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Heteroclinic Solutions of Singular $\Phi-$Laplacian Boundary Value Problems on Infinite Time Scales

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Abstract

In this paper, we derive sufficient conditions for the existence of heteroclinic solutions to the singular Φ-Laplacian boundary value problem,

$$\left[\Phi(y^{\Delta}(t))\right]^{\Delta} = f(t, y(t), y^{\Delta}(t)), \quad t \in \mathbb{T}$$
$$y(-\infty) = -1, \quad y(+\infty) = +1,$$

on infinite time scales by using the Brouwer invariance domain theorem. As an application we demonstrate our result with an example.

Key words: Heteroclinic solution, Φ-Laplacian, singular boundary value problem, time scale, Brouwer invariance domain theorem.

AMS Subject Classification: 34A40, 34B15, 34L30.

1 Introduction

In the study of qualitative theory of differential equations, a prominent role is played by special class of solutions, like positive solutions, periodic solutions, symmetric solutions, chaotic solutions. When a system of dynamic equations has equilibria, i.e. constant solution, whose stability properties are known, it is significant to study the connections between them by trajectories of solutions for the given system. These are called homoclinic or heteroclinic solutions, sometimes pulse or kinks, according to whether they describe a loop based at one stable equilibrium or two stable equilibriums.

Recently, heteroclinic solutions for boundary value problems on the whole real line had a certain impulse, motivated by applications in various disciplines of biological, physical, mechanical and chemical models, such as phase transition, physical processes in which the variable transits from an unstable equilibrium to a stable one, and front propagation in reaction diffusion equation.

Indeed, heteroclinic solutions are often called transitional solutions. These connections provide important information on the dynamics of the system. For autonomous second order system, for instance, homoclinic and heteroclinics separate, in the phase portrait, region where solutions behave differently. When they appear as connection between saddles they prevent structural stability. Due to the importance in both theory and applications, the study of heteroclinic solutions gained momentum on real intervals, we list a few; Avrameseu and Vladimirecu [1], Cabada and Cid [3, 4], Cabada and Tersion [5], and Marcelli and Papalini [7, 8].

A time scale is an arbitrary nonempty closed subset of the real numbers and we denote the time scale by the symbol $\mathbb{T}$. A time scale $\mathbb{T}$ is said to be infinite time scale, we mean, if it has no infimum and no supremum. (i.e., $\inf \mathbb{T} = -\infty$ and $\sup \mathbb{T} = +\infty$.) A time scale $\mathbb{T}$ is said to be semi infinite time scale, we mean, if it has either infimum but no supremum or supremum but no infimum. (i.e., $\inf \mathbb{T} = a$ and $\sup \mathbb{T} = +\infty$ or $\inf \mathbb{T} = -\infty$ and $\sup \mathbb{T} = b$)and we denote as $\mathbb{T}_a^+ = [a, +\infty)$ and $\mathbb{T}_b^- = (-\infty, b]$ respectively. For example if we consider time scale $\mathbb{T} = \{-(a)^n\}_{n\in\mathbb{N}} \cup \{0\} \cup \{b^n\}_{n\in\mathbb{N}}, (a, b > 1)$ then it is an infinite time scale and if we remove either negative terms or positive terms then it is an example of semi infinite time scale. By an interval $[a, b]_{\mathbb{T}}$ means the intersection of the real interval with a given time scale.

$$i.e \quad [a, b]_{\mathbb{T}} = [a, b] \cap \mathbb{T}.$$

For basic definitions and results on time scales we refer to the text books by Bohner and Peterson [2] and Lakshmikantham, Sivasundaram and Kaymakcalan [6].

In this paper, we are dealing with some special class of time scales namely infinite and semi infinite time scales because continuous time orbits and discrete time orbits are topologically different. We consider the following dynamical equation on infinite time scales

$$\left[\Phi(y^{\Delta}(t))\right]^{\Delta} = f(t, y(t), y^{\Delta}(t)), \quad t \in \mathbb{T} \tag{1}$$

satisfying the boundary conditions

$$y(-\infty) = -1, \quad y(+\infty) = +1, \tag{2}$$

where $\Phi(y)$ is singular Φ-Laplacian operator and $f : \mathbb{T} \times \mathbb{R}^2 \to \mathbb{R}$ is continuous function. We derive sufficient conditions for the existence of heteroclinic solutions to the singular Φ-Laplacian boundary value problem on infinite time scales.

The existence of heteroclinic solution of (1)-(2) connects -1 to $+1$ in the phase-space, is a function $y \in C^1_{rd}(\mathbb{T})$ such that $y^\Delta \in (-a, a)$, $\Phi \circ y^\Delta \in C^1_{rd}(\mathbb{T})$ $(a > 0)$ and y satisfies the dynamical equation (1) and with the property

$$\lim_{t \to \pm\infty} \left(y, y^\Delta\right) = (\pm 1, 0).$$

The main idea is that the existence of loops or connections at some rest points can be prove by analyzing the intersection properties of the stable and unstable manifolds through these equilibrium points.

We assume the following conditions throughout the paper.

(A1). $\Phi : (-a, a) \to \mathbb{R}$ is an increasing homeomorphism with $\Phi(0) = 0$ and $a > 0$.

(A2). $f : \mathbb{T} \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and satisfies the symmetric type condition

$$f(t, x, y) = -f(-t, -x, y) \text{ for all } t \in \mathbb{T},\ x \in (-1, 1),\ y \in \mathbb{R}.$$

(A3). $f(t, 1, y) = 0 = f(t, -1, y)$ for all $(t, y) \in \mathbb{T} \times \mathbb{R}$.

(A4). $f(t, x, y) < 0$ for all $t(> 0) \in \mathbb{T}^+_0$, $-1 < x < 1$ and $y \in \mathbb{R}$.

(A5). for every compact set of the form $K = [-r, r] \times [-\epsilon, \epsilon]$, where $0 < r < 1$ and $0 < \epsilon < 1$, there exist $t_l \geq 0$ and a continuous function $h_l : [t_l, \infty) \to \mathbb{R}$ such that $f(t, x, y) \leq h_l(t)$ for all $t \geq t_l$ and $(x, y) \in K$, and $\int_{t_l}^{+\infty} h_l(s)\Delta s = -\infty$.

(A6). if $\{y_n(t)\}$ is a sequence of solutions of (1) for which for each $n \in \mathbb{T}^+_0$ an arbitrary compact interval $[0, \sigma^2(n)]_{\mathbb{T}}$ and there exists an $M > 0$ such that $y_n(t) \leq M$ for all $t \in [0, \sigma^2(n)]_{\mathbb{T}}$ and for all $t \in \mathbb{N}$, then there exists a subsequence $\{y_{n_j}(t)\}$, such that $\{y^{\Delta^i}_{n_j}\}$ converges uniformly on $[0, \sigma^{(2-i)}(n)]_{\mathbb{T}}$, $i = 0, 1$.

The rest of the paper is organized as follows, in Section 2, we prove some lemmas which are needed in the main result. In Section 3, we derive sufficient conditions for the existence of heteroclinic solutions to the singular Φ-Laplacian boundary value problem on infinite time scales. Finally, as an application, we give an example to demonstrate our result.

2 Existence of solution on semi infinite time scales

In this section, we establish a lemma and theorem on semi infinite time scales which are needed to establish heteroclinic solution on infinite time scales.

Lemma 2.1 *Suppose that* $\Phi : (-a, a) \to \mathbb{R}$ *satisfies* $(A1)$. *Then, for all* $g : [0, \sigma^2(n)]_{\mathbb{T}} \to \mathbb{R}$ *the following problem*

$$\left[\Phi(y^{\Delta}(t))\right]^{\Delta} = g(t), \quad t \in [0, n]_{\mathbb{T}} \tag{3}$$

$$y(0) = 0 = y(\sigma^2(n)) \tag{4}$$

has a unique solution and $\| y^{\Delta} \|_{\infty} < a$.

Proof: We can construct a time scale integral equation equivalent to the boundary value problem (3)-(4),

$$y_g(t) = \int_0^t \Phi^{-1}\left(\tau_g + \int_0^{t_1} g(s)\Delta s\right)\Delta t_1, \quad \text{for all } t \in [0, \sigma^2(n)]_{\mathbb{T}}$$

τ_g being a solution of the following equation,

$$F_g(\tau) = \int_0^{\sigma^2(n)} \Phi^{-1}\left(\tau + \int_0^{t_1} g(s)\Delta s\right)\Delta t_1 = 0.$$

The fact that the function $F_g : \mathbb{R} \to \mathbb{R}$ has a unique real root is deduced from the following facts:

1. the continuity of Φ^{-1} implies the continuity of the function F_g
2. since Φ^{-1} is strictly increasing implies the same hold for function F_g.
3. $\lim_{\tau \to +\infty} F_g(\tau) = \int_0^{\sigma^2(n)} a\Delta t_1 = \sigma^2(n).a > 0$
4. $\lim_{\tau \to -\infty} F_g(\tau) = -\int_0^{\sigma^2(n)} a\Delta t_1 = -\sigma^2(n).a < 0.$

Thus we can conclude that τ_g is unique, and hence the boundary value problem (3)-(4) has a unique solution y_g. We have that for all $t \in [0, \sigma(n)]_{\mathbb{T}}$,

$$y^{\Delta}(t) = \Phi^{-1}\left(\tau_y + \int_0^t f(s, y(s), y^{\Delta}(s))\Delta s\right),$$

and hence $y^\Delta \in (-a,a)$ and also

$$\| y^\Delta \|_\infty = \sup_{t\in[0,\sigma(n)]} |y^\Delta(t)| < a.$$

□

Theorem 2.2 *Suppose that $\Phi : (-a,a) \to \mathbb{R}$ satisfies $(A1)$ and $(A6)$, then the problem*

$$\left[\Phi(y^\Delta(t))\right]^\Delta = f(t,y(t),y^\Delta(t)), \quad t\in[0,n]_{\mathbb{T}} \tag{5}$$

$$y(0) = 0 = y(\sigma^2(n)) \tag{6}$$

has at least one solution, and this solution satisfies $\| y^\Delta \|_\infty < a$, where $f : [0,\sigma^2(n)]_{\mathbb{T}} \times \mathbb{R}^2 \to \mathbb{R}$ is continuous function.

Proof: Let $A : \mathbb{T} \to \mathbb{R}$ defined as

$$Ay(t) = \int_0^t \Phi^{-1}\left(\tau_y + \int_0^{t_1} f(s,y(s),y^\Delta(s))\Delta s\right)\Delta t_1, \quad t\in[0,\sigma^2(n)]_{\mathbb{T}} \tag{7}$$

with τ_y the unique solution of the following equation

$$g_y(\tau) = \int_0^{\sigma^2(n)} \Phi^{-1}\left(\tau + \int_0^{t_1} f(s,y(s),y^\Delta(s))\Delta s\right)\Delta t_1 = 0. \tag{8}$$

From Lemma 2.1, the operator A satisfies the problem (7) and (8), [i.e. A is well defined and that the fixed points of the operator A are the solutions of the problem (7) and (8)]

First we prove that operator A is continuous. Suppose $y_m \to y$ in $[0,\sigma^2(n)]\times\mathbb{R}$ and let τ_m be the corresponding value for y_m given by (8) and τ_y associated to y. Let us see that

$$\lim_{m\to\infty} \tau_m = \tau_y.$$

By construction of τ_m and τ_y we have that for all $m\in\mathbb{N}$.

$$\begin{aligned} 0 &= \int_0^{\sigma^2(n)} \Phi^{-1}\left(\tau_y + \int_0^{t_1} f(s,y(s),y^\Delta(s))\Delta s\right)\Delta t_1 \\ &= \int_0^{\sigma^2(n)} \Phi^{-1}\left(\tau_m + \int_0^{t_1} f(s,y_m(s),y_m^\Delta(s))\Delta s\right)\Delta t_1. \end{aligned} \tag{9}$$

Since $\{y_m\}$ is a convergent sequence to y we have that

$$\left\{(t,y_m(t),y_m^\Delta(t)), m\in\mathbb{N}\right\} \cup \left\{(t,y(t),y^\Delta(t))\right\}$$

is a compact set in $[0,\sigma^2(n)]_{\mathbb{T}} \times \mathbb{R}^2$. As consequence $\left\{f(t, y_m(t), y_m^{\Delta}(t))\right\}_{m\in\mathbb{N}}$ is a bounded sequence in $\mathbb{R}$, from (9) and $(A6)$ the sequence $\{\tau_m\}$ is bounded too, and we conclude that there exists a subsequence $\{\tau_{m_k}\}$ converging to a real number $\gamma = \limsup\{\tau_m\}$. Thus, from the continuity of Φ^{-1} and f, we arrive that

$$\int_0^{\sigma^2(n)} \Phi^{-1}\left(\tau_m + \int_0^{t_1} f(s, y(s), y^{\Delta}(s))\Delta s\right)\Delta t_1 =$$

$$\int_0^{\sigma^2(n)} \Phi^{-1}\left(\gamma + \int_0^{t_1} f(s, y(s), y^{\Delta}(s))\Delta s\right)\Delta t_1,$$

and since Φ^{-1} is a strictly increasing function, we conclude that $\tau_y = \gamma$, Analogously, $\tau_y = \liminf\{\tau_m\}$. Now, by the continuity of f it follows that

$$\lim_{m\to\infty} Ay_m(t) = Ay(t) \text{ for all } t \in [0,\sigma^2(n)]_{\mathbb{T}}$$

which is equivalent to say that A is a continuous operator in $\mathbb{R}$. Finally, it is clear that

$$\| Ay \|_{\infty} \leq \sigma^2(n)a,$$

for each $y \in \mathbb{R}$ and then Brouwer fixed point theorem ensures the existence of at least one fixed point of the operator A and as a consequence, the existence of at least one solution of the problem (5),(6), by using equations (7),(8), we have that for all $t \in [0,\sigma(n)]_{\mathbb{T}}$ that

$$y^{\Delta}(t) = \Phi^{-1}\left(\tau_y + \int_0^t f(s, y(s), y^{\Delta}(s))\Delta s\right),$$

and hence $y^{\Delta} \in (-a, a)$. □

3 Existence of solution on infinite time scales

In this section, we derive the sufficient conditions for the existence of the heteroclinic solutions of singular Φ-Laplacian boundary value problem (1) and (2).

Theorem 3.1 *If conditions* $(A1) - (A6)$ *hold then boundary value problem* (1)-(2) *has an odd increasing solution* $y : \mathbb{T} \to \mathbb{R}$.

Proof: By condition $(A2)$ it suffices to prove the existence of a solution $y : \mathbb{T}_0^+ \to \mathbb{R}$ of (1) satisfies $y(0) = 0$ and $\lim_{t\to+\infty} y(t) = 1$, since its odd extension solves $(1) - (2)$. Firstly, we prove that, for each $n \in \mathbb{T}_0^+ - \{0\}$, the boundary value problem

$$\left[\Phi(y^\Delta(t))\right]^\Delta = f(t, y(t), y^\Delta(t)) \tag{10}$$

$$y(0) = 0 = y(\sigma^2(n)), \tag{11}$$

has a solution $y_n : [0, \sigma^2(n)]_\mathbb{T} \to \mathbb{R}$ satisfying, $0 \le y_n(t) \le 1$, and $\| y_n^\Delta \|_\infty < a$.

Consider the continuous function

$$\widetilde{f}(t, x, y) = \begin{cases} f(t, x, y), & \text{if} -1 \le x \le 1 \\ 0, & \text{other wise} \end{cases}$$

For each $n \in \mathbb{T}_0^+ - \{0\}$, the modified problem of (10)-(11) is

$$\left[\Phi(y^\Delta(t))\right]^\Delta = \widetilde{f}(t, y(t), y^\Delta(t)) \tag{12}$$

$$y(0) = 0 = y(\sigma^2(n)) \tag{13}$$

has a solution $y_n : [0, \sigma^2(n)]_\mathbb{T} \to \mathbb{R}$ with $\| y_n^\Delta \|_\infty < a$ by Theorem 2.2. Moreover it is easy to show that $-1 \le y_n \le 1$ and therefore y_n is also a solution of (10),(11) on the other hand $(A4)$ implies that y_n is concave and then $0 \le y_n \le 1$. Secondly we show that, there exists a bounded nondecreasing solution $y : \mathbb{T}_0^+ \to \mathbb{R}$ of (1) such that $y(0) = 0$ and $0 \le y(t) \le 1$.

Since y_n, y_n^Δ are uniformly bounded and from $(A6)$ it is easy to prove that a subsequence of y_n converges uniformly on compact sets to a solution $y : \mathbb{T}_0^+ \to \mathbb{R}$ of (1). Clearly $y(0) = 0$ and $0 \le y(t) \le 1$.

On the other hand, from the uniform continuity of function Φ^{-1} on compact set it follows that the sequence $\{y_n^\Delta\}$ is an equicontinuous family, and as consequence it is easy to verify that $\left[\Phi(y^\Delta(t))\right]^\Delta = f(t, y(t), y^\Delta(t)) \le 0$. So we deduce that y^Δ is nonincreasing. If $y^\Delta(t_0) < 0$ at some point $t_0 \ge 0$ then $y^\Delta(t) \le y^\Delta(t_0) < 0$ for all $t \ge t_0$ and consequently $\lim_{t\to\infty} y(t) = -\infty$ a contradiction. Thus $y^\Delta(t) \ge 0$ for all $t \ge 0$ and hence y is nondecreasing. Since y^Δ is decreasing there exists,

$$\lim_{t\to+\infty} y^\Delta(t) \in \mathbb{R} \cup \{-\infty\}.$$

But as y is bounded we deduce that

$$\lim_{t\to+\infty} y^\Delta(t) = 0.$$

Since y is concave and bounded then there exists $l \in (0,1]$ such that

$$\lim_{t \to +\infty} y(t) = l.$$

Suppose that $l < 1$. From $(A4), (A5)$ and the facts that $0 \le y(t) \le l < 1$ and $\lim_{t \to +\infty} y^{\Delta}(t) = 0$ it follows that there exist a suitable compact set $K \subset (-1,1) \times \mathbb{R}$, $t_k > 0$ for which $0 < y^{\Delta}(t_k) < 1$ and a continuous function h_k such that

$$\left[\Phi(y^{\Delta}(t))\right]^{\Delta} = f(t, y(t), y^{\Delta}(t)) \le h_k(t) \quad \text{for all } t \ge t_k,$$

and $\int_{t_k}^{\infty} h_k(t) = -\infty$. But in this case $\Phi(y^{\Delta}) \to -\infty$ and then $y^{\Delta}(t) \to -a < 0$, which is a contradiction. Thus $l = 1$ and the proof is over. □

4 Example

In this section, as an application, we give an example to demonstrate our result.

Consider the following dynamical equation on an infinite time scale $\mathbb{T} = \{-(2)^n\}_{n \in \mathbb{N}} \cup \{0\} \cup \{3^n\}_{n \in \mathbb{N}}$,

$$\left[\Phi(y^{\Delta}(t))\right]^{\Delta} = f(t, y(t), y^{\Delta}(t)), \quad t \in \mathbb{T} \tag{14}$$

satisfying the boundary conditions

$$y(-\infty) = -1, \quad y(+\infty) = +1, \tag{15}$$

where $\Phi : (-a, a) \to \mathbb{R}$ is the increasing homeomorphism and is given by

$$\Phi(y) = \frac{sign(y)|y|^{\mu}}{(a^2 - y^2)^k}$$

with μ, k are positive real numbers, $f : \mathbb{T} \times \mathbb{R}^2 \to \mathbb{R}$ is continuous function and let us define as

$$f\left(t, y(t), y^{\Delta}(t)\right) = t^5 \left(y^2(t) - 1\right) \left(y^{\Delta^4}(t) + 1\right),$$

and the function $h_l(t) : \mathbb{T}_{t_l}^{+} \to \mathbb{R}$, $t_l \ge 0$ with $h_l(t) = \frac{-|t|^{\lambda}}{t}$, $\lambda > 0$. All the conditions of Theorem 3.1 are satisfied. Hence the boundary value problem (14)-(15) has a heteroclinic solution which connects -1 and $+1$.

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References

[1] C. Avrameseu and C. Vladimirecu, Existence of solution to second order differential equations having finite limits at $\pm\infty$, *Elec. J. Diff. Equa.,* Vol.**18**(2004), pp. 1-12.

[2] M. Bohner and A. C. Peterson, *Dynamic Equations on Time scales, An Introduction with Applications*, Birkhauser, Boston,MA,(2001).

[3] A. Cabada and J. A. Cid, Solvability of some Φ-Laplacian singular difference equation defined on integers, *The Arab. J. Scie. Engi.,* Vol **34**(2009), No.ID, pp.75-81.

[4] A. Cabada and J. A. Cid, Heteroclinic solution for non-autonomous boundary value problem with singular Φ-Laplacian operator, *Discr. Cont. Dyna. Syst.,* Vol **18**(2009), pp. 118-112.

[5] A. Cabada and S. Tersion, Existence of heteroclinic solutions for discrete p-Laplacian problems with a parameter, *Nonlinear; Real world appl.,* doi:10.1016/j.nonwrg.2011.02.022.

[6] V. Lakshmikantham, S. Sivasundaram and B. Kaymakcalan, *Dynanic Systems on Measure Chains*, Mathematics and its Applications, **370**, Kluwer, Dordrecht, 1996.

[7] C. Marcelli and F. Papalini, Heteroclinic solution for second order non-autonomous boundary value problem on real line, *Diff. Equa. Dyna. Syst.,* Vol**11**(2003), pp.333-353.

[8] C. Marcelli and F. Papalini, Heteroclinic connection for fully nonlinear non-autonomous second order differential equations, *J. Diff. Equa.,* **241**(2007), pp.160-183.

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