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Periodic solutions for a class of second order differential equations

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Abstract New results about the existence of periodic solutions for second order differential equations are provided. The method of proof relies on the Schauder's fixed point theorem. Some examples are presented to illustrate the main results.

Keywords Periodic solution; Schauder's fixed point theorem; fixed point.
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1 Introduction

The existence and multiplicity of periodic solutions are an important aspect in differential equations qualitative analysis. Much work about periodic solutions for second order differential equations has been done by using various theorems and methods of nonlinear functional analysis, see [1, 3, 4, 5, 6, 8, 9, 16, 17, 18] and the references therein. In this paper, we investigate the existence of periodic solutions of the following differential equation

$$-x''(t) + a(t)x'(t) = g(t, x(t)) - f(t, x(t), x'(t)), \tag{1.1}$$

where a is a continuous ω-periodic function, $g(t, u), f(t, u, v)$ are ω-periodic in t and $\omega > 0$.

Equation (1.1) includes many important models, for example,

$$x''(t) + \mu \sin x(t) = h(t), \tag{1.2}$$

$$x''(t) + cx'(t) + \mu \sin x(t) = h(t), \tag{1.3}$$

$$x''(t) + f(x'(t)) + g(x(t)) = e(t), \tag{1.4}$$

$$x''(t) + f(x(t))x'(t) + g(x(t)) = e(t), \tag{1.5}$$

$$x''(t) = -\frac{1}{x^{\lambda}(t)} + e(t),\ \ \lambda > 0. \tag{1.6}$$

The above equations arise in many fields, such as, physics, mechanics and engineering. We refer the reader to [2, 10, 11, 13, 14, 15] for recent results of those models.

The main purpose of this article is to discuss the existence of periodic solutions of equation (1.1) by means of Schauder's fixed point theorem. The method of proof is in a simple idea and is composed of two steps: The first step is to transform the original equation into a first order integro-differential equation through a linear integral operator and the second step is an application of the Schauder's fixed point theorem. The existence of single periodic solution

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for (1.1) has been established under suitable behavior of g and f on some closed sets. So some information on the location of periodic solution is also obtained, leading to multiplicity results. Our results are new for (1.2)-(1.5) (see Corollary 3.2, Corollary 3.3, Theorem 3.3), which seems not be found in the literature.

2 Preliminaries

Let $X = \{x \in C(R,R) : x(t+\omega) = x(t) \text{ for all } t \in R\}$ with the norm $\|x\| = \max_{t\in[0,\omega]} |x(t)|$. Clearly, X is a Banach space.

Let $p,\ q \in X$ and consider the following two differential equations

$$x'(t) = -p(t)x(t) + q(t), \tag{2.1}$$

$$x'(t) = p(t)x(t) - q(t). \tag{2.2}$$

Lemma 2.1. *Assume that $\int_0^\omega p(t)dt \neq 0$, then (2.1) has a unique ω-periodic solution*

$$\widetilde{x}(t) = \int_t^{t+\omega} \frac{\exp\left(\int_t^s p(r)dr\right)}{\exp\left(\int_0^\omega p(r)dr\right) - 1} q(s)ds$$

and (2.2) has a unique ω-periodic solution

$$\overline{x}(t) = \int_t^{t+\omega} \frac{\exp\left(\int_s^{t+\omega} p(r)dr\right)}{\exp\left(\int_0^\omega p(r)dr\right) - 1} q(s)ds.$$

Proof. Here we only consider (2.1). Obviously, the periodic solution of (2.1) is unique if $\int_0^\omega p(t)dt \neq 0$ and we show that $\widetilde{x}(t)$ is the periodic solution of (2.1). Differentiating $\widetilde{x}(t)$, we obtain that

$$\begin{aligned}
\widetilde{x}'(t) &= \frac{\exp\left(\int_t^{t+\omega} p(r)dr\right)}{\exp\left(\int_0^\omega p(r)dr\right) - 1} q(t+\omega) - \frac{1}{\exp\left(\int_0^\omega p(r)dr\right) - 1} q(t) \\
&\quad - \int_t^{t+\omega} \frac{p(t)\exp\left(\int_t^s p(r)dr\right)}{\exp\left(\int_0^\omega p(r)dr\right) - 1} q(s)ds \\
&= q(t) - p(t)\widetilde{x}(t)
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{x}(t+\omega) &= \int_{t+\omega}^{t+2\omega} \frac{\exp\left(\int_{t+\omega}^s p(r)dr\right)}{\exp\left(\int_0^\omega p(r)dr\right) - 1} q(s)ds \\
&= \int_t^{t+\omega} \frac{\exp\left(\int_{t+\omega}^{u+\omega} p(r)dr\right)}{\exp\left(\int_0^\omega p(r)dr\right) - 1} q(u+\omega)d(u+\omega) \\
&= \int_t^{t+\omega} \frac{\exp\left(\int_t^u p(r)dr\right)}{\exp\left(\int_0^\omega p(r)dr\right) - 1} q(u)du = \widetilde{x}(t).
\end{aligned}$$

Hence, $\widetilde{x}(t)$ is unique ω-periodic solution of (2.1). $\square$

The following well-known Schauder's fixed point theorem is crucial in our arguments.

Lemma 2.2. *Let X be a Banach space with $D \subset X$ closed and convex. Assume that $T : D \to D$ is a completely continuous operator, then T has a fixed point in D.*

Define an operator J on X by

$$(Ju)(t) = \int_t^{t+\omega} \frac{e^{(s-t)p}}{e^{p\omega}-1} u(s)ds, \quad u \in X,$$

where $p > 0$ is a constant which is determined later. For any $u \in X$, $Ju \in X \cap C^1(R)$ and

$$(Ju)'(t) = -p(Ju)(t) + u(t). \tag{2.3}$$

If $u \in X \cap C^1(R)$, then $Ju \in X \cap C^2(R)$ and

$$(Ju)''(t) = -p(Ju)'(t) + u'(t) = p^2(Ju)(t) - pu(t) + u'(t). \tag{2.4}$$

We transform (1.1) to

$$p^2(Ju)(t) - pu(t) + u'(t) - a(t)(-p(Ju)(t) + u(t)) = f(t, (Ju)(t), u(t) - p(Ju)(t)) - g(t, (Ju)(t)),$$

that is

$$u'(t) = (p + a(t))u(t) - [p^2 Ju + pa(t)Ju + g(t, Ju) - f(t, Ju, u - pJu)]. \tag{2.5}$$

By Lemma 2.1, we obtain that if u is a periodic solution of (2.5), u satisfies

$$u(t) = \int_t^{t+\omega} \frac{\exp \int_s^{t+\omega}(p + a(r))dr}{\exp \int_0^{\omega}(p + a(r))dr - 1}(Hu)(s)ds$$

if $\int_0^{\omega}(p + a(r))dr \neq 0$, where

$$(Hu)(s) = p^2(Ju)(s) + pa(s)(Ju)(s) + g(s, (Ju)(s)) - f(t, (Ju)(s), u(s) - p(Ju)(s)).$$

In order to put more emphasis on the above facts, we summarize them in the following lemma.

Lemma 2.3. *Define an operator T on X by*

$$(Tu)(t) = \int_t^{t+\omega} \frac{\exp \int_s^{t+\omega}(p + a(r))dr}{\exp \int_0^{\omega}(p + a(r))dr - 1}(Hu)(s)ds, \tag{2.6}$$

here $\int_0^{\omega}(p + a(r))dr \neq 0$. Then the fixed point u of T on X is the periodic solution of (2.5) *and Ju is the periodic solution of* (1.1).

Proof. Since $Tu = u$ and

$$(Tu)'(t) = (p + a(t))(Tu)(t) - [p^2 Ju + pa(t)Ju + g(t, Ju) - f(t, Ju, u - pJu)],$$

we obtain that u is the periodic solution of (2.5). In order to prove that Ju is the periodic solution of (1.1), we only show that Ju satisfies (1.1). Form (2.3)-(2.5), this result follows immediately. □

3 Main results

The following theorems are the main results of this paper.

Theorem 3.1. *Assume that there exist constants* $m < M, p > 0$ *such that*

(H_1) $g \in C(R \times [m, M], R)$, *and* $p(p + a(t))u + g(t, u)$ *is nondecreasing in* $u \in [m, M]$.

(H_2) $f(t, u, v) \in C(R \times [m, M] \times [p(m - M), p(M - m)], R)$ *and*

$$g(t, M) \le f(t, u, v) \le g(t, m)$$

for any $(t, u, v) \in R \times [m, M] \times [p(m - M), p(M - m)]$.

Then (1.1) *has at least one* ω*-periodic solution* x *with* $m \le x \le M$.

Proof. Let $\Omega = \{x \in X : mp \le x(t) \le pM \text{ for } t \in [0, \omega]\}$, then Ω is a closed and convex set in X. For any $u \in \Omega$, we compute to obtain that $m \le Ju \le M$ and $pm - pM \le u - pJu \le pM - pm$. Moreover, according to (H_1), we have

$$(p^2+pa(t))m+g(t,m) \le (p^2+pa(t))Ju+g(t,Ju) \le (p^2+pa(t))M+g(t,M) \text{ for } u \in \Omega. \quad (3.1)$$

Using (H_2), we obtain that for any $u \in \Omega$,

$$\begin{aligned}
(Hu)(t) &= p^2(Ju)(t) + pa(t)(Ju)(t) + g(t, (Ju)(t)) - f(t, (Ju)(t), u(t) - p(Ju)(t)) \\
&\le (p^2 + pa(t))M + g(t, M) - g(t, M) \\
&= (p^2 + pa(t))M, \\
(Hu)(t) &= p^2(Ju)(t) + pa(t)(Ju)(t) + g(t, (Ju)(t)) - f(t, (Ju)(t), u(t) - p(Ju)(t)) \\
&\ge (p^2 + pa(t))m + g(t, m) - g(t, m) \\
&= (p^2 + pa(t))m,
\end{aligned}$$

which imply that $p + a(t) \ge 0$ for all $t \in R$. If $p + a(t) \equiv 0$, according to (H_1) and (H_2), we easily to check that $g(t, u) \equiv f(t, u, v)$ for any $(t, u, v) \in R \times [m, M] \times [p(m - M), p(M - m)]$. Thus any constant $C \in [m, M]$ is the periodic solution of (1.1). We assume that $p + a(t) \ge 0$ and $p + a(t) \not\equiv 0$. The operator T is well defined. Now we show that T satisfies all conditions of Lemma 2.2. Noting that

$$\int_t^{t+\omega} \frac{\exp \int_s^{t+\omega}(p + a(r))dr}{\exp \int_0^{\omega}(p + a(r))dr - 1}(p + a(s))ds \equiv 1,$$

$$\frac{\exp \int_s^{t+\omega}(p + a(r))dr}{\exp \int_0^{\omega}(p + a(r))dr - 1} > 0, \text{ for } t \le s \le t + \omega,$$

we obtain that for any $u \in \Omega$,

$$\begin{aligned}
(Tu)(t) &= \int_t^{t+\omega} \frac{\exp \int_t^{s}(p + a(r))dr}{\exp \int_0^{\omega}(p + a(r))dr - 1}(Hu)(s)ds \\
&\le \int_t^{t+\omega} \frac{\exp \int_s^{t+\omega}(p + a(r))dr}{\exp \int_0^{\omega}(p + a(r))dr - 1}(p^2 + pa(t))Mds = pM, \\
(Tu)(t) &= \int_t^{t+\omega} \frac{\exp \int_s^{t+\omega}(p + a(r))dr}{\exp \int_0^{\omega}(p + a(r))dr - 1}(Hu)(s)ds \\
&\ge \int_t^{t+\omega} \frac{\exp \int_s^{t+\omega}(p + a(r))dr}{\exp \int_0^{\omega}(p + a(r))dr - 1}(p^2 + pa(t))mds = pm,
\end{aligned}$$

which imply that $T(\Omega) \subseteq \Omega$.

Next, we show that $T : \Omega \to \Omega$ is completely continuous. Obviously, $T(\Omega)$ is a uniformly bounded set and T is continuous on Ω, so it suffices to show $T(\Omega)$ is equicontinuous by Ascoli-Arzela theorem. For any $u \in \Omega$, we have

$$(Tu)'(t) = (p + a(t))(Tu)(t) - [p^2 Ju + pa(t)Ju + g(t, Ju) - f(t, Ju, u - pJu)]$$

Since $T(\Omega)$ is bounded and f, g, a are continuous, there exists $\rho > 0$ such that

$$|(Tu)'(t)| \leq \rho,\ u \in \Omega,,$$

which implies that $T(\Omega)$ is equicontinuous. So T is a completely continuous operator on Ω. By Lemma 2.2, there exists $u \in \Omega$ with $Tu = u$. Moreover, $Ju \in [m, M]$ is the periodic solution of (1.1). The proof is complete. □

Analogously, we have the following theorem.

Theorem 3.2. *Assume that there exist constants $m < M, p > 0$ such that*
(H_3) $g \in C(R \times [m, M], R)$, *and* $p(p + a(t)u + g(t, u)$ *is nonincreasing in* $u \in [m, M]$.
(H_4) $f(t, u, v) \in C(R \times [m, M] \times [p(m - M), p(M - m)])$ *and*

$$g(t, m) \leq f(t, u, v) \leq g(t, M)$$

for any $(t, u, v) \in R \times [m, M] \times [p(m - M), p(M - m)]$.
Then (1.1) *has at least one* ω*-periodic solution* x *with* $m \leq x \leq M$.

In Theorem 3.1, if $g_u(t, u)$ is continuous on $[0, \omega] \times [m, M]$, (H_1) and (H_2) are fulfilled for any sufficiently large $p > 0$.

Corollary 3.1. *Let $f(t, u, v) \equiv f(t, u)$. Assume that there exist constants $m < M$ such that $g_u(t, u), f(t, u) \in C(R \times [m, M], R)$ and*

$$g(t, M) \leq f(t, u) \leq g(t, m)$$

for any $(t, u) \in R \times [m, M]$. *Then* (1.1) *has at least one* ω*-periodic solution* x *with* $m \leq x \leq M$.

Remark 3.1 Let $c, \mu > 0$ and h be ω-periodic continuous function. In [12], by using critical point theorem, authors proved that (1.3) has at least two periodic solutions if $\|h\| < \mu$ and one periodic solutions if $\|h\| = \mu$.

Corollary 3.2. *Assume that c, μ are constants and h is ω-periodic continuous function with $\|h\| \leq |\mu|$. Then* (1.2) *or* (1.3) *has at least one ω-periodic solution. Further suppose that $c \geq 2\sqrt{|\mu|}$ and $h \not\equiv \pm\mu$, then* (1.3) *has least two ω-periodic solutions.*

Proof. Here we only consider (1.3). If $\mu = 0$, then $h \equiv 0$ and any constant Λ is the periodic solution of (1.3). Now, we assume that $\mu \neq 0$. Let $a(t) = -c$, $g(u) = \mu \sin u$ and $f(t, u, v) = h(t)$. Put $p_1 = (|c| + 1)(|\mu| + 1)$, then $p_1(p_1 - c)u + g(u)$ is increasing in R and (H_1) is fulfilled. If $\mu > 0$, choosing $m_1 = 0.5\pi, M_1 = 1.5\pi$; if $\mu < 0$, choosing $m_1 = 1.5\pi, M_1 = 2.5\pi$, we obtain that

$$g(M_1) \leq h(t) \leq g(m_1), \forall t \in R.$$

Hence, (1.3) has at least one periodic solution x_1 with $m_1 \leq x_1 \leq M_1$.

Further suppose that $c \geq 2\sqrt{|\mu|}$ and $h \not\equiv \pm\mu$. Put $p_2 = c/2$, then $p_2(p_2 - c)u + g(u)$ is nonincreasing in R and (H_3) is fulfilled. If $\mu > 0$, choosing $m_2 = -0.5\pi, M_2 = 0.5\pi$; if $\mu < 0$, choosing $m_2 = 0.5\pi, M_2 = 1.5\pi$, we obtain that

$$g(m_2) \leq h(t) \leq g(M_2), \forall t \in R.$$

(1.3) has at least one periodic solution x_2 with $m_2 \le x_2 \le M_2$. Since $h \not\equiv \pm\mu$, $x_i \not\equiv m_j, x_i \not\equiv M_j (i,j = 1,2)$, $x_1 \not\equiv x_2$. $\square$

Corollary 3.3. *Let e be ω-periodic continuous function and f a bounded continuous function on R. Further suppose that there exist constants $m < M$ such that $g \in C^1([m,M],R)$ and*

$$g(M) + \alpha \le e(t) \le g(m) + \beta, \quad t \in [0,\omega],$$

where $\alpha = \sup_{u\in R} f(u)$ and $\beta = \inf_{u\in R} f(u)$. Then (1.4) has at least one ω-periodic solution x with $m \le x \le M$.

Assume that e is ω-periodic continuous function, for (1.5), we have the following result.

Theorem 3.3. *Assume that there exist constants $m < M$ such that $g, f \in C^1([m,M],R)$ and*

$$g(M) \le e(t) \le g(m) \quad \text{for } \forall t \in [0,\omega].$$

Then (1.5) has at least one ω-periodic solution x with $m \le x \le M$.

Proof. Define the function

$$\Gamma(t,v) = pv + \frac{g(v) - e(t)}{p - f(v)}, t \in R, v \in [m,M],$$

where $p > 0$ is sufficiently large which is determined later. By computation, we have

$$\Gamma_v(t,v) = p + \frac{g'(v)}{p - f(v)} + \frac{f'(v)(g(v) - e(t))}{(p - f(v))^2}.$$

Put

$$\begin{aligned} p &= \max_{m\le u\le M} |f(u)| + \max_{m\le u\le M} |g'(u)| \\ &\quad + \max_{m\le u\le M} |f'(u)| \times \left(\max_{m\le u\le M} |g(u)| + \max_{t\in R} |e(t)| \right) + 2, \end{aligned}$$

then $\Gamma_v(t,v) > 0$ for $t \in R, v \in [m,M]$ and Γ is nondecreasing in $v \in [m,M]$ for any fixed $t \in R$. Hence,

$$pm + \frac{g(m) - e(t)}{p - f(m)} \le \Gamma(t,v) \le pM + \frac{g(M) - e(t)}{p - f(M)}, \ t \in R, \ v \in [m,M].$$

Similar to (2.5), using the operator Ju, we transform (1.5) to the equation

$$u'(t) = (p - f(Ju))u(t) - [p^2 Ju + g(Ju) - e(t) - p(Ju)f(Ju)]. \tag{3.2}$$

Define an operator K on the closed, convex set Ω by

$$(Ku)(t) = \int_t^{t+\omega} \frac{\exp \int_s^{t+\omega} (p - f(Ju))dr}{\exp \int_0^{\omega} (p - f(Ju))dr - 1} (Lu)(s)ds,$$

$$(Lu)(t) = p^2 (Ju)(t) + g((Ju)(t)) - e(t) - p(Ju)(t)f((Ju)(t)),$$

$$\Omega = \{x \in X : mp \le x(t) \le pM \text{ for } t \in [0,\omega]\}.$$

Hence, the fact that K has fixed point on Ω implies that (1.5) has the periodic solution.

For any $u \in \Omega$, $m \le Ju \le M$, $p > f(Ju)$ and

$$\begin{aligned}
(Ku)(t) &= \int_t^{t+\omega} \frac{\exp \int_s^{t+\omega}(p-f(Ju))dr}{\exp \int_0^{\omega}(p-f(Ju))dr - 1}(p-f(Ju)(s))\Gamma(s,(Ju)(s))ds \\
&\le \int_t^{t+\omega} \frac{\exp \int_s^{t+\omega}(p-f(Ju))dr}{\exp \int_0^{\omega}(p-f(Ju))dr - 1}(p-f(Ju)(s))\left(pM + \frac{g(M)-e(s)}{p-f(M)}\right)ds \\
&\le pM \int_t^{t+\omega} \frac{\exp \int_s^{t+\omega}(p-f(Ju))dr}{\exp \int_0^{\omega}(p-f(Ju))dr - 1}(p-f(Ju)(s))ds = pM,
\end{aligned}$$

$$\begin{aligned}
(Ku)(t) &\ge \int_t^{t+\omega} \frac{\exp \int_s^{t+\omega}(p-f(Ju))dr}{\exp \int_0^{\omega}(p-f(Ju))dr - 1}(p-f(Ju)(s))\left(pm + \frac{g(m)-e(s)}{p-f(M)}\right)ds \\
&\ge pm \int_t^{t+\omega} \frac{\exp \int_s^{t+\omega}(p-f(Ju))dr}{\exp \int_0^{\omega}(p-f(Ju))dr - 1}(p-f(Ju)(s))ds = pm.
\end{aligned}$$

Hence, $K(\Omega) \subset \Omega$.

The rest proof is similar to that of Theorem 3.1. □

Remark 3.2 Using the same idea, for the following differential equation

$$x''(t) + f(t,x(t))x'(t) + g(t,x(t)) = e(t), \tag{3.3}$$

we have the following result.

Corollary 3.4. *Assume that e is ω-periodic continuous function and there exist constants $m < M$ such that $g_u(t,u), f_u(t,u) \in C(R \times [m,M], R)$,*

$$g(t,M) \le e(t) \le g(t,m) \quad \textit{for } \forall t \in [0,\omega].$$

Then (3.3) has at least one ω-periodic solution x with $m \le x \le M$.

4 Some examples

In this section, three examples are provided to highlight our results obtained in previous section.

Example 4.1. Consider the differential equation with singularity

$$x''(t) = \frac{G(t)}{x^{\mu}(t)} - \frac{H(t)}{x^{\lambda}(t)} + F(t), \tag{4.1}$$

where G, H, F are ω-periodic continuous functions, $\mu, \lambda > 0$.

In recent paper [7], R. Hakl and P J. Torres discussed the existence of positive periodic solution of (4.1) when $G, H \in L^+$ and $F \in L$, where L is the Banach space of ω-periodic Lebesgue integrable function and $L^+ = \{p \in L : p \ge 0 \text{ for a.e. } t \in [0,\omega]\}$. By using method of upper and lower functions, authors established several existence results. By Corollary 3.1, we obtain the following new result.

Proposition 4.1 Assume that there exist $0 < m < M$ such that for $\forall t \in [0,\omega]$,

$$m^{\lambda-\mu}G^+(t) + \frac{m^{\lambda}}{M^{\mu}}G^-(t) + m^{\lambda}F(t) \le H(t) \le M^{\lambda-\mu}G^+(t) + \frac{M^{\lambda}}{m^{\mu}}G^-(t) + M^{\lambda}F(t)\ , \tag{4.2}$$

here $G^+(t) = \max\{G(t), 0\}$ and $G^-(t) = \min\{G(t), 0\}$. Then (4.1) has at least one positive ω-periodic solution.

Proof. By the condition (4.2), we have

$$\frac{G^+(t)}{m^\mu} + \frac{G^-(t)}{M^\mu} + F(t) \le \frac{H(t)}{m^\lambda}, \quad \frac{H(t)}{M^\lambda} \le \frac{G^+(t)}{M^\mu} + \frac{G^-(t)}{m^\mu} + F(t).$$

On the other hand, for any $u \in [m, M]$,

$$\frac{G^+(t)}{M^\mu} + \frac{G^-(t)}{m^\mu} \le \frac{G(t)}{u^\mu} \le \frac{G^+(t)}{m^\mu} + \frac{G^-(t)}{M^\mu}.$$

Hence for any $u \in [m, M]$, the inequality

$$\frac{H(t)}{M^\lambda} \le \frac{G(t)}{u^\mu} + F(t) \le \frac{H(t)}{m^\lambda}$$

holds. By Corollary 3.1, (4.1) has at least one positive ω-periodic solution. □

The condition (4.2) admits that G changes sign, which is different with those of [7] . For example, using (4.2), one can check that the differential equation

$$x''(t) = \frac{1 + 2\sin t}{x^{0.8}(t)} - \frac{2}{x(t)} + 15 \tag{4.3}$$

has one 2π-periodic solution x with $0.1 \le x \le 10$.

When $G \equiv 0$, (4.2) reduces to

$$m^\lambda F(t) \le H(t) \le M^\lambda f(t) \text{ for } t \in [0, \omega], \tag{4.4}$$

which guarantee that the equation

$$x''(t) = -\frac{H(t)}{x^\lambda(t)} + F(t) \tag{4.5}$$

has at least one positive ω-periodic solution. The new condition (4.4) in which H is possibly zero on the set of a positive measure gives partial answer to the open problem 4.2 in [7].

Example 4.2. Consider the differential equation

$$x''(t) + \frac{1}{8}(x'(t))^2 - x^2(t) = \sin t - 1. \tag{4.6}$$

We claim that (4.6) has at least two 2π-periodic solutions. In fact, $g(u) = -u^2$,$f(t, u, v) = \sin t - 1 - v^2/8$.

Put $p = 2, m = 0, M = 2$, then for $(t, u, v) \in [0, 2\pi] \times [0, 2] \times [-4, 4]$,

$$g(M) = -4 \le f(t, u, v) \le g(m) = 0.$$

By Theorem 3.1, (4.6) has at least one 2π-periodic solution x_1 with $0 \le x_1 \le 2$. Similarly, one easily check that (4.6) has one 2π-periodic solution x_2 with $-2 \le x_2 \le 0$.

Example 4.3. Consider the differential equation

$$-x''(t) + a(t)x'(t) = x^\alpha(t) \sin x(t) - f(t, x(t)), \tag{4.7}$$

where $\alpha > 0$, a is a continuous ω-periodic function, $f(t, u) : R \times R \to R$ is continuous and ω-periodic in t.

We claim that (4.7) has infinitely many periodic solutions if the following condition is fulfilled:

$$\lim_{u\to+\infty}\frac{|f(t,u)|}{u^\alpha}<1 \tag{4.8}$$

uniformly with respect to $t\in R$.

In fact, there exist constant $\rho>0$ such that

$$|f(t,u)|\le u^\alpha \quad u\ge\rho.$$

Choose integer n such that $2\pi n>\rho$. Let $m=2\pi n+0.5\pi$ and $M=2\pi n+1.5\pi$, then $g(u)=u^\alpha\sin u\in C^1([m,M],R)$ and

$$g(M)=-M^\alpha\le f(t,u)\le m^\alpha=g(m) \;\text{ for }\; (t,u)\in R\times[m,M].$$

By Corollary 3.1, (4.7) has at least one ω-periodic solution x with $m\le x\le M$. Since n is arbitrary sufficiently large integer, (4.7) has infinitely many periodic solution.

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