Self–similar solutions to a convection–diffusion processes

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Zaïna, in memorium

Abstract

Geometric properties of self–similar solutions to the equation $u_t = u_{xx} + \gamma(u^q)_x,\ x > 0,\ t > 0$ are studied, q is positive and $\gamma \in \mathbb{R} \setminus \{0\}$. Two critical values of q (namely 1 and 2) appear the corresponding shapes are of very different nature.

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1. Introduction

In this paper we shall derive properties of solutions to the equation

$$\begin{cases} u_t = u_{xx} + \gamma(u^q)_x, & \text{for } (x,t) \in (0,+\infty) \times (0,+\infty), \\ u_x(0,t) = 0, & \text{for } t > 0, \end{cases} \tag{1.1}$$

having the form

$$u(x,t) = t^\alpha g(xt^{-1/2}) =: t^\alpha g(\xi), \tag{1.2}$$

where $q > 0,\ \alpha = -\frac{1}{2(q-1)},\ \gamma \in \mathbb{R} \setminus \{0\}$, and $u > 0$ in the half space for appropriate nonnegative initial data.

If we substitute (1.2) into (1.1) we obtain for $q \neq 1$, the ODE

$$g'' + \gamma(g^q)' = \alpha g - \frac{1}{2}\xi g', \quad \xi > 0, \tag{1.3}$$

subject to the condition

$$g'(0) = 0. \tag{1.4}$$

Setting $\gamma = \pm 1$ we are led to the problem

$$\begin{cases} g'' + \varepsilon(g^q)' + \frac{1}{2}\xi g' - \alpha g = 0, & \xi > 0, \\ g'(0) = 0, \quad g(0) = \lambda, \end{cases} \tag{1.5}$$

where $\varepsilon = \pm 1, \lambda > 0$ and $q \neq 1$ is a positive number. This problem has a unique local solution for every $\lambda > 0$. We shall be interested in possible extension of solutions and their properties. A more general equation with $\gamma \neq 0$ can be transformed to (1.5) by introducing a new function $|\gamma|^{\frac{1}{q-1}} g$ which solves (1.5) with $\lambda|\gamma|^{\frac{1}{q-1}}$ instead of λ. When $\varepsilon = 1$ and $q > 1$ problem (1.5) was investigated in detail by Peletier and Serafini [12]. It is shown that there exists λ_c such that problem (1.5) has a unique global solution $g > 0$ such that $\xi^{-2\alpha} g(\xi)$ goes to 0 if and only if $1 < q < 2$ and $g(0) = \lambda_c$ and its asymptotic behavior at infinity is given by

$$g(\xi) = L\xi^{-2\alpha-1} e^{-\xi^2/4} \Big\{ 1 + 2(2\alpha+1)(\alpha-1)\xi^{-2} + o(\xi^{-2}) \Big\}$$

as $\xi \to +\infty$, for some positive constant L. The paper by Biler and Karch [3] is devoted to study the large-time behavior of solutions to (1.1) with $(u|u|^{q-1})_x, q > 1$ instead of $\gamma(u^q)_x$, where initial data satisfying $\lim_{x\to\infty} x^\beta u(x,0) = A$ for some $A \in \mathbb{R}$ and $0 < \beta < 1$. In this paper we shall show that if $\varepsilon = -1$ and $1 < q < 2$ the solution of (1.5) changes the sign for every $\lambda > 0$. Also we are interested on values on q and $\lambda > 0$ which guarantee that problem (1.5) has a global positive solution with given behavior at infinity. The basic method used here is due to [5]. We analyze problem (1.5) in the phase plane. Somes results can be found in [3]
Equation (1.3) does not belong to the class of well–studied second order nonlinear ODE's. If we write it in the standard (from point of view of nonlinear oscillation theory) form

$$g'' + \left(q\varepsilon g^{q-1} + \frac{1}{2}\xi \right) g' - \alpha g = 0, \tag{1.6}$$

we can see that the "friction coefficient" which depends nonlinearly on g and on position, can change sign if $\varepsilon = -1$. The sign of α depends on q: if $q < 1$ then $\alpha > 0$ and $\alpha < 0$ for $1 < q$.

Remark 1.1. The function $w(x,t) = x^a h(tx^{-b}) =: x^a h(\eta)$ satisfies (1.1) if and only if $b = 2$ and $a = (q-2)/(q-1)$. The corresponding ODE is

$$h'' = (a - 3/2)\frac{h'}{\eta} + \frac{\varepsilon}{2}(h^q)' + \frac{h'}{\eta^2} + \frac{h}{\eta}(1 + qa\gamma\varepsilon h^{q-1}), \quad \eta > 0.$$

We shall not deal here with it.

The plan of the paper is the following:
Section 2 : Preliminary results.
Section 3 : Large ξ behaviour of all global solutions for $q > 2$ and $\varepsilon = -1$.
Section 4 : The case $0 < q < 1$.

2. Preliminaries

Rather than studying (1.5), we will deal here with the slightly more general ODE

$$g'' + q\varepsilon|g|^{q-1}g' = \alpha g - \frac{1}{2}\xi g', \quad \xi > 0, \tag{2.1}$$

$$g(0) = \lambda, \quad g'(0) = 0, \tag{2.2}$$

in which the nonnegative number q is not equal to 1, $\alpha = -\frac{1}{2(q-1)}$, $\lambda > 0$ and $\varepsilon \in \{-1, 1\}$.
Problem (1.5) is a special case of (2.1)–(2.2) in which $g > 0$. As it was mentioned before, for any $\lambda > 0$, problem (2.1)–(2.2) has a unique maximal solution $g(., \lambda) \in C^2([0, \xi_{max}))$. Furthermore $g(\xi, \lambda) > 0$ for small $\xi > 0$. An important objective is to find values of λ and q which insure that $g(., \lambda)$ is global, nonnegative and to give the asymptotic behavior as ξ tends to infinity. In this section we shall derive some properties of g which will be useful in the proof of the main results.

LEMMA 2.1. *Assume that $\alpha < 0$. Let g be a solution to (2.1)–(2.2) such that $g > 0$ on $[0, \xi_0)$. Then $g'(\xi) < 0$, for all $0 < \xi < \xi_0$.*

PROOF. As $g''(0) = \alpha\lambda < 0$ and $g'(0) = 0$, the function g is decreasing for small ξ. Suppose that there exists $\xi_1 \in (0, \xi_0)$ such that $g'(\xi) < 0$ on $(0, \xi_1)$ and $g'(\xi_1) = 0$. Using (2.1) one sees $g''(\xi_1) < 0$. Therefore we get a contradiction. □

LEMMA 2.2. *Assume that $\varepsilon = -1$ and $\alpha \leq -\frac{1}{2}$. Then $g(., \lambda)$ changes the sign for every $\lambda > 0$.*

PROOF. Suppose in the contrary that $g(., \lambda) > 0$. Then $g'(., \lambda) < 0$ (and then $g(., \lambda)$ is global). On the other hand Equation (2.1) can be written as

$$g'' + \frac{1}{2}(\xi g)' = (\alpha + \frac{1}{2})g + (g^q)'.$$

Thus we have

$$g'(\xi) + \frac{1}{2}\xi g(\xi) = (\alpha + \frac{1}{2}) \int_0^\xi g(\eta)d\eta + g^q(\xi) - \lambda^q.$$

This implies that $g(\xi) \leq e^{-\frac{\xi^2}{4}}$, for all $\xi \geq 0$. Then passing to the limit, $\xi \to +\infty$, we infer

$$0 = (\alpha + \frac{1}{2}) \int_0^\infty g(\eta)d\eta - \lambda^q.$$

This is impossible.

□

REMARK 2.1. The situation is different if $\varepsilon = 1$. Peletier and Serafini [12] showed that if $\alpha < -\frac{1}{2}$ the solution changes the sign for λ sufficiently small. And if $0 > \alpha \geq -\frac{1}{2}$, the solution g is nonnegative on $[0, +\infty[$.

Finally a standard analysis gives the following

LEMMA 2.3. *Assume that $\alpha > 0$. Let g be a solution to (2.1)–(2.2) defined on $[0, \xi_0[$, where $\varepsilon = \pm 1$. Then $g'(\xi) > 0$ for all $0 < \xi < \xi_0$.*

In fact we shall show in Section 4 that the solution g cannot blow-up for finite ξ. In the next sections we shall give the asymptotic behavior of all possible positive solutions to (2.1)–(2.2) in the following cases : $\varepsilon = -1$ and $q > 2$ and $\varepsilon = \pm 1$ and $0 < q < 1$.

3. Global behavior for $q > 2$ and $\varepsilon = -1$

The first simple consequence of the fact that $q > 2$ is that $0 > \alpha > -\frac{1}{2}$, and then if $g(\xi) > 0$ on $(0, +\infty)$ we have $g'(\xi) < 0$ for all $\xi > 0$. It is also clear that

$$g(\xi) \leq \lambda, \quad \forall\, \xi \geq 0. \tag{3.1}$$

Actually $g(\xi) \leq \lambda$, for small ξ, in order to be bigger that λ, the solution g has to return at some $\xi_1 > 0$, and at this point $g'(\xi_1) = 0$ and $g''(\xi_1) \geq 0$ in contradiction with (2.1) for $g(\xi_1) \geq 0$. If $g(\xi_1) < 0$ then g cannot cross the line $\xi = 0$ again : suppose "yes" at point ξ_2 : $g(\xi_2) = 0$. Here we have that $g < 0$ on (ξ_1, ξ_2) and by a uniqueness argument we may conclude that $g'(\xi_1) < 0$ and $g'(\xi_2) > 0$. Now observe that (2.1) can be written as

$$g'' + (|g|^q)' + \frac{1}{2}(\xi g)' = (\alpha + \frac{1}{2})g, \quad \xi \in (\xi_1, \xi_2).$$

Integrate the last equation over (ξ_1, ξ_2) :

$$g'(\xi_2) = (\alpha + \frac{1}{2}) \int_{\xi_1}^{\xi_2} g d\xi + g'(\xi_1) < 0,$$

while the left hand side is positive. We get a contradiction. So $g(\xi)$ is bounded from above by λ. And we can conclude that

LEMMA 3.1. *For any $\lambda > 0$, and $\varepsilon = \pm 1$, the solution $g(., \lambda)$ to (2.1)–(2.2) can have at most one zero on $(0, \infty)$.*

Peletier and Serafini proved in fact that for $\varepsilon = 1$ any solution is nonnegative.

The following lemma shows that all global positive solutions decay to 0.

LEMMA 3.2. *Let g be the solution to* (2.1)–(2.2) *where $q > 2$. Assume that $g(\xi) > 0$ for any $\xi > 0$. Then*

$$\lim_{\xi \to +\infty} g(\xi) = 0, \quad \lim_{\xi \to +\infty} g'(\xi) = 0.$$

PROOF. Since $g' < 0$ and g is bounded below by 0 g has a finite limit at ∞; say g_0. First there exists (ξ_n) such that $g'(\xi_n)$ goes to 0 as ξ_n tends to ∞ with n.
Now as the energy $E = (g')^2 - \alpha g^2$ is monotone decreasing for a large ξ we deduce that g' tends to 0 as $\xi \to \infty$. Now suppose that $g_0 > 0$. Equation (2.1) gives

$$g'' + \frac{1}{2}\xi g' < \alpha g_0.$$

Multiply this by $e^{\frac{\xi^2}{4}}$ and integrate :

$$g'(\xi) < \alpha g_0 e^{-\frac{\xi^2}{4}} \int_0^{\xi} e^{\frac{\tau^2}{4}} d\tau. \tag{3.2}$$

Since

$$\lim_{\xi\to+\infty}\frac{\displaystyle\int_0^\xi e^{\frac{\tau 2}{4}}d\tau}{\frac{1}{\xi}e^{\frac{\xi^2}{4}}}=2,$$

thanks to l'Hôpital's rule, we infer

$$g'(\xi)<\alpha g_0\frac{1}{\xi},\ \text{for } \xi \text{ large},$$

which implies that g goes to $-\infty$ as $\xi\to+\infty$, this is impossible. □

In [3] it is shown that

$$g(\xi)\le\lambda\xi^{2\alpha}\left\{1-2\alpha\int_0^{+\infty}\tau^{-2\alpha-1}e^{-\frac{\tau^2}{4}}d\tau\right\}, \tag{3.3}$$

for all $\xi>0$. Therefore $g(\xi)$ goes to 0 as $\xi\to+\infty$ since $\alpha<0$.

LEMMA 3.3. *Assume $\varepsilon=-1$. Then there exists $\lambda_1>0$ such that the solution $g(.,\lambda)$ to (2.1)–(2.2), where $q>2$ and $\lambda>\lambda_1$, has exactly one positive zero.*

PROOF. Assume that for all $\lambda>0$ the solution, $g(.,\lambda)$, to (2.1)–(2.2) is positive on $(0,+\infty)$ and then $g'(.,\lambda)<0$.
Set $g=g(.,\lambda)$.
Integrating of (2.1) over $(0,\xi)$ yields

$$g'(\xi)+\frac{1}{2}\xi g(\xi)=(\alpha+\frac{1}{2})\int_0^\xi g(\tau)d\tau+g^q(\xi)-\lambda^q.$$

Then

$$g'(\xi)\le(\alpha+\frac{1}{2})\lambda\xi-\lambda^q+g^q(\xi).$$

From the last inequality and (3.3), we deduce that

$$g(\xi)\le\lambda+\frac{1}{2}(\alpha+\frac{1}{2})\lambda\xi^2-\lambda^q\xi+C\lambda^q,$$

for some positive constant C, which is independent of λ.
Setting $\xi=2C$ we infer

$$g(2C)\le\lambda+2(\alpha+\frac{1}{2})C^2\lambda-C\lambda^q.$$

This shows that $g(2C)<0$ if λ is large, a contradiction. □

Let us now investigate in more detail how (g,g') behaves in the phase plane as ξ increases. We proceed as in [5]. Set $h=g'$, equation (2.1) is reduced to the following first order system

$$\left\{\begin{array}{rcl} g' &=& h,\\ \\ h' &=& \alpha g+q|g|^{q-1}h-\frac{1}{2}\xi h,\end{array}\right. \tag{3.4}$$

with the initial condition

$$g(0) = \lambda, \quad h(0) = 0. \tag{3.5}$$

This system has only one critical point $(0,0)$ and since $q > 1$, problem (3.4)–(3.5) has a unique local solution (g, h) for every $\lambda > 0$.
For each $\gamma > 0$ we define

$$P_\gamma = \Big\{(g, h); g > 0, h < 0, h \geq -\gamma g\Big\},$$

and we introduce

$$\xi(\lambda, \gamma) = 2(-\frac{\alpha}{\gamma} + \gamma) + 2q\lambda^{q-1}.$$

Arguing as in [5, 9] we obtain

LEMMA 3.4. *For any fixed $\gamma > 0$, the set P_γ is positively invariant for $\xi_0 > \xi(\lambda,\gamma)$; that is if $\xi_0 > \xi(\lambda, \gamma)$ and $(g(\xi_0), h(\xi_0)) \in P_\gamma$, then $(g(\xi), h(\xi)) \in P_\gamma$ for all $\xi \geq \xi_0$.*

According to Lemmas 3.2 and 3.4 we have

LEMMA 3.5. *Let g be the solution to (2.1). Assume that $g(\xi) > 0$ for all $\xi > 0$. Then*

$$\text{either} \quad \lim_{\xi\to+\infty} \frac{g'(\xi)}{g(\xi)} = 0 \quad \text{or} \quad \lim_{\xi\to+\infty} \frac{g'(\xi)}{g(\xi)} = -\infty. \tag{3.6}$$

The proof is similar to the proof of the corresponding results in [5, 9, 12].

Setting

$$L^\star(\lambda) = \lambda \left\{1 - 2\alpha \int_0^{+\infty} \tau^{-2\alpha-1} e^{-\frac{\tau^2}{4}} d\tau\right\}.$$

PROPOSITION 3.1. *Let g be the solution to (2.1)–2.2) such that $g > 0$. Then the limit*

$$L(\lambda) = \lim_{\xi\to\infty} \xi^{-2\alpha} g(\xi),$$

exists in $[0, L^\star(\lambda)]$ and we have

$$\begin{aligned} \lim_{\xi\to\infty} \frac{g'(\xi)}{g(\xi)} &= -\infty \quad \Rightarrow \quad L(\lambda) = 0, \\ \lim_{\xi\to\infty} \frac{g'(\xi)}{g(\xi)} &= 0 \qquad \Rightarrow \quad L(\lambda) > 0. \end{aligned}$$

PROOF. If $\lim_{\xi\to\infty} \frac{g'(\xi)}{g(\xi)} = -\infty$, then $g(\xi) = O(e^{-k\xi})$ as $\xi \to \infty$, so that $\xi^{-2\alpha} g(\xi)$ goes to 0 as $\xi \to \infty$. Now assume that

$$\lim_{\xi\to\infty} \frac{g'(\xi)}{g(\xi)} = 0.$$

Set

$$u(\xi) = \frac{g'(\xi)}{g(\xi)}.$$

Thus

$$u' + \frac{1}{2}\xi u = -\frac{1}{2}\frac{2-q}{q-1} + \varphi(\xi), \quad u(0) = 0, \tag{3.7}$$

where $\varphi(\xi) = qg^{q-1}u - u^2$.
Note that φ goes to 0 as $\xi \to +\infty$ and u can be defined by

$$u(\xi) = e^{-\frac{\xi^2}{4}} \int_0^\xi \{\alpha + \varphi(\tau)\} e^{\frac{\tau^2}{4}} d\tau, \tag{3.8}$$

hence

$$\xi u(\xi) = \frac{\displaystyle\int_0^\xi \{\alpha + \varphi(\tau)\} e^{\frac{\tau^2}{4}} \} d\tau}{\frac{1}{\xi} e^{\frac{\xi^2}{4}}}, \quad \forall\, \xi > 0. \tag{3.9}$$

Applying the l'Hôpital's rule to the right–hand side of (3.9), we infer

$$\lim_{\xi \to \infty} \xi u(\xi) = 2\alpha. \tag{3.10}$$

This shows in particular that for any $\tau > 0$ there exists $K_\tau > 0$ such that

$$g(\xi) \leq K_\tau \xi^{(2\alpha+\tau)}, \text{ for all } \xi \geq 0. \tag{3.11}$$

Now given $1 \leq k < 2 - \tau(q-1)$. Since

$$\xi^k \varphi(\xi) = qg^{q-1}\xi^k u - \xi^k u^2,$$

we get

$$\xi^k |\varphi(\xi)| \leq qK_\tau^{q-1} \xi^{k-2+\tau(q-1)} + \xi^{k-2}(\xi u)^2.$$

According to the choice of k and to (3.10) we deduce $\lim_{\xi \to +\infty} \xi^k \varphi(\xi) = 0$.

On the other hand u satisfies:

$$\xi^k \{\xi u(\xi) - 2\alpha\} = \frac{\displaystyle\int_0^\xi (\alpha + \varphi(\tau))\, e^{\frac{\tau^2}{4}} - 2\alpha e^{\frac{\xi^2}{4}} \xi^{-1}}{e^{\frac{\tau^2}{4}} \xi^{-1-k}}.$$

Then, by l'Hôpital's rule, we get that

$$\lim_{\xi \to +\infty} \xi^k \{\xi u(\xi) - 2\alpha\} = 2 \lim_{\xi \to +\infty} \xi^k \varphi(\xi) = 0.$$

It follows from this that

$$\frac{g'}{g} = 2\alpha\frac{1}{\xi} + \frac{\epsilon(\xi)}{\xi^{k+1}},$$

for all $\xi > \xi_0$.
Hence

$$g(\xi) = L(\lambda)\xi^{2\alpha}\left\{1 + o(\frac{1}{\xi})\right\}, L(\lambda) > 0.$$

$\square$

Now we are in position to give the asymptotic behavior of $g(., \lambda)$.

THEOREM 3.1. *Let g be the solution to (2.1)–(2.2) such that $g(\xi) > 0$ for all $\xi > 0$.*

1. *If $L(\lambda) = 0$, there exists $A > 0$ such that*

$$g(\xi) = Ae^{-\frac{\xi^2}{4}}\xi^{\frac{2-q}{q-1}}\left\{1 - \frac{b}{\xi^2} + o(\frac{1}{\xi^2})\right\},$$

2. *if $L(\lambda) > 0$, then*

$$g(\xi) = L(\lambda)\xi^{-\frac{1}{q-1}}\left\{1 - \frac{c}{\xi^2} + o(\frac{1}{\xi^2})\right\},$$

as $\xi \to \infty$, where $b = \dfrac{(2q-3)(q-2)}{(q-1)^2}$ and $c = 2q\alpha(L(\lambda))^{q-1} + 2\alpha(1-2\alpha)$.

PROOF. For the proof of item 2 it is sufficient to settle $\lim_{\xi\to+\infty} \xi^2(\xi u(\xi) - 2\alpha)$. Same as above we have

$$\lim_{\xi\to+\infty} \xi^2\left[\xi u(\xi) - 2\alpha\right] = 2\lim_{\xi\to+\infty} \xi^2\varphi(\xi) + 4\alpha.$$

This yields that

$$\lim_{\xi\to+\infty} \xi^2\left[\xi u(\xi) - 2\alpha\right] = 4q\alpha(L(\lambda))^{q-1} - 2(2\alpha)^2 + 4\alpha.$$

Consequently

$$\frac{g'}{g} = -2\frac{c}{\xi^3} - \frac{1}{q-1}\frac{1}{\xi} + \frac{\epsilon(\xi)}{\xi^3}, \tag{3.12}$$

where $c = 2q\alpha(L(\lambda))^{q-1} + 2\alpha(1-2\alpha)$.
A simple integration of (3.12) yields the desired asymptotic.

Now we shall prove item 1. Here we assume that $L(\lambda) = 0$. By Equation (2.1) one sees

$$\frac{g''}{\xi g'(\xi) + g} = \frac{-\dfrac{1}{2} + \alpha\dfrac{g}{\xi g'} + \dfrac{qg^{q-1}}{\xi}}{1 + \dfrac{g}{\xi g'}}.$$

Now using the l'Hôpital's rule and the fact that $g/g' \to 0$ at infinity we obtain

$$\lim_{\xi\to\infty} \frac{g'}{\xi g} = -\frac{1}{2}. \tag{3.13}$$

Next define

$$G(\xi) = \xi g' + \frac{1}{2}\xi^2 g, \quad F(\xi) = \xi^2 G - a\xi^2 g,$$

where

$$a = -(2\alpha+1) = \frac{q-2}{q-1}.$$

A simple computation shows that

$$\frac{G'}{g'} = 1 + \frac{\xi g}{g'} + q\xi g^{q-1} + \alpha\frac{\xi g}{g'}, \tag{3.14}$$

and

$$\frac{F'(\xi)}{g'(\xi)} = 2(\alpha+1)\xi\frac{g}{g'}\frac{G}{g} + q\xi^3 g^{q-1} + 2\frac{\xi g}{g'}\left[\frac{G}{g} - a\right]. \tag{3.15}$$

Applying again the l'Hôpital's rule to (3.14)–(3.15) we deduce successively

$$\lim_{\xi\to+\infty} \frac{G(\xi)}{g(\xi)} = a,$$

and

$$\lim_{\xi\to+\infty} \frac{F(\xi)}{g(\xi)} = 2b,$$

where $b = \frac{(2q-3)(q-2)}{(q-1)^2}$. Same as above results we get item 1 by an easy integration. $\square$

Remark 3.1. The results of Theorem 3.1 have been recently obtained independently by P. Biler and G. Karch in [3].

Remark 3.2. It follows from Theorem 3.1 that if $g \in L^1((0,+\infty))$ and is positive then g satisfies item 1;

$$g(\xi) = Ae^{-\frac{\eta^2}{4}}\xi^{\frac{2-q}{q-1}}\left\{1 - \frac{b}{\xi^2} + o(\frac{1}{\xi^2})\right\}.$$

Integrating (2.1) over $(0,\xi)$ yields

$$g'(\xi) + \frac{1}{2}\xi g(\xi) - g^q(\xi) + \lambda^q = (\alpha + \frac{1}{2})\int_0^\xi g(\eta)d\eta.$$

Passing to the limit, as $\xi \to \infty$, we deduce

$$\lambda^q \frac{2(q-1)}{q-2} = \int_0^\infty g(\xi)d\xi.$$

This shows in particular the following uniqueness result.

Proposition 3.2. *Let $q > 2$. Let f and h be two solutions to*

$$\begin{cases} g'' - (g^q)' = \alpha g - \frac{1}{2}\xi g', \text{ on } (0,+\infty), \\ \\ g'(0) = 0, \quad g(\xi) > 0, \text{ for all } \xi \geq 0, \end{cases}$$

such that

$$\int_0^\infty f(\xi)d\xi = \int_0^\infty h(\xi)d\xi.$$

Then $f \equiv h$.

Now we shall show that problem (2.1)–(2.2) has a positive solution satisfying item 2 provided that the initial data $g(0)$ is sufficiently small. To this end we set

$$f = g/\lambda.$$

Therefore f satisfies

$$\begin{cases} f'' + \frac{1}{2}\xi f' - q\lambda^{q-1}|f|^{q-1}f' - \alpha f = 0, \\ \\ f'(0) = 0, \quad f(0) = 1. \end{cases} \tag{3.16}$$

If we now let $\lambda \to 0$, we formally obtain

$$\begin{cases} f'' + \frac{1}{2}\xi f' - \alpha f = 0, \\ \\ f'(0) = 0, \quad f(0) = 1. \end{cases} \tag{3.17}$$

Since the energy function $H = (f')^2 - \alpha f^2$ is nonincreasing and uniformly bounded by $-\alpha > 0$, f is global and goes to 0. Moreover $f > 0, f' < 0$ and satisfies item 2 of Theorem 3.1 (otherwise we get $0 = \|f\|_1$, a contradiction -see Remark 3.1-). Since (3.16) is a regular perturbation of (3.17) it follows that the solution to (3.16) is global, positive and satisfies item 1 for λ sufficiently small. Results of the present section gives us quite a good picture of the main properties of solutions to (2.1)–(2.2). We have one of the following properties:

a) $g(\xi) > 0$ on some $(0, \xi_0)$ and $g(\xi_0) = 0$,

b) $g(\xi) > 0$ for all $\xi \geq 0$ and $g(\xi) = L(\lambda)\xi^{-\frac{1}{q-1}}\left\{1 - \frac{c}{\xi^2} + o(\frac{1}{\xi^2})\right\}$,

c) $g(\xi) > 0$ for all $\xi \geq 0$ and $g(\xi) = Ae^{-\frac{\xi^2}{4}}\xi^{\frac{2-q}{q-1}}\left\{1 - \frac{b}{\xi^2} + o(\frac{1}{\xi^2})\right\}$.

Returning to the original variables u and x we can see that the asymptotics behavior given in a) and b) yield the following two possibilities

a1) either

$$\int_0^\infty u(x,t)dx = +\infty, \qquad \text{for any } t > 0,$$

b1) or

$$\int_0^\infty u(x,t)dx = Mt^{\frac{1}{2}\frac{q-2}{q-1}}, \qquad \text{for any } t > 0.$$

4. Case $0 < q < 1$ and $\varepsilon = \pm 1$

In this section we consider

$$g'' + \varepsilon q|g|^q g' = \alpha g - \frac{1}{2}\xi g', \quad \xi > 0, \tag{4.1}$$

$$g(0) = \lambda > 0, \quad g'(0) = 0, \tag{4.2}$$

in which $\alpha = -\frac{1}{2}\frac{1}{q-1}, 0 < q < 1$ and $\varepsilon = \pm 1$. We study the asymptotic behavior of global solutions to (4.1)–(4.2). Note that $\alpha > 0$ and the standard theory of initial value problems implies the existence and uniqueness of such solutions in a neighbourhood of the origin. At $\xi = 0$ $g''(0) = \alpha\lambda > 0$. So in a small neighbourhood of 0 g is increasing. In order to show that problem (4.1)–(4.2) has a unique global solution, it is sufficient to show the following

LEMMA 4.1. *The solution $g(\xi)$ to (4.1)–(4.2) cannot blow-up for finite ξ; moreover $g'(\xi) > 0$ for all $\xi > 0$.*

PROOF. Let $\xi_0 > 0$ be the first positive zero for g'. At this point $g > 0$ so is g'' which is impossible in a small left neighbourhood of ξ_0.
Now assume that g blows-up at $\bar{\xi}$. Set

$$E = (g')^2 - \alpha g^2. \tag{4.3}$$

Using (4.1)–(4.2) one sees that $E'(\xi) = -2(g')^2(\xi)\left[\frac{1}{2}\xi + \varepsilon q g^{q-1}\right]$. Since $g^{q-1}(\xi)$ goes to 0 as $\xi \to \bar{\xi}$ we deduce that the limit $\lim_{\xi \to \bar{\xi}} E(\xi) = L$ exits in $[-\infty, A], A < +\infty$. This implies that

$$\frac{g'}{g} \le \sqrt{\alpha} + \gamma, \gamma > 0$$

for all $\xi \in (\xi_\gamma, \bar{\xi})$. And the last inequality yields that

$$g(\xi) \le g(\xi_\gamma) e^{(\sqrt{\alpha}+\gamma)(\xi - \xi_\gamma)}.$$

Therefore we get a contradiction. This means that g is bounded and then is global. □

LEMMA 4.2. $\lim_{\xi \to +\infty} g(\xi) = +\infty$.

PROOF. Suppose to the contrary that g is bounded. In that case, because of the monotonicity of g, we have $g(\xi) \to g_0, 0 < g_0 < +\infty$ and $g'(\xi_m) \to 0$ for some sequence ξ_m converging to $+\infty$ with m. Using E we can see that $\lim_{\xi \to +\infty} g'(\xi) = 0$. Therefore

$$\lim_{\xi \to +\infty} g'' + \frac{1}{2}\xi g' = \alpha g_0,$$

thanks to equation (4.1).
Arguing as in the proof of Lemma 3.2 we get

$$g' > \frac{C}{\xi}, \quad \text{for large } \xi,$$

and then g goes to infinity which leads to a contradiction. $\square$

Now we shall study the large ξ behaviour of g. First we prove that $u = g'/g$ decays to 0 as $\xi \to \infty$. Recall that u is bounded and satisfies

$$u' + \frac{1}{2}\xi u = \alpha + \varphi(\xi), \tag{4.4}$$

where

$$\varphi(\xi) = \varepsilon q u g^{q-1} - u^2. \tag{4.5}$$

A standard analysis of (4.4) implies that $u(\xi)$ converges to 0 as $\xi \to \infty$, and then $\varphi(\xi) \to 0$.

THEOREM 4.1. *Assume that* $0 < q < 1$. *Let* g *be the solution to* (4.1)–(4.2). *Then there exists* $L(\lambda) > 0$ *such that*

$$g(\xi) = L(\lambda)\xi^{2\alpha}\left\{1 - \frac{c}{\xi^2} + o(\frac{1}{\xi^2})\right\}, \qquad as \;\; \xi \to +\infty, \tag{4.6}$$

where $c = 2\alpha(1-2\alpha) + 2\varepsilon q\alpha(L(\lambda))^{q-1}$.

The proof is similar as in Section 3. We show that $u = \dfrac{g'}{g}$ satisfies

$$\xi u = 2\alpha + 2\frac{c}{\xi^2} + o(\frac{1}{\xi^2}), \tag{4.7}$$

which leads to (4.6). $\square$

The following result gives a more precise estimate of g as ξ goes to infinity.

PROPOSITION 4.1. *Let* g *be the solution to* (4.1)–(4.2). *Assume that* $0 < q < 1$, *then*

$$g(\xi) = L(\lambda)\xi^{2\alpha}\left\{1 - \frac{c}{\xi^2} - \frac{d}{\xi^4} + o(\frac{1}{\xi^4})\right\}, \qquad as \;\; \xi \to +\infty, \tag{4.8}$$

where

$$c = 2\alpha(1-2\alpha) + 2\varepsilon q\alpha(L(\lambda))^{q-1} \quad \textit{and} \quad d = 3 - 4\alpha - 2\varepsilon q L^{q-1}(\lambda)c.$$

PROOF. It is sufficient to calculate

$$\lim_{\xi\to+\infty} \xi^2\Big[\xi^2(\xi u(\xi) - 2\alpha) - 2c\Big].$$

In fact by (4.4) we deduce that

$$\xi^2\Big[\xi^2(\xi u(\xi) - 2\alpha) - 2c\Big] = \frac{\displaystyle\int_0^\xi (\alpha + \varphi(s)e^{\frac{s^2}{4}}ds - 2\alpha\xi^{-1}e^{\frac{\xi^2}{4}} - 2c\xi^{-3}e^{\frac{\xi^2}{4}}}{e^{\frac{\xi^2}{4}}\xi^{-5}}.$$

Thus

$$\lim_{\xi\to+\infty} \xi^2\Big[\xi^2(\xi u(\xi) - 2\alpha) - 2c\Big] = 12c + 2\lim_{\xi\to+\infty} \xi^2\Big[\xi^2\varphi(\xi) + 2\alpha - c\Big],$$

thanks to l'Hôpital's rule. Define

$$A(\xi) = \xi^2 \varphi(\xi) + 2\alpha - c.$$

Thus

$$\begin{aligned} A(\xi) &= (2\alpha)^2 - (\xi u)^2 - \varepsilon q \left\{ \xi^2 g^{q-1} u - 2\alpha (L(\lambda))^{q-1} \right\}, \\ A(\xi) &= (2\alpha - \xi u)(2\alpha + \xi u) - \varepsilon q \left\{ \xi^2 g^{q-1} u - 2\alpha (L(\lambda))^{q-1} \right\}. \end{aligned}$$

By (4.7) and (4.8), we conclude that

$$\xi^2 A(\xi) = -8\alpha c - 4\varepsilon q (L(\lambda))^{q-1} c + o(1),$$

as $\xi \to 0$. Therefore

$$\lim_{\xi \to +\infty} \xi^2 \Big[\xi^2 (\xi u(\xi) - 2\alpha) - 2c \Big] = \Big(12 - 16\alpha - 8\varepsilon q (L(\lambda))^{q-1} \Big) c =: 4d.$$

The proof is completed as in the proof of the Theorem 3.1. □

In what follows we give some properties of $L(\lambda)$ in the case where $\varepsilon = 1$. We shall establish in particular that $L(\lambda)$ is strictly increasing with respect to λ, $L(\lambda)$ goes to 0 with λ and

$$L(\lambda) = l.\lambda + o(1), \; l > 0, \qquad \text{as } \lambda \to +\infty.$$

This is a consequence of the following

Theorem 4.2. *The function $\lambda \to L(\lambda)$ is continuous. Moreover for any $\lambda_2 > \lambda_1$ we have*

$$\frac{L(\lambda_2)}{\lambda_2} \geq \frac{L(\lambda_1)}{\lambda_1}$$

and there exists $L^\star > 0$ such that $L(\lambda) < \lambda L^\star$, for any $\lambda > 0$.

Proof. First we claim that if g_1 and g_2 are two solutions to problem (4.1)–(4.2) with initial values $\lambda_1 < \lambda_2$, then

$$\frac{g_2(\xi)}{g_1(\xi)} \geq \frac{\lambda_2}{\lambda_1}.$$

This leads in particular to

$$\frac{L(\lambda_2)}{L(\lambda_1)} \geq \frac{\lambda_2}{\lambda_1}.$$

Proof of the claim.

We show that the quotient $v = \dfrac{g_2}{g_1}$ is an increasing function. To this end we study the sign of the Wronskian

$$W = g_1 g_2' - g_1' g_2.$$

Using (4.1)–(4.2) one sees that W satisfies

$$\left(e^{h(\xi)}W\right)' = -qg_2'g_1e^{h(\xi)}\left[g_2^{q-1} - g_1^{q-1}\right], \quad W(0) = 0, \tag{4.9}$$

where

$$h(\xi) := \frac{\xi^2}{4} + q\int_0^{\xi} g_1^{q-1}(\tau)d\tau.$$

By assumption $\lambda_2 > \lambda_1$ the number

$$\xi_0 := \sup\left\{\xi, g_2(\tau) > g_1(\tau) \text{ on } [0,\xi]\right\}$$

is nonnegative. Suppose that $\xi_0 < +\infty$. It is clear that $g_1(\xi_0) = g_2(\xi_0)$ and $g_1'(\xi_0) > g_2'(\xi_0)$, so $W(\xi_0) < 0$. But since $q < 1$ we have

$$\left(e^{h(\xi)}W\right)' > 0,$$

on $(0, \xi_0)$. This implies that

$$e^{h(\xi)}W(\xi) > W(0) = 0,$$

for any $0 < \xi < \xi_0$. By continuity of W we deduce that $W(\xi_0) \geq 0$. We get a contradiction. □

This means that $\xi_0 = +\infty$ and $W(\xi) > 0$ for any $\xi > 0$. Therefore v is increasing. Now to prove that $L(\lambda)/\lambda$ is bounded, we consider problem (3.16) :

$$\begin{cases} f'' + \frac{1}{2}\xi f' + q\lambda^{q-1}|f|^{q-1}f' - \alpha f = 0, \\ \\ f'(0) = 0, \quad f(0) = 1. \end{cases} \tag{4.10}$$

If we now let $\lambda \to \infty$, we get

$$\begin{cases} f'' + \frac{1}{2}\xi f' - \alpha f = 0, \\ \\ f'(0) = 0, \quad f(0) = 1. \end{cases} \tag{4.11}$$

Let f_0 be the solution to (4.11). Arguing as above we deduce that

$$f_0(\xi) = L^\star\xi^{2\alpha}\left\{1 - \frac{c^\star}{\xi^2} + o(\frac{1}{\xi^2})\right\}.$$

Thus we conclude that $L(\lambda) < \lambda L^\star$ for any $\lambda > 0$.

Now we are in position to prove the continuity of the function $\lambda \to L(\lambda)$. We follow an idea due to [12]. Fix $\lambda_0 > 0, \xi_0 > 0$ and let $\delta > 0$ be a constant to be specified later.
Set $\lambda_1 = \lambda_0 - \delta,\ \lambda_2 = \lambda_0 + \delta$. For any $\lambda_1 \leq \lambda \leq \lambda_2$ we have

$$\frac{g'(\xi,\lambda)}{g(\xi,\lambda}) = \frac{2\alpha}{\xi} + r(\xi,\lambda), \quad \xi \geq \xi_0,$$

where

$$r(\xi,\lambda) = 2\frac{c}{\xi^3} + o(\frac{1}{\xi^3}), c = 2\alpha(1-2\alpha) + 2q\alpha(L(\lambda))^{q-1}$$

thanks to (4.7). As $L(\lambda)$ is bounded on $[\lambda_1, \lambda_2]$ there exists $\bar{c}$, which depends only on λ_1, λ_2 and ξ_0 such that

$$|r(\xi,\lambda)| \leq \bar{c}\frac{1}{\xi^3}, \quad \forall\ \xi \geq \xi_0.$$

This yields that

$$\xi^{-2\alpha} g(\xi,\lambda) = \xi_0^{-2\alpha} g(\xi_0,\lambda) \exp\left(\int_{\xi_0}^{+\infty} r(\tau,\lambda)d\tau\right),$$

and for any $\beta > 0$

$$\left|\exp\left(\int_{\xi_0}^{+\infty} r(\tau,\lambda)d\tau\right) - 1\right| < \beta,$$

if $\xi_0 > \xi_1(\beta)$. This implies that for $\xi_0 > \xi_1(\beta)$ and $\lambda_1 \leq \lambda \leq \lambda_2$

$$\left|L(\lambda) - \xi_0^{-2\alpha} g(\xi_0,\lambda)\right| < \beta \xi_0^{-2\alpha} g(\xi_0,\lambda),$$

therefore

$$\xi_0^{-2\alpha} g(\xi_0,\lambda) < \frac{L(\lambda)}{1-\beta} \leq \frac{L(\lambda_2)}{1-\beta}.$$

Consequently if $\beta = \dfrac{\varepsilon}{8L(\lambda_2)} < \dfrac{1}{2}$, for ε small, we get

$$\left|L(\lambda) - \xi_0^{-2\alpha} g(\xi_0,\lambda)\right| < \frac{\varepsilon}{4},$$

for any $\lambda_1 \leq \lambda \leq \lambda_2$.
Hence

$$\left|L(\lambda) - L(\lambda_0)\right| \leq \left|L(\lambda) - \xi_0^{-2\alpha} g(\xi_0,\lambda)\right| + \left|\xi_0^{-2\alpha} g(\xi_0,\lambda) - \xi_0^{-2\alpha} g(\xi_0,\lambda_0)\right| + \left|L(\lambda_0) - \xi_0^{-2\alpha} g(\xi_0,\lambda_0)\right|.$$

Now if we choose for fixed $\xi_0 > \xi_1$ a $\delta > 0$ such that

$$\left|g(\xi_0,\lambda) - g(\xi_0,\lambda_0)\right| < \frac{\varepsilon}{2}\xi_0^{-2\alpha},$$

for any $|\lambda - \lambda_0| < \delta$ we infer

$$\left|L(\lambda) - L(\lambda_0)\right| < \varepsilon,$$

if $|\lambda - \lambda_0| < \delta$.

This completes the proof of Theorem 4.2. $\square$

Corollary 4.1. *For any $L > 0$ the problem*

$$\begin{cases} g'' + \frac{1}{2}\xi g' + q|g|^{q-1}g' - \alpha g = 0, & \textit{on } (0,+\infty), \\ g'(0) = 0, g > 0, \quad \xi^{-2\alpha}g(\xi) \to L, \end{cases}$$

has a unique solution.

Corollary 4.2. *Let $\alpha > -\frac{1}{2}$. For any $A > 0$ the function $f = \dfrac{A}{L^\star} f_0$ is the unique solution to*

$$\begin{cases} f'' + \frac{1}{2}\xi f' - \alpha f = 0, \\ f'(0) = 0, \quad \lim_{\xi\to+\infty} \xi^{2\alpha} f(\xi) = A, \end{cases} \tag{4.12}$$

where f_0 is the solution to (4.11).

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