^0 q_j(z)(0) dx, \; 1 \leqslant j \leqslant m.$$

The function $u_0 \in \mathcal{U}_\theta$ given by (20) being unique, we conclude that

$$u_0 = x. \tag{147}$$

Since $\tilde{u}_{n_k} \in \mathcal{U}(z_{n_k})^\perp$, we have

$$\int_G \tilde{u}_{n_k} q_j(z_{n_k}) dx dt = 0, \; 1 \leqslant j \leqslant m.$$

Passing to the limit $k \to +\infty$ in the latter identity, we obtain according to (137), (142) and (145),

$$\int_G u q_j(z) dx dt = 0, \; 1 \leqslant j \leqslant m.$$

We deduce that $u \in \mathcal{U}^\perp$.

Since $\tilde{\rho}(z_{n_k}) \in V$ satisfies (126) and (128), we can apply (25) to $\tilde{\rho}(z_{n_k})$ and deduce that $\tilde{\rho}(z_{n_k})$ is bounded in $L^2(]\beta, T-\beta[; H^2(\Omega)), \forall \beta > 0$. Then for any $\beta > 0$,

$$\tilde{\rho}(z_{n_k}) \rightharpoonup \rho \text{ weakly in } L^2(]\beta, T-\beta[\times\Omega),$$

$$\tilde{\rho}(z_{n_k}) \rightharpoonup \rho \text{ weakly in } L^2(]\beta, T-\beta[\times\Gamma).$$

Consequently,

$$\tilde{\rho}(z_{n_k}) \to \rho \text{ in } \mathcal{D}'(Q),$$
$$\tilde{\rho}(z_{n_k}) \to \rho \text{ in } \mathcal{D}'(\Sigma).$$

Therefore,

$$-\frac{\partial\tilde{\rho}(z_{n_k})}{\partial t} - \Delta\tilde{\rho}(z_{n_k}) + a_0(z_{n_k})\tilde{\rho}(z_{n_k}) \rightharpoonup L^*\rho = -\frac{\partial\rho}{\partial t} - \Delta\rho + a_0(z)\rho \text{ weakly in } \mathcal{D}'(Q).$$

Hence from (126), we have

$$\begin{cases} L^*\rho &= 0 \quad \text{in } Q, \\ \rho|_\Sigma &= 0. \end{cases}$$

Using (125) and (131), there exists a positive constant $C = C(\Omega, \omega, K, T, \sum_{j=1}^m ||e_j||_{H_0^1(\Sigma)})$ such that

(148) $$||\tilde{\rho}(z_{n_k})\chi_\omega - P_{n_k}\tilde{\rho}(z_{n_k})||_{L^2(G)} \leqslant C||y^0||_{L^2(\Omega)}.$$

Now applying (32) to $\tilde{\rho}(z_{n_k})$, we get

(149) $$\left|\left|\frac{1}{\theta}\tilde{\rho}(z_{n_k})\right|\right|_{L^2(Q)} \leqslant C||y^0||_{L^2(\Omega)}.$$

Arguing as in the proof of Proposition 3.5, we deduce from (148) and (149) that

(150) $$||P_{n_k}\tilde{\rho}(z_{n_k})||_{L^2(G)} \leqslant C||y^0||_{L^2(\Omega)}.$$

Consequently, $P_{n_k}\tilde{\rho}(z_{n_k})$ being in $\mathcal{U}(z_{n_k})$, we can apply Lemma 3.4 with $H = L^2(G)$, $h_n = P_{n_k}\tilde{\rho}(z_{n_k})$, $p_{ni} = q_j(z_{n_k})$, $p_i = q_j$, according to (142), (145) and (150). We conclude that,

(151)
$$P_{n_k}\tilde{\rho}(z_{n_k}) \to \tau \in \mathcal{U}(z) = \text{Span}(\{q_1(z)\chi_\omega, \ldots, q_m(z)\chi_\omega\}) \text{ strongly in } L^2(G).$$

Now in view of (125), (134), (137) and (151), we get

(152) $$\tilde{u}_{n_k} = \tilde{\rho}(z_{n_k})\chi_\omega - P_{n_k}\tilde{\rho}(z_{n_k}) \rightharpoonup \rho\chi_\omega - \tau = u \text{ weakly in } L^2(G).$$

Since $u \in \mathcal{U}^\perp$ and $\tau \in \mathcal{U}$, we have $Pu = 0$ and $P\tau = \tau$. From (152), it follows that $P\rho - \tau = 0$. Then $\tau = P\rho$ and $u = \rho\chi_\omega - P\rho = \tilde{u}$. Using (138) and (147),

$$v = u_0 + u = \tilde{v}.$$

It results that $(\tilde{v}, \tilde{y})$ satisfies (12), (7), (8).

5.2. **Compactness of $\mathcal{S}$.** The arguments above shows that when z lies in a bounded subset B of $L^2(Q)$, $\tilde{y}(\tilde{v}) = \mathcal{S}(z)$ also lies in a bounded set of $W(0,T)$. As a consequence of Aubin-Lions compactness Lemma, $W(0,T)$ is a compact set of $L^2(Q)$. Then, $\mathcal{S}(B)$ is relatively compact in $L^2(Q)$. This completes the proof of the compactness of $\mathcal{S}$.

5.3. **Boundedness of the range of $\mathcal{S}$.** Let $z \in L^2(Q)$. Since $\tilde{y}(\tilde{v}) = \mathcal{S}(z)$ solves (12) with $\tilde{v}$ satisfying (9), we have

$$||\tilde{y}(\tilde{v})||_{L^2(0,T;H_0^1(\Omega))} \leqslant C||y^0||_{L^2(\Omega)}$$

with $C = C(\Omega, \omega, K, T, \sum_{j=1}^m ||e_j||_{H_0^1(\Sigma)})$. The embedding of $L^2(0,T;H_0^1(\Omega))$ into $L^2(Q)$ being continuous, it follows that

$$||\tilde{y}(\tilde{v})||_{L^2(Q)} \leqslant C||y^0||_{L^2(\Omega)}.$$

This concludes the proof of Theorem 1.1.

References

[1] O. Nakoulima, Contrôlabilité à zéro avec contraintes sur le contrôle, C. R. Acad. Sci. Paris, Ser. I 339 (2004), 405-410.

[2] O. Nakoulima, Optimal control for distributed systems subject to null-controllability. Application to discriminating sentinels, ESAIM: COCV, Vol. 13, No. 4, 2007, pp. 623-638.

[3] G. Massengo Mophou and O. Nakoulima, Sentinels with given sensitivity, Euro. Jnl of Applied Mathematics, vol. 19, pp. 21-40 (2008).

[4] G. Massengo Mophou and O. Nakoulima, Null controllability with constraints on the state for the semilinear heat equation, J Optim Theory Appl 143: 539-565 (2009).

[5] G. Massengo Mophou and J.-P. Puel, Boundary sentinels with given sensitivity, Rev. Mat. Complut. 22, No. 1, 165-185 (2009).

[6] G. M. Mophou, Null controllability with constraints on the state for nonlinear heat equations, Forum. Math. 23 (2011), 285-319.

[7] A. V. Fursikov and O. Yu. Imanuvilov, Controllability of Evolution Equations, Lecture Notes, Research Institute of Mathematics, Seoul National University, Korea, 1996.

[8] O. Yu. Imanuvilov, Controllability of parabolic equations, Sbornik: Mathematics 186:6, 879-900 (1995).

[9] J. L. Lions, Contrôle optimal de systèmes gouvernés par des équations aux dérivées partielles, Dunod, Gauthier-Villars, Paris (1968).

[10] E. Zuazua, Approximate controllability for semilinear heat equations with globally Lipschitz nonlinearities, Control and Cybernetics 26 (1999), pp.665-683.

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