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Observability of string vibrations

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Abstract. Transversal vibrations $u = u(x,t)$ of a string of length l under three essential boundary conditions are studied, where u is governed by the Klein–Gordon equation:

$$u_{tt}(x,t) = a^2 u_{xx}(x,t) - cu(x,t), \qquad (x,t) \in [0,l] \times \mathbb{R};\ 0 < a, c \in \mathbb{R}.$$

Sufficient conditions are obtained that guarantee the unique solvability of a general observation problem with the given state functions $f,\ g \in D^s(0,l),\ s \in \mathbb{R}$ at two distinct instants of time $-\infty < t_1 < t_2 < \infty$:

$$\begin{aligned} A_1 u|_{t=t_1} + B_1 u_t|_{t=t_1} &= f, \qquad & |A_1| + |B_1| > 0,\ A_1 B_1 \geq 0, \\ A_2 u|_{t=t_2} + B_2 u_t|_{t=t_2} &= g, \qquad & |A_2| + |B_2| > 0,\ A_2 B_2 \leq 0. \end{aligned}$$

Here s is arbitrary, the space $D^s(0,l)$ (see [2] and [13]) is some subspace of the Sobolev space $H^s(0,l)$. The essential condition of the solvability is that $(t_2 - t_1)a/l$ is a rational number.
In fact, this result is a consequence of a general observability result related to the vibration $u = u(x,t)$ governed by the equation

$$u_{tt} = (p(x)u_x)_x - q(x)u, \qquad (x,t) \in [0,l] \times \mathbb{R}, \quad 0 < p, q \in C^\infty([0,l]),$$

subject to some initial data and linear boundary conditions (see in Proposition 1 below). This time the main restrictions are some Diophantine conditions and asymptotic properties of the eigenfrequencies ω_n as $n \to \infty$. Some other results without these restrictions are also presented.

Keywords: observation problems, string vibrations, generalized solutions, method of Fourier expansions.

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1. INTRODUCTION

In control theory which is closely related to the subject of this paper, a number of monographs and articles dealt with the accessability of a given complete final state

$$u|_{t=T} = f,\ u_t|_{t=T} = g$$

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of oscillations (in particular, string oscillations) in the time interval $0 \le t \le T < \infty$; see, e.g., in [1–10].

In observation problems for the oscillations $u = u(x,t)$ one looks for the initial functions

$$u|_{t=0} = \varphi,\ u_t|_{t=0} = \psi$$

only from the partial states, e.g.,

$$u|_{t=t_1} = f,\ u|_{t=t_2} = g,\ -\infty < t_1 < t_2 < \infty,$$

so the aim is the same (roughly speaking) as in the control theory: to find φ and ψ occurring above. Although, only the short communication [12] dealt with observability of the string oscillations u on $[0,l] \times [0,T]$ governed by $u_{tt} = a^2 u_{xx}$; and another paper [13] where u is described by the Klein–Gordon equation

$$(1) \qquad u_{tt}(x,t) = a^2 u_{xx}(x,t) - cu(x,t), \qquad (x,t) \in \Omega := [0,l] \times \mathbb{R};\ 0 < a, c \in \mathbb{R},$$

with the initial conditions

$$(2) \qquad u|_{t=0} = \varphi(x),\ u_t|_{t=0} = \psi(x),$$

and boundary conditions

$$(3) \qquad u|_{x=0} = u|_{x=l} = 0,$$

with $u,\ \varphi,\ \psi$ from the corresponding generalized function spaces $D^s,\ s \in \mathbb{R}$. In [12] these functions φ and ψ were constructed only for small t_1 and t_2: $0 \le t_1 < t_2 \le 2l/a$ under the additional assumption that the initial functions φ and ψ in question are known on some subinterval $[h_1, h_2] \subset [0,l]$. In [13] we dealt with the case of arbitrary $t_1, t_2 \in \mathbb{R}$, $-\infty < t_1 < t_2 < \infty$, under the restriction that $(t_2 - t_1)a/l$ is a rational number, and we considered the following observation conditions:

$$(a) \qquad u|_{t=t_1} = f, \quad u|_{t=t_2} = g,$$

$$(b) \qquad u_t|_{t=t_1} = f, \quad u|_{t=t_2} = g,$$

$$(c) \qquad u|_{t=t_1} = f, \quad u_t|_{t=t_2} = g,$$

$$(d) \qquad u_t|_{t=t_1} = f, \quad u_t|_{t=t_2} = g.$$

Now, our goal is to prove similar results related to the problem (1)–(2) with the following general observation conditions:

$$(e) \qquad \begin{aligned} A_1 u|_{t=t_1} + B_1 u_t|_{t=t_1} &= f, \qquad |A_1| + |B_1| > 0, \\ A_2 u|_{t=t_2} + B_2 u_t|_{t=t_2} &= g, \qquad |A_2| + |B_2| > 0, \end{aligned}$$

under the following boundary conditions: either

$$u(0,t) = u_x(l,t) = 0; \tag{4}$$

or

$$u_x(0,t) = u_x(l,t) = 0; \tag{5}$$

or the Sturm–Liouville boundary conditions

$$\begin{aligned} u(0,t)\cos\alpha + u_x(0,t)\sin\alpha = 0, &\qquad \cot\alpha < 0, \\ u(l,t)\cos\beta + u_x(l,t)\sin\beta = 0, &\qquad \cot\beta > 0. \end{aligned} \tag{6}$$

Thus, we deal with observation problems related to the mixed problems $(1),(2),(4)$; $(1),(2),(5)$ and $(1),(2),(6)$ respectively for the specific properties for all three mixed problems. We emphasize some common ideas that may be useful even for the study of some other problems (see Propositions 1, 2 and Lemma 1 below), e.g., for the class of the following mixed problems:

$$u_{tt} = (p(x)u_x)_x - q(x)u \equiv Lu, \qquad (x,t) \in \Omega := [0,l] \times \mathbb{R}, \quad 0 < p, q \in C^\infty([0,l]), \tag{7}$$

$$u|_{t=0} = \varphi(x), \quad u_t|_{t=0} = \psi(x), \tag{8}$$

$$\mathcal{U}_i[u] \equiv \mathcal{U}_i(u|_{x=0},\ u|_{x=l},\ u_x|_{x=0},\ u_x|_{x=l}) = 0, \qquad i = 1,2, \tag{9}$$

where $\mathcal{U}_1$, $\mathcal{U}_2$ are independent, self-adjoint (see, e. g., [14]) linear expressions (they may be different from $(4),(5),(6)$), where u, φ, ψ are from the generalized function spaces D^s, and the conditions (e) and (9) are understood in the sense of generalized functions. It can be supposed without loss of the generality, that the ordinary homogeneous BVP for

$$y = y(x), \quad L \sim (p(x)y')' - q(x)y$$

$$Ly = 0, \qquad \mathcal{U}_i[y] = 0, \quad i = 1,2,$$

has only the trivial solution. The ordinary differential operator L is formally self-adjoint, therefore the operator $\mathcal{L} \sim (L, \mathcal{U}_1, \mathcal{U}_2)$ is also self-adjoint.

We use the definition of the spaces $D^s(S)$, $s \in \mathbb{R}$ given in [2] and [13]. Let the system $\{X_n(x)\}_{n=0}^\infty$ be a complete orthonormal basis in $L_2(S)$. Given arbitrary real number s, we consider on the linear span D of the functions $X_n(x)$, $n \in \mathbb{N}_0 = \mathbb{N}\bigcup\{0\}$, $x \in \overline{S}$, the following Euclidean norm:

$$\left\| \sum_{n=0}^\infty c_n X_n(x) \right\|_s := \left(\sum_{n=0}^\infty n^{2s} |c_n|^2 \right)^{\frac{1}{2}}.$$

Completing D with respect to this norm, we obtain a Hilbert space denoted by D^s. We use the notation $S = (0, l)$ associated with string vibrations.

We will need that for every $s \in \mathbb{R}$ and for every $(\varphi, \psi) \in D^{s+1}(0, l) \times D^s(0, l)$, the mixed problem (7), (8), (9) possesses the following good properties:

$$(10) \qquad \exists! \text{ solution } u \text{ and } u \in C(D^{s+1}, \mathbb{R}) \cap C^1(D^s, \mathbb{R}) \cap C^2(D^{s-1}, \mathbb{R}),$$

moreover, u can be written as

$$(11) \qquad \begin{gathered} u(x,t) = \sum_{n=0}^{\infty} \left[\alpha_n \cos(\omega_n t) + \beta_n \sin(\omega_n t)\right] X_n(x), \qquad (x,t) \in \overline{\Omega}, \text{ where} \\ LX_n = -\omega_n^2 X_n, \qquad \mathcal{U}_i X_n = 0, \quad i = 1, 2. \end{gathered}$$

Remark 1. The set of problems satisfying our restrictions on $L, \mathcal{U}_1, \mathcal{U}_2$ and (10), (11) is not empty. It contains, e.g. the problem considered in [13], the self-adjoint problems in Sections 3, 4 and the problem under the Sturm–Liouville boundary condition in Section 5. The self-adjointness of $\mathcal{L} \sim (L, \mathcal{U}_1, \mathcal{U}_2)$ is also valid in the case where

$$\begin{aligned} \mathcal{U}_1 : &\quad a_{11} y'(0) + a_{12} y'(l) = 0 \\ \mathcal{U}_2 : &\quad a_{21} y(0) \ + a_{22} y(l) \ = 0 \end{aligned}$$

if and only if $a_{12} a_{22} p(0) = a_{11} a_{21} p(l)$.

2. MAIN RESULTS

We will need the following lemma for certain estimates below.

Lemma 1. *Let the sequence r_n be such that $0 < M/n < |r_n| \to 0$ with a positive constant M and let $x_0 > 0$ be a rational number. Then for any fixed $d \in \mathbb{R}$, there exists a number N such that*

$$|\sin(n\pi x_0 + d + r_n)| > \frac{M}{2n}, \qquad \forall n > N.$$

Proof. Since x_0 is rational, it can be written as $x_0 = p/q$; p, $q \in \mathbb{N}$, and then $\sin(n\pi x_0 + d)$ can assume at most q different values. Let $\mathbb{N}_1$ and $\mathbb{N}_2$ be two disjoint sets of numbers, $\mathbb{N}_1 \bigcup \mathbb{N}_2 = \mathbb{N}$ such that $\sin(n\pi x_0 + d) = 0$ if $n \in \mathbb{N}_1$ and $\sin(n\pi x_0 + d) \neq 0$ if $n \in \mathbb{N}_2$. We will prove Lemma 1 according to these two cases.

First, if $n \in \mathbb{N}_1$, then we have

$$|\sin(n\pi x_0 + d + r_n)| = |\sin(r_n)| > \frac{1}{2}|r_n| > \frac{M}{2n}$$

for all large $n \in \mathbb{N}_1$ since $r_n \to 0$ as $n \to \infty$.

Second, if $n \in \mathbb{N}_2$, then this means that there is a constant $d_1 > 0$ such that $|\sin(n\pi x_0 + d)| > d_1$ for all $n \in \mathbb{N}_2$. Due to the uniform continuity of the sine function, it follows that

$$|\sin(n\pi x_0 + d + r_n)| > \frac{d_1}{2} > \frac{M}{2n}$$

for all large $n \in \mathbb{N}_2$.

Combining the two cases we get the statement of Lemma 1. □

Proposition 1. *Let*

$$f \in D^{s+2},\ g \in D^{s+2}, \qquad s \in \mathbb{R}, \tag{12}$$

and suppose that there are constants $0 < A \in \mathbb{Q},\ B \in \mathbb{R},\ 0 < M_1 \in \mathbb{R}$ *and a sequence* $C_n \in \mathbb{R}\backslash\{0\}$ *such that*

$$\begin{gathered} \omega_n(t_2 - t_1) + \gamma_n - \delta_n = n\pi A + B + C_n, \\ 0 < \frac{M_1}{n} < |C_n|,\ \forall n \in \mathbb{N} \quad \text{and} \quad C_n \to 0 \ \text{as}\ n \to \infty, \end{gathered} \tag{13}$$

and

$$\sin(\omega_n(t_2 - t_1) + \gamma_n - \delta_n) \neq 0, \qquad n \in \mathbb{N}_0. \tag{14}$$

Here the angles $\gamma_n,\ \delta_n \in [0, 2\pi)$ *are uniquely determined by the following relationships:*

$$\sin\gamma_n = \frac{A_1}{\sqrt{A_1^2 + B_1^2\omega_n^2}}, \quad \cos\gamma_n = \frac{B_1\omega_n}{\sqrt{A_1^2 + B_1^2\omega_n^2}},$$

$$\sin\delta_n = \frac{A_2}{\sqrt{A_2^2 + B_2^2\omega_n^2}}, \quad \cos\delta_n = \frac{B_2\omega_n}{\sqrt{A_2^2 + B_2^2\omega_n^2}}.$$

If the solution of the mixed problem (7), (8), (9) *satisfies* (10), (11) *and conditions* (12), (13), (14) *hold, then the observation problem posed for* (7), (9) *under the observation condition* (e) *has a unique solution for* $(\varphi, \psi) \in D^{s+1} \times D^s$.

Remark 2. According to condition (13), we can use Lemma 1 for the estimation of

$$\sin(\omega_n(t_2 - t_1) + \gamma_n - \delta_n) = \sin(n\pi A + B + C_n),$$

which means that $\sin(\omega_n(t_2 - t_1) + \gamma_n - \delta_n) \neq 0$ automatically holds for all large n, say when n is greater than a threshold number N.

Proof of Proposition 1. Clearly, the initial functions can be uniquely expanded into Fourier series with respect to the system $\{X_n\}$ with – yet unknown – coefficients $\alpha_n,\ \omega_n\beta_n;\ n \in \mathbb{N}_0$:

$$\varphi(x) = u(x,0) = \sum_{n=0}^{\infty} \alpha_n X_n(x), \tag{15}$$

$$\psi(x) = u_t(x,0) = \sum_{n=0}^{\infty} \omega_n\beta_n X_n(x). \tag{16}$$

Since the solution u of the problem (7), (8), (9) has the representation (11) with some coefficients $\alpha_n,\ \beta_n;\ n \in \mathbb{N}_0$, the observation problem can be reduced to the problem of finding the appropriate choices of α_n and β_n such that (e) is satisfied. For this reason, we substitute t_1 and t_2 into (11), and use both conditions in (e). As a result, we get the following necessary conditions for α_n and β_n:

$$f(x) = A_1 u(x,t_1) + B_1 u_t(x,t_1) = \tag{17}$$

$$= \sum_{n=0}^{\infty} [\alpha_n(A_1\cos(\omega_n t_1) - B_1\omega_n\sin(\omega_n t_1)) + \beta_n(A_1\sin(\omega_n t_1) + B_1\omega_n\cos(\omega_n t_1))]X_n(x),$$

$$g(x) = A_2 u(x,t_2) + B_2 u_t(x,t_2) = \tag{18}$$

$$= \sum_{n=0}^{\infty} [\alpha_n(A_2\cos(\omega_n t_2) - B_2\omega_n\sin(\omega_n t_2)) + \beta_n(A_2\sin(\omega_n t_2) + B_2\omega_n\cos(\omega_n t_2))]X_n(x).$$

Since $f \in D^{s+2}$ and $g \in D^{s+2}$, the coefficients of the Fourier expansions (with respect to the system $\{X_n\}$) of the functions $f(x),\ g(x)$ are unambiguously determined, and comparing these Fourier series with (17) and (18), we get the following conditions for α_n and β_n, $n \in \mathbb{N}_0$:

$$\begin{aligned} &\alpha_n(A_1\cos(\omega_n t_1) - B_1\omega_n\sin(\omega_n t_1)) + \beta_n(A_1\sin(\omega_n t_1) + B_1\omega_n\cos(\omega_n t_1)) = f_n,\\ &\alpha_n(A_2\cos(\omega_n t_2) - B_2\omega_n\sin(\omega_n t_2)) + \beta_n(A_2\sin(\omega_n t_2) + B_2\omega_n\cos(\omega_n t_2)) = g_n. \end{aligned} \tag{19}$$

where

$$f_n = \int_0^l f(x)X_n(x)dx \quad \text{and} \quad g_n = \int_0^l g(x)X_n(x)dx.$$

By substituting γ_n and δ_n into (19) and using trigonometric identities, we get that

$$\begin{aligned} -\alpha_n\sin(\omega_n t_1 - \gamma_n) + \beta_n\cos(\omega_n t_1 - \gamma_n) &= \frac{f_n}{\sqrt{A_1^2 + B_1^2\omega_n^2}},\\ -\alpha_n\sin(\omega_n t_2 - \delta_n) + \beta_n\cos(\omega_n t_2 - \delta_n) &= \frac{g_n}{\sqrt{A_2^2 + B_2^2\omega_n^2}}. \end{aligned}$$

This linear system can be uniquely solved for the unknown coefficients α_n and β_n, due to the fact that $\sin(\omega_n(t_2-t_1)+\gamma_n-\delta_n)\neq 0,\ n\in\mathbb{N}_0$:

$$
\begin{aligned}
\alpha_n &= \frac{\sqrt{A_2^2+B_2^2\omega_n^2}\cos(\omega_n t_2-\delta_n)f_n-\sqrt{A_1^2+B_1^2\omega_n^2}\cos(\omega_n t_1-\gamma_n)g_n}{\sqrt{A_1^2+B_1^2\omega_n^2}\sqrt{A_2^2+B_2^2\omega_n^2}\sin(\omega_n(t_2-t_1)+\gamma_n-\delta_n)},\\
\beta_n &= \frac{\sqrt{A_2^2+B_2^2\omega_n^2}\sin(\omega_n t_2-\delta_n)f_n-\sqrt{A_1^2+B_1^2\omega_n^2}\sin(\omega_n t_1-\gamma_n)g_n}{\sqrt{A_1^2+B_1^2\omega_n^2}\sqrt{A_2^2+B_2^2\omega_n^2}\sin(\omega_n(t_2-t_1)+\gamma_n-\delta_n)}.
\end{aligned}
\tag{20}
$$

Thus, the unknown initial functions φ and ψ are uniquely determined in the form of (15) and (16). It remains to show that $\varphi,\ \psi$ are from the classes $D^{s+1},\ D^s$, respectively, i.e., to show that the following inequality holds:

$$
\max\{\|\varphi\|_{s+1},\|\psi\|_s\}=\max\left\{\sum_{n=0}^{\infty}n^{2s+2}|\alpha_n|^2,\sum_{n=0}^{\infty}n^{2s}|\omega_n\beta_n|^2\right\}<\infty. \tag{21}
$$

We note that condition (13) involves that $|\omega_n|=O(n)$, so by using Lemma 1 and (20), we get that there is a constant M such that

$$
|\alpha_n|\leq\left|M\frac{n}{1+B_1n}f_n\right|+\left|M\frac{n}{1+B_2n}g_n\right|,\quad \forall n>N,
$$

$$
|\beta_n|\leq\left|M\frac{n}{1+B_1n}f_n\right|+\left|M\frac{n}{1+B_2n}g_n\right|,\quad \forall n>N,
$$

which means that

$$
\max\{|\alpha_n|,|\beta_n|\}<c_1n\max\{|f_n|,|g_n|\},\qquad n\in\mathbb{N}_0, \tag{22}
$$

for a suitable constant c_1. Obviously, the inequality (22) (and thus condition (12)) can be improved when $B_1\neq 0$ or $B_2\neq 0$ (see Remark 4 below).

By using (22), we get that

$$
\max\left\{\sum_{n=0}^{\infty}n^{2s+2}|\alpha_n|^2,\sum_{n=0}^{\infty}n^{2s}|\omega_n\beta_n|^2\right\}\leq\sum_{n=0}^{\infty}C^2n^{2s+2}\max\{|\alpha_n|^2,|\beta_n|^2\}<
$$
$$
<c_1^2C^2\sum_{n=0}^{\infty}n^{2s+4}\max\{|f_n|^2,|g_n|^2\},
$$

where

$$
\sum_{n=0}^{\infty}n^{2s+4}\max\{|f_n|^2,|g_n|^2\}\leq\|f\|_{s+2}^2+\|g\|_{s+2}^2<\infty
$$

and $(f,g)\in D^{s+2}\times D^{s+2}$ according to the definition of the $s+2$ norm. □

Remark 3. The above examinations and estimates show that the operator $\mathcal{A}$, which assigns to the partial states (f,g) at t_1 and t_2 the couple (φ,ψ), is a continuous (bounded) operator from $D^{s+2}\times D^{s+2}$ into $D^{s+1}\times D^s$.

On the other hand, we get from the system of equations (19) that

$$\alpha_n \sin(\omega_n(t_2-t_1)+\gamma_n-\delta_n) = \frac{f_n\cos(\omega_n t_2-\delta_n)}{v_n} - \frac{g_n\cos(\omega_n t_1-\gamma_n)}{w_n},$$

$$\beta_n \sin(\omega_n(t_2-t_1)+\gamma_n-\delta_n) = \frac{f_n\sin(\omega_n t_2-\delta_n)}{v_n} - \frac{g_n\sin(\omega_n t_1-\gamma_n)}{w_n},$$

where

$$v_n = \sqrt{A_1^2+B_1^2\omega_n^2}, \qquad w_n = \sqrt{A_2^2+B_2^2\omega_n^2}$$

and obviously

$$v_n = |A_1| \text{ if } B_1=0, \quad w_n = |A_2| \text{ if } B_2 = 0,$$

$$v_n = O(n) \text{ if } B_1\neq 0, \quad w_n = O(n) \text{ if } B_2\neq 0, \quad n\to\infty.$$

Based on the behaviour of the Fourier coefficients $\alpha_n,\ \beta_n$ as $n\to\infty$ we get that

$$\sum_{n=0}^{\infty}\left(n^{2s+4}|\alpha_n|^2+n^{2s+2}|\omega_n\beta_n|^2\right)\sin^2(\omega_n(t_2-t_1)+\gamma_n-\delta_n)\leq$$

$$\leq C\sum_{n=0}^{\infty}\left(n^{2s+4}|f_n|^2+n^{2s+4}|g_n|^2\right)\leq C\left(\|f\|_{s+2}^2+\|g\|_{s+2}^2\right),$$

which indicates a slightly stronger smoothness for the couple $(\varphi,\psi)=\mathcal{A}(f,g)$.

This suggests to prove a slightly sharper statement than the one in Proposition 1. To this effect, let us introduce the subspaces $D_0^s\subset D^s$ that contain the functions $f\in D^s$ whose Fourier coefficients f_n for $n\in\mathbb{N}_1$ (the set $\mathbb{N}_1$ depends on the problem, see its definition in Lemma 1) have the following property:

$$\sum_{n\in\mathbb{N}_1}|f_n|^2|n|^{2s+2}<\infty.$$

Certainly this involves that $D^{s+1}\subset D_0^s$.

Proposition 2. *If the conditions of Proposition* 1 *hold and* $f,\ g\in D_0^{s+1}$ *(instead of* D^{s+2}*), then the observation problem posed for* (7), (8), (9) *under condition* (e) *has a unique solution for* $(\varphi,\psi)\ \in D^{s+1}\times D^s$.

In other terms, the operator $\mathcal{A}:\ \mathcal{A}(f,g):=(\varphi,\psi)$ *maps* $D_0^{s+1}\times D_0^{s+1}$ *into* $D^{s+1}\times D^s$ *as a continuous (bounded) operator.*

Proof of Proposition 2. We can do the same steps as in the proof of Proposition 1. It remains only to show that (21) also holds for $(f,g)\in D_0^{s+1}\times D_0^{s+1}$. Indeed, we have

$$\max\left\{\sum_{n=0}^{\infty}n^{2s+2}|\alpha_n|^2,\sum_{n=0}^{\infty}n^{2s}|\omega_n\beta_n|^2\right\}\leq\sum_{n=0}^{\infty}C^2n^{2s+2}\max\{|\alpha_n|^2,|\beta_n|^2\}=$$

$$= \sum_{n\in\mathbb{N}_1} C^2 n^{2s+2} \max\{|\alpha_n|^2, |\beta_n|^2\} + \sum_{n\in\mathbb{N}_2} C^2 n^{2s+2} \max\{|\alpha_n|^2, |\beta_n|^2\}.$$

Using inequality (22) for the first sum, we get

$$\sum_{n\in\mathbb{N}_1} C^2 n^{2s+2} \max\{|\alpha_n|^2, |\beta_n|^2\} \le c_1^2 C^2 \sum_{n\in\mathbb{N}_1} n^{2s+4} \max\{|f_n|^2, |g_n|^2\},$$

which is finite by the definition of the spaces D_0^{s+1}.

For $n \in \mathbb{N}_2$ we can improve inequality (22). Namely, for these values of n (if n is large enough) we have $\sin(\omega_n(t_2 - t_1) + \gamma_n - \delta_n) > d_1/2$ as it was shown in the proof of Lemma 1, whence it follows that

$$\max\{|\alpha_n|, |\beta_n|\} < c_2 \max\{|f_n|, |g_n|\} \qquad \forall n \in \mathbb{N}_2$$

with a suitable constant c_2. So, we get, that

$$\sum_{n\in\mathbb{N}_2} C^2 n^{2s+2} \max\{|\alpha_n|^2, |\beta_n|^2\} \le c_2^2 C^2 \sum_{n\in\mathbb{N}_2} n^{2s+2} \max\{|f_n|^2, |g_n|^2\} < \infty,$$

due to the fact that $D_0^{s+1} \subset D^{s+1}$. □

Remark 4. As we mentioned before, the inequality (22) can be improved when $B_1 \neq 0$ or $B_2 \neq 0$. More precisely, if $B_1 \neq 0$ then $f \in D^{s+1}$; while if $B_2 \neq 0$ then $g \in D^{s+1}$ is sufficient in condition (12).

In the special case when one of A_1, B_1 equals zero and one of A_2, B_2 equals zero, the observation condition (e) simplifies to one of the condition (a), (b), (c), (d) (after suitably dividing with A_1, A_2, B_1, or B_2). In this case, we have $\gamma_n \equiv 0$ or $\gamma_n \equiv \pi/2$; $\delta_n \equiv 0$ or $\delta_n \equiv \pi/2$, and conditions (13) and (14) are simplified accordingly. These terms correspond to our previous results in [13].

Remark 5. Condition (13) is suitable for the usage of Lemma 1 to have the estimation

$$\frac{1}{\sin(\omega_n(t_2 - t_1) + \gamma_n - \delta_n)} < \frac{n}{c_3}, \qquad \forall n > N.$$

However condition (13) of Proposition 1 is not necessary, it can be replaced, e.g., with the assumption that $f_n = g_n = 0$ for all $n \in \mathcal{K}(\epsilon)$, where for the arbitrary small fixed $\epsilon > 0$, $\mathcal{K}(\epsilon) \subset \mathbb{N}$ is the collection of all n such that

$$-\epsilon < \omega_n(t_2 - t_1) + \gamma_n - \delta_n < \epsilon \mod (\pi).$$

Now, we will show three applications of Proposition 1 in the following Sections 3–5.

3. THE VIBRATING STRING WITH FIXED LEFT AND FREE RIGHT ENDS

Let $\Omega = \{(x,t) \ : \ 0 < x < l, \ t \in \mathbb{R}\}$. Consider the problem of the vibrating $[0,l]$ string with a fixed left end and a free right end under the assumption that there is an elastic withdrawing force proportional to the transversal deflection $u(x,t)$ of the point x of the string at the instant denoted by t. This phenomenon is described by the Klein–Gordon equation as follows:

$$u_{tt}(x,t) = a^2 u_{xx}(x,t) - cu(x,t), \qquad (x,t) \in \overline{\Omega}, \ 0 < a, c \in \mathbb{R}, \tag{1}$$

with the initial conditions

$$u(x,0) = \varphi(x), \ u_t(x,0) = \psi(x), \qquad 0 \le x \le l, \tag{2}$$

and the homogeneous boundary conditions

$$u(0,t) = 0, \ u_x(l,t) = 0, \qquad t \in \mathbb{R}. \tag{4}$$

Some of the results of [2] (see Section 1.1–1.3) and [10] say that for arbitrary $s \in \mathbb{R}$ with $(\varphi, \psi) \in D^{s+1} \times D^s$, the solution of the mixed problem $(1), (2), (4)$ satisfies (10), and (11); and using $q(x) \equiv c > 0$ we also have

$$X_n = \sqrt{\frac{2}{l}} \sin\left(\frac{(n+\frac{1}{2})\pi}{l} x\right), \qquad n \in \mathbb{N}_0,$$

and

$$u(x,t) = \sum_{n=0}^{\infty} \left[\alpha_n \cos(\omega_n t) + \beta_n \sin(\omega_n t)\right] \sqrt{\frac{2}{l}} \sin\left(\frac{(n+\frac{1}{2})\pi}{l} x\right), \qquad (x,t) \in \overline{\Omega}.$$

Here

$$\omega_n = \sqrt{\left(\frac{(n+\frac{1}{2})\pi a}{l}\right)^2 + c}, \qquad n \in \mathbb{N}_0,$$

thus we have

$$\omega_n = \frac{(n+\frac{1}{2})\pi a}{l} + \left(\omega_n - \frac{(n+\frac{1}{2})\pi a}{l}\right) = \frac{(n+\frac{1}{2})\pi a}{l} + \frac{\omega_n^2 - \left(\frac{(n+\frac{1}{2})\pi a}{l}\right)^2}{\omega_n + \frac{(n+\frac{1}{2})\pi a}{l}} =$$

$$= n\frac{\pi a}{l} + \frac{\pi a}{2l} + \frac{c}{\omega_n + \frac{(n+\frac{1}{2})\pi a}{l}}.$$

Let us consider the following observation conditions:

$$\begin{aligned} A_1 u|_{t=t_1} + B_1 u_t|_{t=t_1} &= f, \qquad |A_1| + |B_1| > 0, \\ A_2 u|_{t=t_2} + B_2 u_t|_{t=t_2} &= g, \qquad |A_2| + |B_2| > 0, \end{aligned} \tag{e}$$

and let us suppose that $A_1B_1 \geq 0$ and $A_2B_2 \leq 0$. Without loss of the generality, we can take $A_1, B_1, B_2 \geq 0$ and $A_2 \leq 0$. Hence it follows that $\sin\gamma_n \to 0^+,\ \gamma_n \to 0^+,\ \sin\delta_n \to 0^-,\ \delta_n \to 0^-$ as $n \to \infty$. Therefore, there exists $N \in \mathbb{N}$, such that $\forall n > N$,

$$0 \leq \frac{2}{\pi}\gamma_n \leq \sin\gamma_n \leq \gamma_n, \qquad \delta_n \leq \sin\delta_n \leq \frac{2}{\pi}\delta_n \leq 0.$$

Due to the facts that

$$\sin\gamma_n = \frac{A_1}{\sqrt{A_1^2 + B_1^2\omega_n^2}}, \qquad \sin\delta_n = \frac{A_2}{\sqrt{A_2^2 + B_2^2\omega_n^2}}, \qquad \omega_n = O(n),$$

there is a constant $M \in \mathbb{R}^+ \bigcup \{0\}$ such that

$$0 \leq \frac{M}{n} \leq \gamma_n \to 0, \qquad 0 \leq \frac{M}{n} \leq -\delta_n \to 0.$$

So, the condition (13) is fulfilled with

$$A = (t_2 - t_1)\frac{a}{l}, \qquad B = (t_2 - t_1)\frac{\pi a}{2l}, \qquad C_n = (t_2 - t_1)\frac{c}{\omega_n + \frac{(n+\frac{1}{2})\pi a}{l}} + \gamma_n - \delta_n,$$

provided $(t_2 - t_1)\dfrac{a}{l} \in \mathbb{Q}$.

Consequently, we can apply Proposition 1 to the vibrating string with fixed left end and free right end as follows.

Theorem 1. *Let*

$$f \in D^{s+2},\ g \in D^{s+2}, \qquad s \in \mathbb{R},$$

and suppose that

$$\sin((t_2 - t_1)\omega_n + \gamma_n - \delta_n) \neq 0, \qquad n \in \mathbb{N}_0.$$

Then the observation problem posed for (1), (2), (4) *under the observation conditions*

$$\begin{aligned} A_1 u|_{t=t_1} + B_1 u_t|_{t=t_1} &= f, \qquad & |A_1| + |B_1| > 0,\ A_1B_1 \geq 0, \\ A_2 u|_{t=t_2} + B_2 u_t|_{t=t_2} &= g, \qquad & |A_2| + |B_2| > 0,\ A_2B_2 \leq 0, \end{aligned}$$

has a unique solution for $(\varphi, \psi) \in D^{s+1} \times D^s$ *provided the elapsed time between* t_1 *and* t_2 *is a rational multiple of* l/a.

4. THE VIBRATING STRING WITH FREE ENDS

Let $\Omega = \{(x,t)\ :\ 0 < x < l,\ t \in \mathbb{R}\}$. Consider the problem of the vibrating $[0,l]$ string with free ends when there is an elastic withdrawing force proportional to the transversal deflection $u(x,t)$ of the point x of the string at the instant denoted by t. This phenomenon is described by the Klein–Gordon equation as follows:

(1) $$u_{tt}(x,t) = a^2 u_{xx}(x,t) - cu(x,t), \qquad (x,t) \in \overline{\Omega},\ 0 < a, c \in \mathbb{R},$$

with the initial conditions

$$u(x,0)=\varphi(x),\ u_t(x,0)=\psi(x),\qquad 0\le x\le l, \tag{2}$$

and the homogeneous boundary conditions of the second kind

$$u_x(0,t)=0,\ u_x(l,t)=0,\qquad t\in\mathbb{R}. \tag{5}$$

From the same results of [2] and [10] as in Section 3 it follows that for arbitrary $s\in\mathbb{R}$ with $(\varphi,\psi)\in D^{s+1}\times D^s$ the solution of the mixed problem (1), (2), (5) satisfies (10), (11), and due to the fact that $q(x)\equiv c>0$, we have

$$X_0=\sqrt{\frac{1}{l}} \text{ and } X_n=\sqrt{\frac{2}{l}}\cos\left(\frac{n\pi}{l}x\right),\qquad n\in\mathbb{N},$$

and

$$u(x,t)=\sum_{n=0}^{\infty}\left[\alpha_n\cos(\omega_n t)+\beta_n\sin(\omega_n t)\right]X_n(x),\qquad (x,t)\in\overline{\Omega}.$$

Here

$$\omega_n=\sqrt{\left(\frac{n\pi a}{l}\right)^2+c},\qquad n\in\mathbb{N}_0,$$

so

$$\omega_n=\frac{n\pi a}{l}+\left(\omega_n-\frac{n\pi a}{l}\right)=n\frac{\pi a}{l}+\frac{\omega_n^2-\left(\frac{n\pi a}{l}\right)^2}{\omega_n+\frac{n\pi a}{l}}=n\frac{\pi a}{l}+\frac{c}{\omega_n+\frac{n\pi a}{l}}.$$

If $A_1B_1\ge 0$ and $A_2B_2\le 0$, then the same arguments for γ_n,δ_n as in Section 3 give that condition (13) is fulfilled with

$$A=(t_2-t_1)\frac{a}{l},\qquad B=0,\qquad C_n=(t_2-t_1)\frac{c}{\omega_n+\frac{n\pi a}{l}}+\gamma_n-\delta_n,$$

provided $(t_2-t_1)\frac{a}{l}\in\mathbb{Q}$.

Consequently, we can apply Proposition 1 to the vibrating string with free ends as follows.

Theorem 2. *Let*

$$f\in D^{s+2},\ g\in D^{s+2},\qquad s\in\mathbb{R},$$

and suppose that

$$\sin((t_2-t_1)\omega_n+\gamma_n-\delta_n)\neq 0,\qquad n\in\mathbb{N}_0.$$

Then the observation problem posed for (1), (2), (5) *under observation conditions*

$$\begin{aligned} A_1u|_{t=t_1}+B_1u_t|_{t=t_1}&=f, &\qquad |A_1|+|B_1|>0,\ A_1B_1\ge 0,\\ A_2u|_{t=t_2}+B_2u_t|_{t=t_2}&=g, &\qquad |A_2|+|B_2|>0,\ A_2B_2\le 0,\end{aligned}$$

has a unique solution for $(\varphi,\psi)\in D^{s+1}\times D^s$ provided the elapsed time between t_1 and t_2 is a rational multiple of l/a.

5. THE VIBRATING STRING WITH BOUNDARY CONDITION OF THE THIRD KIND (STURM–LIOUVILLE BOUNDARY CONDITIONS)

Let $\Omega = \{(x,t) \;:\; 0 < x < l,\; t \in \mathbb{R}\}$. Consider the problem of the vibrating $[0,l]$ string when there exists a varying withdrawing force proportional to the transversal deflection $u(x,t)$ of the point x of the string at the instant denoted by t. This phenomenon is described by the Klein–Gordon equation as follows:

$$(1') \quad u_{tt}(x,t) = a^2 u_{xx}(x,t) - c(x)u(x,t), \quad (x,t) \in \overline{\Omega},\; 0 < a \in \mathbb{R},\; 0 < c(x) \in C^2[0,l],$$

$$(2) \qquad u(x,0) = \varphi(x),\; u_t(x,0) = \psi(x), \qquad 0 \le x \le l,$$

and the boundary conditions

$$(6) \qquad \begin{aligned} u(0,t)\cos\alpha + u_x(0,t)\sin\alpha &= 0, & \cot\alpha < 0, \\ u(l,t)\cos\beta + u_x(l,t)\sin\beta &= 0, & \cot\beta > 0. \end{aligned}$$

The conditions for the sign of $\cot\alpha$ and $\cot\beta$ in (6) are sufficient to ensure the conservation of energy, hence guaranteeing the uniqueness of the solution of the mixed problem $(1'), (2), (6)$ in the classical case. Moreover, this condition also ensures that the (later introduced) constant γ is strictly positive.

It follows again from [2] and [10] that for arbitrary $s \in \mathbb{R}$ with $(\varphi, \psi) \in D^{s+1} \times D^s$, the solution of the mixed problem $(1'), (2), (6)$ satisfies (10) and (11). The orthonormal basis $\{X_n\}$, $n \in \mathbb{N}_0$, in (11) can be derived from [11] as follows:

$$X_n(x) = \sqrt{\frac{2}{\pi}} \left[\cos\left(\frac{n\pi}{l}x\right) + \frac{\beta(x)}{n}\sin\left(\frac{n\pi}{l}x\right)\right] + O\left(\frac{1}{n^2}\right),$$

$$\beta(x) = -\gamma\frac{\pi}{l}x - \frac{l}{\pi}\cot\alpha + \frac{1}{2a^2}\int_0^x \frac{l}{\pi}c(\tau)d\tau,$$

$$u(x,t) = \sum_{n=0}^{\infty} \left[\alpha_n \cos(\omega_n t) + \beta_n \sin(\omega_n t)\right] X_n(x), \qquad (x,t) \in \overline{\Omega}.$$

Here we have

$$\omega_n = \frac{a\pi}{l}\left(n + \frac{\gamma}{n} + O\left(\frac{1}{n^3}\right)\right), \qquad \gamma = \frac{l}{\pi^2}\left(-\cot\alpha + \cot\beta + \frac{1}{2a^2}\int_0^l c(\tau)d\tau\right),$$

where ω_n can be written in the form of:

$$\omega_n = n\frac{\pi a}{l} + \left(\frac{\gamma a\pi}{nl} + O\left(\frac{1}{n^3}\right)\right).$$

If $A_1B_1 \geq 0$ and $A_2B_2 \leq 0$, with the same arguments for γ_n, δ_n as in Section 3, we get that condition (13) is fulfilled with

$$A = (t_2 - t_1)\frac{a}{l}, \qquad B = 0, \qquad C_n = (t_2 - t_1)\frac{\gamma a\pi}{nl} + \gamma_n - \delta_n + O\left(\frac{1}{n^3}\right),$$

provided $(t_2 - t_1)\dfrac{a}{l} \in \mathbb{Q}$.

Consequently, we can apply Proposition 1 to the vibrating string with fixing (6) as follows.

Theorem 3. *Let*

$$f \in D^{s+2}, \ g \in D^{s+2}, \qquad s \in \mathbb{R},$$

and suppose that

$$\sin((t_2 - t_1)\omega_n + \gamma_n - \delta_n) \neq 0, \qquad n \in \mathbb{N}_0.$$

Then the observation problem posed for $(1'), (2), (6)$ *under the observation conditions*

$$\begin{aligned} A_1u|_{t=t_1} + B_1u_t|_{t=t_1} &= f, \qquad & |A_1| + |B_1| > 0,\ A_1B_1 \geq 0,\\ A_2u|_{t=t_2} + B_2u_t|_{t=t_2} &= g, \qquad & |A_2| + |B_2| > 0,\ A_2B_2 \leq 0, \end{aligned}$$

has a unique solution for $(\varphi, \psi) \in D^{s+1} \times D^s$ *provided the elapsed time between* t_1 *and* t_2 *is a rational multiple of* l/a.

6. OBSERVABILITY RESULTS FOR ARBITRARY $t_1 < t_2$

Considering the observation problem posed in Proposition 1 without conditions (13), (14) and for randomly chosen $t_1 < t_2$, we have to investigate the necessity of the description of all possibilities of the observability. We shall give an alternative treatment, and for this reason we introduce the following notations: for every $n \in \mathbb{N}_0$, let d_n denote the distance of $h_n = \omega_n(t_2 - t_1) + \gamma_n - \delta_n$ to the nearest zero of the sine function, i.e.,

$$d_n := \rho(h_n, \mathbb{N}_0\pi),\ n \in \mathbb{N}_0, \qquad \mathbb{N}_0\pi := \bigcup_{k=0}^{\infty}\{k\pi\}.$$

The promised alternative is the following:

1. If $d_n > 0$ for all $n \in \mathbb{N}_0$ then the observation problem is (formally) uniquely solvable, since the coefficients $\alpha_n,\ \beta_n$ for the definition of $\varphi,\ \psi$ can be uniquely finded from the linear system (19) for every $n \in \mathbb{N}_0$. Although, for the estimates of $\alpha_n,\ \beta_n$ (and for the belonging of $\varphi,\ \psi$ to the corresponding spaces $D^{s+1},\ D^s$, respectively), some

special tools are needed (e.g., we used Diophantine ones in Sections 1–5). A general result can be the following: The formal series:

$$\varphi(x) \sim \sum_{n=0}^{\infty} \alpha_n X_n(x), \quad \psi(x) \sim \sum_{n=0}^{\infty} \omega_n \beta_n X_n(x)$$

have the following properties:

$$\sum_{n=0}^{\infty} \alpha_n d_n X_n(x) \in D^{s+1}, \qquad \sum_{n=0}^{\infty} \omega_n \beta_n d_n X_n(x) \in D^s. \tag{24}$$

Moreover, the relations in (24) also hold in the case if we replace d_n by $\widetilde{d}_n$, where

$$\widetilde{d}_n = \begin{cases} 1, & d_n \geq \delta > 0, \\ b_n, & d_n < \delta. \end{cases} \qquad n \in \mathbb{N}_0,$$

with arbitrarily small fixed δ and arbitrary $|b_n| \leq d_n$.

2. Let us denote by $\mathbb{N}(0)$ the set of all $n \in \mathbb{N}_0$ such that $d_n = 0$. Then for every $n \in \mathbb{N}(0)$, the linear system (19) can be solved for (α_n, β_n) if and only if

$$\frac{f_n}{g_n} = \frac{A_1 \cos(\omega_n t_1) - B_1 \omega_n \sin(\omega_n t_1)}{A_2 \cos(\omega_n t_2) - B_2 \omega_n \sin(\omega_n t_2)} = \left(\frac{A_1 \sin(\omega_n t_1) + B_1 \omega_n \cos(\omega_n t_1)}{A_2 \sin(\omega_n t_2) + B_2 \omega_n \cos(\omega_n t_2)}\right), \quad n \in \mathbb{N}(0),$$

but the solution (α_n, β_n) as well as (φ, ψ) are not unique, and the belonging of the constructed (φ, ψ) to $D^{s+1} \times D^s$ is not guaranteed.

We emphasize, that all remarks of the present section are also valid for the observation problems posed for the case $c = 0$ of the Klein–Gordon equation, i.e., for the standard vibrating string.

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