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On contact problems for nonlinear parabolic functional differential equations

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Abstract

The results of [3] by W. Jäger and N. Kutev on a nonlinear elliptic transmission problem are extended (in a modified way) to nonlinear parabolic problems with nonlinear and nonlocal contact conditions.

Introduction

In [3] W. Jäger and N. Kutev considered the following nonlinear transmission (contact) problem for nonlinear elliptic equations:

$$\sum_{i=1}^{n} D_i[a_i(x,u,Du)] + b(x,u,Du) = 0 \text{ in } \Omega \tag{0.1}$$

$$u = g \text{ on } \partial\Omega \tag{0.2}$$

$$\left[\sum_{i=1}^{n} a_i(x,u,Du)\nu_i\right]\Big|_S = 0 \tag{0.3}$$

$$u_1 = \Phi(u_2) \text{ on } S \tag{0.4}$$

where $\Omega \subset R^n$ is a bounded domain with sufficiently smooth boundary $\partial\Omega$ which is divided into two subdomains Ω_1, Ω_2 by means of a smooth surface S which has no intersection point with $\partial\Omega$, the boundary of Ω_1 is S and the boundary of Ω_2 is $S \cup \partial\Omega$. Further, $[f]|_S$ denotes the jump of f on S in the direction of the normal ν, Φ is a smooth strictly increasing function and u_j denotes the restriction of u to Ω_j $(j = 1, 2)$. The coefficients of the equation are smooth in $\overline{\Omega_j}$ and satisfy standard conditions but they have jump on the surface S. The problem was motivated e.g. by reaction-diffusion phenomena in porous medium. The authors formulated conditions which implied comparison principles, existence and uniqueness of the weak and the classical solution, respectively.

The aim of this paper is to consider nonlinear parabolic functional differential equations with a modified contact condition on S: with boundary conditions of

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third type, containing delay. In [7] we studied parabolic differential equations with contact conditions, considered in [3]. In Section 1 we shall prove existence and uniqueness theorems and in Section 2 we shall formulate a theorem on boundedness of the solutions and a stabilization result.

1 Existence and uniqueness of solutions

Let $\Omega \subset R^n$ be a bounded domain having the uniform C^1 regularity property (see [1]) which is divided into two subdomains Ω_1, Ω_2 by means of a smooth surface S which has no intersection point with $\partial\Omega$, the boundary of Ω_1 is S and the boundary of Ω_2 is $S \cup \partial\Omega$ (such that Ω_1 and Ω_2 have the C^1 regularity property).

We shall consider weak solutions of the problem

$$D_t u^j - \sum_{i=1}^{n} D_i[a_i^j(t,x,u^j,Du^j)] + b^j(t,x,u^j,Du^j) + G^j(u^1,u^2) = F^j(t,x), \tag{1.5}$$

$$(t,x) \in Q_T^j = (0,T) \times \Omega_j, \quad j = 1,2$$

$$u^2 = 0 \text{ on } \Gamma_T = [0,T] \times \partial\Omega \tag{1.6}$$

$$\sum_{i=1}^{n} a_i^j(t,x,u^j,Du^j)\nu_i^j|_{S_T} = H^j(u^1,u^2), \quad S_T = [0,T] \times S, \quad j = 1,2 \tag{1.7}$$

$$u(0,x) = 0, \quad x \in \Omega_1 \cup \Omega_2 \tag{1.8}$$

where $u^j = u|_{Q_T^j}$, G^j, H^j are operators (which will be defined below as well as functions F^1, F^2), $\nu^j = (\nu_1^j, ..., \nu_n^j)$ is the normal unite vector on S ($\nu^1 = -\nu^2$), a_i^j, b^j have certain polynomial growth in u^j, Du^j.

Let $p \geq 2$ be a real number. For any domain $\Omega_0 \subset R^n$ denote by $W^{1,p}(\Omega_0)$ the usual Sobolev space of real valued functions with the norm

$$\| u \| = \left[\int_{\Omega_0} (|Du|^p + |u|^p) \right]^{1/p}.$$

Let $V_1 = W^{1,p}(\Omega_1)$, $V_2 = \{w \in W^{1,p}(\Omega_2) : w|_{\partial\Omega} = 0\}$ and $V = V_1 \times V_2$. Denote by $L^p(0,T;V)$ the Banach space of the set of measurable functions $u = (u^1,u^2) : (0,T) \to V$ such that $\| u \|^p$ is integrable and define the norm by

$$\| u \|^p_{L^p(0,T;V)} = \int_0^T \| u(t) \|_V^p \, dt.$$

The dual space of $L^p(0,T;V)$ is $L^q(0,T;V^\star)$ where $1/p + 1/q = 1$ and $V^\star$ is the dual space of V (see, e.g., [4], [8]).

Now we formulate the conditions with respect to the problem (1.5) - (1.8) and the existence theorem on the weak solutions of this problem where $F = (F^1,F^2) \in L^q(0,T;V^\star)$.

Assume that

I. The functions $a_i^j, b^j : Q_T^j \times R^{n+1} \to R$ satisfy the Carathéodory conditions, i.e. $a_i^j(t,x,\eta,\zeta)$, $b^j(t,x,\eta,\zeta)$ are measurable in $(t,x) \in Q_T^j = (0,T) \times \Omega_j$ for each fixed $(\eta,\zeta) \in R^{n+1}$ and they are continuous in $(\eta,\zeta) \in R^{n+1}$ for a.e. $(t,x) \in Q_T^j$.

II. $|a_i^j(t,x,\eta,\zeta)| \le c_1[|\eta|^{p-1} + |\zeta|^{p-1}] + k_1^j(x)$, for a.e. $(t,x) \in Q_T^j$, each $(\eta,\zeta) \in R^{n+1}$ with some constant c_1 and a function $k_1^j \in L^q(\Omega_j)$,

$|b^j(t,x,\eta,\zeta)| \le c_1[|\eta|^{p-1} + |\zeta|^{p-1}] + k_1^j(x)$.

III. $\sum_{i=1}^n [a_i^1(t,x,\eta,\zeta) - a_i^1(t,x,\eta,\zeta^\star)](\zeta_i - \zeta_i^\star) > 0$ if $\zeta \ne \zeta^\star$.

IV. $\sum_{i=1}^n a_i^j(t,x,\eta,\zeta)\zeta_i + b^j(t,x,\eta,\zeta)\eta \ge c_2[|\zeta|^p + |\eta|^p] - k_2^1(x)$, $(t,x) \in Q_T^1$

$\sum_{i=1}^n a_i^2(t,x,\eta,\zeta)\zeta_i + b^2(t,x,\eta,\zeta)\eta \ge c_2|\zeta|^p - k_2^2(x)$, $(t,x) \in Q_T^2$

with some constant $c_2 > 0$, $k_2^j \in L^1(\Omega_j)$.

V. $G^j : L^p(Q_T^1) \times L^p(Q_T^2) \to L^q(Q_T^j)$ are bounded (nonlinear) operators which are demicontinuous (i.e. $(u_k) \to u$ with respect to the norm $L^p(Q_T^1) \times L^p(Q_T^2)$ implies that $G^j(u_k) \to G^j(u)$ weakly in $L^q(Q_T^j)$.

VI. $H^j : L^p(0,T;V) \to L^q(S_T)$ are bounded (nonlinear) operators having the following property: There exists a positive number $\delta < 1 - 1/p$ such that the operators H^j are demicontinuous from $L^p(0,T; W^{1-\delta,p}(\Omega_1) \times W^{1-\delta,p}(\Omega_2))$ into $L^q(S_T)$.

VII. $\lim_{\|u\|\to\infty} \frac{\|G^j(u)\|^q_{L^q(Q_T^j)} + \|H^j(u)\|^q_{L^q(S_T)}}{\|u\|^p_{L^p(0,T;V)}} = 0$

for any $u \in L^p(0,T;V)$.

Then we may define the operators $A^j : L^p(0,T;V_j) \to L^q(0,T;V_j^\star)$ by

$$[A^j(u^j), v^j] = \int_{Q_T^j} \left[\sum_{i=1}^n a_i^j(t,x,u^j,Du^j) D_i v^j + b^j(t,x,u^j,Du^j) v^j \right] dtdx,$$

$$= \int_0^T \langle A^j(u^j)(t), v^j(t) \rangle dt,$$

$$A = (A^1, A^2) : L^p(0,T;V) \to L^q(0,T;V^\star)$$

$$\text{by } [A(u), v] = [A^1(u^1), v^1] + [A^2(u^2), v^2]$$

and the operators $B^j : L^p(0,T;V) \to L^q(0,T;V_j^\star)$ by

$$[B^j(u), v^j] = [B_1^j(u), v^j] - [B_2^j(u), v^j] =$$

$$\int_{Q_T^j} G^j(u) v^j dtdx - \int_{S_T} H^j(u) v^j dtd\sigma_x,$$

$$u = (u^1, u^2) \in L^p(0,T;V), \quad (v^1, v^2) \in L^p(0,T;V).$$

By I, II, V, VI, Hölder's inequality and Vitali's theorem operator

$$A + B = (A^1, A^2) + (B^1, B^2) : L^p(0,T;V) \to L^q(0,T;V^\star)$$

is bounded (i.e. it maps bounded sets of $L^p(0,T;V)$ into bounded sets of $L^q(0,T;V^\star)$) and demicontinuous.

Theorem 1.1 *Assume I - VII. Then for any $F = (F^1, F^2) \in L^q(0,T;V^\star)$ there exists $u = (u^1, u^2) \in L^p(0,T;V)$ such that $D_t u^j \in L^q(0,T;V_j^\star)$,*

$$D_t u^j + A^j(u^j) + B^j(u^1, u^2) = F^j, \quad j = 1, 2 \tag{1.9}$$

$$u^j(0) = 0, \quad j = 1, 2. \tag{1.10}$$

Remark 1 If u satisfies (1.9), (1.10), we say that $u = (u^1, u^2)$ is a weak solution of (1.5) - (1.8).

Proof of Theorem 1.1 Let the operators $L^j : L^p(0,T;V_j) \to L^q(0,T;V_j^\star)$ be defined by

$$D(L^j) = \{u^j \in L^p(0,T;V_j) : D_t u^j \in L^q(0,T;V_j^\star), \quad u^j(0) = 0\},$$

$$[L^j u^j, v^j] = \int_0^T \langle D_t u^j(t,\cdot).v^j(t,\cdot)\rangle dt, \quad u^j \in D(L^j), \quad v^j \in L^p(0,T;V_j)$$

where $D_t u^j$ is the distributional derivative of u^j. It is well known that L^j is a closed linear maximal monotone map (see, e.g., [8]), thus $L = (L^1, L^2) : L^p(0,T;V) \to L^q(0,T;V^\star)$ is a closed linear maximal monotone map, too. Therefore, Theorem 1. will follow from Theorem 4. of [2] if we show that operator $A + B$ is coercive and pseudomonotone with respect to $D(L)$. It is known that A is pseudomonotone with respect to $D(L)$ (see, e.g. [5]). The latter property means that for any sequence (u_k) in $D(L)$ with

$$(u_k) \to u \text{ weakly in } L^p(0,T;V), \tag{1.11}$$

$$(Lu_k) \to Lu \text{ weakly in } L^q(0,T;V^\star), \tag{1.12}$$

$$\limsup_{k\to\infty}[A(u_k), u_k - u] \le 0 \tag{1.13}$$

we have

$$\lim_{k\to\infty}[A(u_k), u_k - u] = 0, \tag{1.14}$$

$$(A(u_k)) \to A(u) \text{ weakly in } L^q(0,T;V^\star). \tag{1.15}$$

Now we prove that $(A + B)$ is pseudomonotone with respect to $D(L)$, too. Assume (1.11), (1.12) and

$$\limsup_{k\to\infty}[(A + B)(u_k), u_k - u] \le 0. \tag{1.16}$$

Since the imbedding $W^{1,p}(\Omega_j) \to W^{1-\delta,p}(\Omega_j)$ is compact, by a well known compactness result (see, e.g., [4]) (1.11), (1.12) imply that there is a subsequence (u_{k_l}) of (u_k) such that

$$(u_{k_l}) \to u \text{ in } L^p(0,T;W^{1-\delta,p}(\Omega_1) \times W^{1-\delta,p}(\Omega_2)). \tag{1.17}$$

Since the trace operators $W^{1-\delta,p}(\Omega_j) \to L^p(S)$ are continuous ($\delta + 1/p < 1$, see, e.g., [1]), we obtain by (1.17), V, VI and Hölder's inequality

$$\lim_{l\to\infty}[B(u_{k_l}), u_{k_l} - u] = 0. \tag{1.18}$$

Further, (1.17), V, VI imply

$$(B(u_{k_l})) \to A(u) \text{ weakly in } L^q(0,T;V^\star). \tag{1.19}$$

From (1.16), (1.18) we obtain

$$\limsup_{l\to\infty}[A(u_{k_l}), u_{k_l} - u] \le 0. \tag{1.20}$$

As A is pseudomonotone with respect to $D(L)$, (1.11), (1.12) and (1.20) imply

$$\lim_{l\to\infty}[A(u_{k_l}), u_{k_l} - u] = 0, \tag{1.21}$$

$$(A(u_{k_l})) \to A(u) \text{ weakly in } L^q(0,T;V^\star). \tag{1.22}$$

Finally, from (1.18), (1.19), (1.21) and (1.22) we obtain

$$\lim_{l\to\infty}[(A+B)(u_{k_l}), u_{k_l} - u] = 0, \tag{1.23}$$

$$((A+B)(u_{k_l})) \to (A+B)(u) \text{ weakly in } L^q(0,T;V^\star) \tag{1.24}$$

which means that $(A+B)$ is pseudomonotone with respect to $D(L)$. (It is easy to show that (1.23), (1.24) hold for the sequence (u_k), too.)

Now we show that $A+B$ is coercive. By assumption IV we have

$$[A(u), u] \ge c_2 \parallel u \parallel^p_{L^p(0,T;V)} - c_2^\star \tag{1.25}$$

with constants $c_2 > 0, c_2^\star$. Further, assumption VII implies

$$\frac{|[B(u), u]|}{\parallel u \parallel^p} \le \frac{\parallel B(u) \parallel}{\parallel u \parallel^{p-1}} = \left[\frac{\parallel B(u) \parallel^q}{\parallel u \parallel^p}\right]^{1/q} \to 0 \tag{1.26}$$

if $\parallel u \parallel \to 0$. Thus by (1.25), (1.26)

$$\frac{[(A+B)(u), u]}{\parallel u \parallel} \ge \frac{[A(u), u]}{\parallel u \parallel} - \frac{|[B(u), u]|}{\parallel u \parallel} \ge$$

$$\frac{c_2 \parallel u \parallel^p - c_2^\star}{\parallel u \parallel} - \frac{|[B(u), u]|}{\parallel u \parallel^p} \parallel u \parallel^{p-1} =$$

$$\parallel u \parallel^{p-1} \left[c_2 - \frac{|[B(u), u]|}{\parallel u \parallel^p}\right] - \frac{c_2^\star}{\parallel u \parallel} \to +\infty$$

if $\parallel u \parallel \to \infty$, i.e. $A+B$ is coercive.

Examples for G^j and H^j

a/ Let

$$[G^1(u)](t,x) = \gamma^1(t, x, u^1(\chi_1(t), x), \int_{\Omega_2} d^2(y) u^2(\chi_2(t), y) dy), \quad (t,x) \in Q_T^1,$$

$$[G^2(u)](t,x) = \gamma^2(t, x, \int_{\Omega_1} d^1(y) u^1(\chi_1(t), y) dy, u^2(\chi_2(t), x)), \quad (t,x) \in Q_T^2$$

where χ_1, χ_2 are C^1 functions satisfying $\chi_j' > 0$, $0 \le \chi_j(t) \le t$; d^1, d^2 are L^∞ functions; the functions γ^j satisfy the Carathéodory conditions and

$$|\gamma_j(t,x,\theta_1,\theta_2)| \le c^j(\theta_1,\theta_2)|\theta|^{p-1} + k_1^j(x) \tag{1.27}$$

with continuous functions c^j having the property

$$\lim_{|(\theta_1,\theta_2)|\to\infty} c^j = 0, \quad k_1^j \in L^q(\Omega_j).$$

By using Hölder's inequality and Vitali's theorem it is not difficult to prove that condition V is fulfilled (see [5], [6]) and by (1.27) one obtains VII.

b/ Similarly can be considered operators

$$[G^1(u)](t,x) = \int_0^t \gamma^1\left(t,\tau,x,u^1(\tau,x), \int_{\Omega_2} d^2(y)u^2(\tau),y)dy\right)d\tau, \quad (t,x)\in Q_T^1,$$

$$[G^2(u)](t,x) = \int_0^t \gamma^2\left(t,\tau,x, \int_{\Omega_1} d^1(y)u^1(\tau,y)dy, u^2(\tau,x)\right)d\tau, \quad (t,x)\in Q_T^2$$

where γ^j satisfy a condition which is analogous to (1.27).

c/ Let

$$[H^j(u)](t,x) = h^j(t,x,u^1(\chi_1(t),x),u^2(\chi_2(t),x)), \quad (t,x)\in S_T,$$

where the functions h^j satisfy a condition analogous to (1.27). By $\delta < 1-1/p$ the trace operator $W^{1-\delta,p}(\Omega) \to L^p(\partial\Omega)$ is bounded, thus by using Hölder's inequality and Vitali's theorem, one can prove that VI and by the condition, analogous to (1.27), VII are satisfied.

Similarly can be treated the following examples:

d/

$$[H^j(u)](t,x) = \int_0^t h^j(t,\tau,x,u^1(\tau,\Phi_1(x)),u^2(\tau,\Phi_2(x)))d\tau, \quad (t,x)\in S_T,$$

where $\Phi_j, (\Phi_j)^{-1}$ are C^1 functions in a neighbourhood of S, $\Phi_j(S) = S$.

e/

$$[H^j(u)](t,x) = h^j(t,x,\int_S u^1(\chi(t),y)d\sigma_y, \int_S u^2(\chi(t),y)d\sigma_y), \quad (t,x)\in S_T,$$

f/

$$[H^j(u)](t,x) = \int_0^t h^j\left(t,\tau,x,\int_S u^1(\tau,y)d\sigma_y, \int_S u^2(\tau,y)d\sigma_y\right)d\tau, \quad (t,x)\in S_T.$$

By using monotonicity arguments one can prove uniqueness of the solution.

Theorem 1.2 *Assume that*

$$\sum_{i=1}^{n}[a_i^j(t,x,\eta,\zeta)-a_i^j(t,x,\eta^\star,\zeta^\star)](\zeta_i-\zeta_i^\star)+ \tag{1.28}$$

$$[b^j(t,x,\eta,\zeta)-b^j(t,x,\eta^\star,\zeta^\star)](\eta-\eta^\star)\geq -c_0(\eta-\eta^\star)^2$$

with some constant c_0 *and*

$$\sum_{j=1}^{n}[H^j(u)-H^j(v),u-v]\geq 0,\quad u,v\in L^p(0,T;V). \tag{1.29}$$

Further, for the operators

$$[\tilde{G}^j(\tilde{u})](t,x)=e^{-\alpha t}[G^j(e^{\alpha t}\tilde{u})](t,x),$$

the inequality

$$\| \tilde{G}^j(\tilde{u})-\tilde{G}^j(\tilde{v}) \|_{L^2(Q_T^j)}\leq \tilde{c} \| \tilde{u}-\tilde{v} \|_{L^2(Q_T^1)\times L^2(Q_T^2)} \tag{1.30}$$

holds where the constant $\tilde{c}$ *is not depending on the positive number* α *and* $\tilde{u},\tilde{v}$. *Then the problem (1.9), (1.10) may have at most one solution.*

Remark 2 It is easy to show that (1.30) holds for the above examples a/ and b/ if functions γ^j satisfy (global) Lipschitz condition with respect to θ_1 and θ_2. Further, (1.29) holds if $[H^j(u)](t,x)=h^j(t,x,u^1(t,x),u^2(t,x))$ and

$$\sum_{j=1}^{2}[h^j(t,x,\theta_1,\theta_2)-h^j(t,x,\theta_1^\star,\theta_2^\star)](\theta_j-\theta_j^\star)\geq 0.$$

(E.g. h^1 is not depending on θ_2, h^2 is not depending on θ_1 and for a.e. fixed (t,x) the functions $\theta_j\to h^j(t,x,\theta_j)$ are monotone increasing.)

The proof of Theorem 1.2 Perform the substitution $u=e^{\alpha t}\tilde{u}$. Then (1.9), (1.10) is equivalent with

$$D_t\tilde{u}^j+\tilde{A}^j(\tilde{u}^j)+\tilde{B}^j(\tilde{u}^1,\tilde{u}^2)+\alpha\tilde{u}^j=\tilde{F}^j, \tag{1.31}$$

$$\tilde{u}^j(0)=0. \tag{1.32}$$

where

$$[\tilde{A}^j(\tilde{u}^j),v^j]=\int_{Q_T^j}\left[\sum_{i=1}^{n}\tilde{a}_i^j(t,x,\tilde{u}^j,D\tilde{u}^j)D_iv^j+\tilde{b}^j(t,x,\tilde{u}^j,D\tilde{u}^j)v^j\right]dtdx,$$

$$\tilde{a}_i^j(t,x,\eta,\zeta)=e^{-\alpha t}a_i^j(t,x,e^{\alpha t}\eta,e^{\alpha t}\zeta),$$

$$\tilde{b}^j(t,x,\eta,\zeta)=e^{-\alpha t}b^j(t,x,e^{\alpha t}\eta,e^{\alpha t}\zeta),$$

$$[\tilde{B}^j(\tilde{u}),v^j]=\int_{Q_T^j}\tilde{G}^j(\tilde{u})v^j dtdx-\int_{S_T}\tilde{H}^j(\tilde{u})v^j dtd\sigma_x,$$

$$[\tilde{G}^j(\tilde{u})](t,x)=e^{-\alpha t}[G^j(e^{\alpha t}\tilde{u})](t,x),[\tilde{H}^j(\tilde{u})](t,x)=e^{-\alpha t}[H^j(e^{\alpha t}\tilde{u})](t,x).$$

The solution of (1.31), (1.32) is unique because by (1.28) - (1.29) the operator $\tilde{A}+\tilde{B}+\alpha I$ is monotone if α is sufficiently large:

$$[(\tilde{A}+\tilde{B})(\tilde{u})+\alpha\tilde{u}-(\tilde{A}+\tilde{B})(\tilde{v})-\alpha\tilde{v},\tilde{u}-\tilde{v}]\geq 0.$$

2 Boundedness and stabilization

One can prove an existence theorem also for the interval $(0,\infty)$. Denote by X_∞ and $X_\infty^\star$ the set of functions

$$u:(0,\infty)\to V, \quad w:(0,\infty)\to V^\star,$$

respectively, such that (for their restrictions to $(0,T)$)

$$u\in L^p(0,T;V), \quad w\in L^q(0,T;V^\star)$$

for any finite $T>0$. Further, let $Q_\infty^j=(0,\infty)\times\Omega_j$, $S_\infty=[0,\infty)\times S$. $L^q_{loc}(Q_\infty^j)$ will denote the set of functions $v^j:Q_\infty^j\to R$ such that $v^j|_{Q_T^j}\in L^q(Q_T^j)$; $L^q_{loc}(S_\infty)$ will denote the set of functions $v:S_\infty\to R$ such that $v|_{S_T}\in L^q(S_T)$.

Theorem 2.1 *Assume that we have functions $a_i^j, b^j : Q_\infty^j\times R^{n+1}\to R$ such that assumptions I - IV are satisfied for any finite T with the same constants c_j and functions k_i^j. Further, operators $G^j:X_\infty\to L^q_{loc}(Q_\infty^j)$ and $H^j:X_\infty\to L^q_{loc}(S_\infty)$ are such that their restrictions to $L^p(0,T;V)$ satisfy V - VII. Assume that G^j,H^j are of Volterra type, which means that $[G^j(u)](t,x)$, $[H^j(u)](t,x)$ depend only on the restrictions of u^j to $(0,t)\times\Omega_j$ ($j=1,2$). Then for any $F\in X_\infty^\star$ there exists $u\in X_\infty$ such that the statement of Theorem 1.1 holds for any finite T.*

Proof Let T_k be a strictly increasing sequence of positive numbers with $\lim(T_k)=+\infty$. For arbitrary k there exists a weak solution $u_k\in L^p(0,T_k;V)$ of (1.9), (1.10) with $T=T_k$. Since G^j,H^j are of Volterra type, the restrictions of u_l^j to $Q_{T_k}^j$ is a solution in $Q_{T_k}^j$ if $l>j$. By using a diagonal process and arguments of the proof of Theorem 1.1 we can select a subsequence of (u_k) which is weakly convergent to a function $u\in X_\infty$ in $L^p(0,T;V)$ for arbitrary finite T and the statement of of Theorem 1.1 holds for u with any finite T.

If some additional conditions are satisfied then one can prove that

$$y(t)=\| u(t)\|^2_{L^2(\Omega_1)\times L^2(\Omega_2)}$$

is bounded in $(0,\infty)$ for a solution u.

Theorem 2.2 *Let the assumptions of Theorem 2.1 be satisfied and assume that $p>2$,*

$$\| F(t)\|_{V^\star} \text{ is bounded },\, t\in[0,\infty), \tag{2.33}$$

for arbitrary $u\in X_\infty$

$$\int_{\Omega_j}|G^j(u)(t,x)|^q dx+\int_S |H^j(u)(t,x)|^q d\sigma_x \le \tag{2.34}$$

$$c_4\sup_{[0,t]}|y|+c_5(t)\sup_{[0,t]}|y|^{p/2}+c_6$$

where c_4, c_6 are constants and c_5 is a continuous function with $\lim_\infty c_5 = 0$. Then y is bounded in $[0, \infty)$ for a solution u. Further,

$$\int_{T_1}^{T_2} \| u(t) \|_V^p \, dt \leq c'(T_1 - T_2) + c", \quad 0 < T_1 < T_2 \tag{2.35}$$

with some constants $c', c"$, not depending on T_1, T_2.

The idea of the proof Applying (1.9) to $u = (u^1, u^2)$ with arbitrary $T_1 < T_2$ we obtain

$$\int_{T_1}^{T_2} \langle D_t u^j(t), u^j(t) \rangle dt + \int_{T_1}^{T_2} \langle [A^j(u^j)](t), u^j(t) \rangle dt + \tag{2.36}$$

$$\int_{T_1}^{T_2} \langle [B^j(u^1, u^2)](t), u^j(t) \rangle dt = \int_{T_1}^{T_2} \langle F^j(t), u^j(t) \rangle dt.$$

Since y is absolutely continuous and

$$y'(t) = 2\langle D_t u^1(t), u^1(t) \rangle + 2\langle D_t u^2(t), u^2(t) \rangle$$

(see, e.g., [8]), by using assumption IV, (2.33), (2.34), Young's inequality and Hölder's inequality, we obtain from (2.36) the inequality

$$y(T_2) - y(T_1) + c_3^\star \int_{T_1}^{T_2} [y(t)]^{p/2} dt \leq \tag{2.37}$$

$$c_4^\star \int_{T_1}^{T_2} \left[\sup_{[0,t]} y + c_5(t) \sup_{[0,t]} y^{p/2} + 1 \right] dt$$

where $c_3^\star, c_4^\star$ are constants. It is not difficult to show that (2.37) and $p > 2$ imply the boundedness of y and (2.35).

Remark 3 The estimation (2.34) is fulfilled for G^j, e.g. if G^j is given in examples a/ or b/ and the functions γ^j satisfy

$$|\gamma^1(t, x, \theta_1, \theta_2)|^q, \quad |\gamma^1(t, \tau, x, \theta_1, \theta_2)|^q \leq c_5^\star(\theta_1^2 + \theta_2^2) + c_6^\star(t)|\theta_2|^p + c_7^\star,$$

$$|\gamma^2(t, x, \theta_1, \theta_2)|^q, \quad |\gamma^2(t, \tau, x, \theta_1, \theta_2)|^q \leq c_5^\star(\theta_1^2 + \theta_2^2) + c_6^\star(t)|\theta_1|^p + c_7^\star,$$

respectively, with some constants $c_7^\star$, $c_9^\star$, $\lim_\infty c_6 = 0$ and there is a positive number ρ such that

$$\gamma^j(t, \tau, x, \theta_1, \theta_2) = 0 \text{ if } \tau \leq t - \rho.$$

The estimation (2.34) is fulfilled for H^j, e.g. if H^j is given in examples c/, d/, e/ or f/, the functins h^j are bounded and

$$h^j(t, \tau, x, \theta_1, \theta_2) = 0 \text{ if } \tau \leq t - \rho.$$

By using monotonicity arguments, similarly to Theorem 2.2, one can prove the following stabilization result.

Theorem 2.3 *Assume that the conditions of Theorem 2.2 are fulfilled and*

$$\sum_{i=1}^{n}[a_i^j(t,x,\eta,\zeta)-a_i^j(t,x,\eta^\star,\zeta^\star)](\zeta_i-\zeta_i^\star)+$$

$$[b^j(t,x,\eta,\zeta)-b^j(t,x,\eta^\star,\zeta^\star)](\eta_i-\eta_i^\star)\geq c[\alpha_j|\eta-\eta^\star|^p+|\zeta-\zeta^\star|^p]$$

with some constant $c>0$ *and* $\alpha_1=1$, $\alpha_2=0$*; for a.e.* $x\in\Omega_j$, *each* $(\eta,\zeta)\in R^{n+1}$

$$\lim_{t\to\infty}a_i^j(t,x,\eta,\zeta)=a_{i,\infty}^j(x,\eta,\zeta),\quad \lim_{t\to\infty}b^j(t,x,\eta,\zeta)=b_\infty^j(x,\eta,\zeta),$$

$a_{i,\infty}^j$, b_∞^j *satisfy the Carathéodory condition. Further, assume that for any* $u\in X_\infty$

$$\int_{\Omega_j}|G^j(u)(t,x)|^q dx+\int_S|H^j(u)(t,x)|^q d\sigma_x \tag{2.38}$$

$$\leq c_4(t)\sup_{[0,t]}|y|+c_5(t)\sup_{[0,t]}|y|^{p/2}+c_6(t),\quad t\in(0,\infty)$$

where

$$y(t)=\| u(t)\|^2_{L^2(\Omega_1)\times L^2(\Omega_2)},\quad \lim_{\infty}c_\nu=0,\quad \nu=4,5,6.$$

Finally, assume that there exists $F_\infty\in V^\star$ *such that*

$$\lim_{t\to\infty}\| F(t)-F_\infty\|_{V^\star}=0.$$

If u *is a solution in* $(0,\infty)$ *then there exists* $u_\infty\in V$ *such that*

$$\lim_{t\to\infty}\| u(t)-u_\infty\|_{L^2(\Omega_1)\times L^2(\Omega_2)}=0$$

and u_∞ *is the (unique) solution of*

$$A_\infty^j(u_\infty^j)=F_\infty^j$$

where A_∞^j *is defined by*

$$\langle A_\infty^j(u_\infty^j),w^j\rangle=\sum_{i=1}^{n}\int_{\Omega_j}a_{i,\infty}^j(x,u_\infty^j,Du_\infty^j)D_iw^j dx+$$

$$\int_{\Omega_j}b_\infty^j(x,u_\infty^j,Du_\infty^j)w^j dx,\quad w^j\in V_j.$$

Remark 4 The assumption (2.38) is satisfied for the examples a/ - f/ if

$$|\gamma^j(t,x,\theta_1,\theta_2)|^q\leq\Phi^\star(t)(\theta_1^2+\theta_2^2)+\tilde\Phi(t),$$

$$|\gamma^j(t,\tau,x,\theta_1,\theta_2)|^q\leq\Phi^\star(t)(\theta_1^2+\theta_2^2)+\tilde\Phi(t),$$

respectively, with $\lim_\infty \Phi^\star = \lim_\infty \tilde{\Phi} = 0$ and there is a positive number ρ such that

$$\gamma^j(t, \tau, x, \theta_1, \theta_2) = 0 \text{ if } \tau \le t - \rho;$$

further,

$$|h^j(t, x, \theta_1, \theta_2)|^q \le \tilde{\Phi}(t), \quad |h^j(t, \tau, x, \theta_1, \theta_2)|^q \le \tilde{\Phi}(t)$$

and

$$h^j(t, \tau, x, \theta_1, \theta_2) = 0 \text{ if } \tau \le t - \rho.$$

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