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EXISTENCE THEORY FOR FUNCTIONAL INITIAL VALUE PROBLEMS OF ORDINARY DIFFERENTIAL EQUATIONS

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Abstract

In this paper the existence of a solution of general nonlinear functional differential equations is proved under mixed generalized Lipschitz and Carathéodory condition. An existence theorem for the extremal solutions is also proved under certain monotonicity and weaker continuity conditions. Examples are provided to illustrate the abstract theory developed in this paper.

Key Words and Phrases: Functional differential equation and existence theorem.
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1 Statement of Problem

Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space with a norm $|\cdot|$ defined by

$$|x| = |x_1| + ... + |x_n|$$

for $x = (x_1, ..., x_n) \in \mathbb{R}^n$. Let $a, r \in \mathbb{R}$ be such that $a > 0$, $r > 0$ and let $I_0 = [-r, 0]$ and $I = [0, a]$ be two closed and bounded intervals in $\mathbb{R}$. Let $C = C(I_0, \mathbb{R}^n)$ denote a Banach space of all continuous $\mathbb{R}^n$-valued functions on I_0 with the usual supremum norm $\|\cdot\|_C$. For every $t \in I$ we define a continuous function $x_t : I_0 \to \mathbb{R}$ by $x_t(\theta) = x(t + \theta)$ for each $\theta \in I_0$. Let $J = [-r, a]$ and let $BM(J, \mathbb{R}^n)$ denote the space of bounded and measurable $\mathbb{R}^n$-valued functions on J. Define a maximum norm $\|\cdot\|$ in $BM(J, \mathbb{R}^n)$ by $\|x\| = \max\limits_{t \in J} |x(t)|$. Given a bounded operator $G : X \subset BM(J, \mathbb{R}^n) \to Y \subset BM(J, \mathbb{R}^n)$, consider the perturbed functional differential equation (in short FDE)

$$\left.\begin{aligned} & x'(t) = f(t, x(t), Sx) \ \text{ a.e. } \ t \in I \\ & x(t) = Gx(t), \ t \in I_0 \end{aligned}\right\} \tag{1.1}$$

where $f : I \times \mathbb{R}^n \times BM(J, \mathbb{R}^n) \to \mathbb{R}^n$ and $S : X \subset BM(J, \mathbb{R}^n) \to Y \subset BM(J, \mathbb{R}^n)$.

By a solution of FDE (1.1) we mean a function $x \in C(J, \mathbb{R}^n) \cap B(I_0, \mathbb{R}^n) \cap AC(I, \mathbb{R}^n)$ that satisfies the equations in (1.1), where $AC(I, \mathbb{R}^n)$ is the space of all absolutely continuous functions on I with $J = I_0 \bigcup I$.

The FDE (1.1) seems to be new and special cases of it have been discussed in the literature since long time. These special cases to FDE (1.1) can be obtained by defining the operators G and S appropriately. The operators G and S are called functional operators of the functional differential equation (1.1). As far as the author is aware there is no previous work on the existence theory for the FDE (1.1) in the framework of Carathéodory as well as monotonicity conditions. Now take $X = BM(I_0, \mathbb{R}) \cap AC(I, \mathbb{R}) \cap BM(J, \mathbb{R}) \subset BM(J, \mathbb{R})$. Let $G : X \to BM(I_0, \mathbb{R})$ and define the operator $S : X \to X$ by $Sx(t) = x(t), \quad t \in J$. Then the FDE (1.1) takes the form

$$\left.\begin{aligned} &x'(t) = f(t, x(t), x) \ \text{ a.e. } \ t \in I \\ &x(t) = Gx(t), \ t \in I_0 \end{aligned}\right\} \tag{1.2}$$

which is the functional differential equation discussed in Liz and Pouso [8] for the existence of solution in the framework of upper and lower solutions. Further as mentioned in Liz and Pouso [8], the FDE (1.2) includes several important classes of functional differential equations as special cases. Again when $S, G : X \to C(I_0, \mathbb{R})$ are two operators defined by $Sx(t) = x_t, \ t \in I$ and $Gx(t) = \phi(t), \ t \in I_0$, the FDE (1.1) reduces to the following FDE

$$\left.\begin{aligned} &x'(t) = f(t, x(t), x_t) \ \text{ a.e. } \ t \in I \\ &x(t) = \phi(t), \ t \in I_0 \end{aligned}\right\} \tag{1.3}$$

where $f : I \times \mathbb{R}^n \times C(I_0, \mathbb{R}^n) \to \mathbb{R}^n$ and $\phi \in C(I_0, \mathbb{R}^n)$.

We note that the FDE (1.3) again covers several important classes of functional differential equations discussed earlier as special cases. See Haddock and Nkashama [4], Lee and O'Regan [6], Leela and Oguztoreli [7], Stepanov [9], Xu and Liz [11] and references therein.

We shall apply fixed point theorems for proving the existence theorems for FDE (1.1) under the generalized Lipschitz and monotonicity conditions.

2 Existence Theorem

An operator $T : X \to X$ is called **compact** if $\overline{T(X)}$ is a compact subset of X. Similarly $T : X \to X$ is called **totally bounded** if T maps a bounded subset of X into the relatively compact subset of X. Finally $T : X \to X$ is called **completely continuous** operator if it is continuous and totally bounded operator on X.

In this paper we shall prove existence theory for the FDE (1.1) via the following nonlinear alternative of Leray- Schauder [2].

Theorem 2.1 *Let X be a Banach space and let $T : X \to X$ be a completely continuous operator. Then either*

(i) the equation $\lambda Tx = x$ has a solution for $\lambda = 1$, or

(ii) the set $\mathcal{E} = \{u \in X \mid \lambda Tu = u, 0 < \lambda < 1\}$ is unbounded.

Let $M(J, \mathbb{R}^n)$ and $B(J, \mathbb{R}^n)$ denote respectively the spaces of measurable and bounded $\mathbb{R}^n$-valued functions on J. We shall seek the solution of FDE (1.1) in the space $C(J, \mathbb{R}^n)$, of all continuous real-valued functions on J. Define a norm $\|\cdot\|$ in $C(J, \mathbb{R}^n)$ by

$$\|x\| = \sup_{t \in J} |x(t)|.$$

Clearly $C(J, \mathbb{R}^n)$ becomes a Banach space with this norm. We need the following definition in the sequel.

Definition 2.1 *A mapping $\beta : J \times \mathbb{R}^n \times C \to \mathbb{R}^n$ is said to satisfy Carathéodory's conditions or simply is called L^1-Carathéodory if*

(i) $t \to \beta(t, x, y)$ is measurable for each $x \in \mathbb{R}^n$ and $y \in BM(J, \mathbb{R}^n)$,

(ii) $(x, y) \to \beta(t, x, y)$ is continuous almost everywhere for $t \in J$, and

(iii) for each real number $k > 0$, there exists a function $h_k \in L^1(J, \mathbb{R})$ such that

$$|\beta(t, x, y)| \le h_k(t), \quad a.e.\ t \in J$$

for all $x \in \mathbb{R}^n$ and $y \in BM(J, \mathbb{R}^n)$ with $|x| \le k$, $\|y\| \le k$.

We will need the following hypotheses:

(A_1) The operator $S : BM(J, \mathbb{R}^n) \to BM(J, \mathbb{R}^n)$ is continuous.

(A_2) The operator $G : BM(J, \mathbb{R}^n) \to C(I_0, \mathbb{R}^n)$ is compact and continuous with $N = \max\{\|Gx\| : x \in BM(J, \mathbb{R}^n)\}$.

(A_3) The function $f(t, x, y)$ is L^1-Carathéodory.

(A_4) There exists a nondecreasing function $\phi : [0,\infty) \to (0,\infty)$ and a function $\gamma \in L^1(J,\mathbb{R})$ such that $\gamma(t) > 0$, a.e. $t \in J$ and

$$|f(t,x,y)| \leq \gamma(t)\phi(|x|), \quad a.e.\ t \in I,$$

for all $x \in \mathbb{R}^n$ and $y \in BM(J,\mathbb{R}^n)$.

Theorem 2.2 *Assume that the hypotheses (A_1)-(A_4) hold. Suppose that*

$$\int_N^\infty \frac{ds}{\phi(s)} > \|\gamma\|_{L^1}. \tag{2.1}$$

Then the FIE (1.1) has a solution on J.

Proof. Now the FDE (1.1) is equivalent to the functional integral equation (in short FIE)

$$x(t) = \begin{cases} Gx(0) + \displaystyle\int_0^t f(s,x(s),Sx)\,ds, & t \in I \\ Gx(t), & t \in I_0. \end{cases} \tag{2.2}$$

Let $X = AC(J,\mathbb{R}^n)$. Define a mappings T on X by

$$Tx(t) = \begin{cases} Gx(0) + \displaystyle\int_0^t f(s,x(s),Sx)\,ds, & t \in I, \\ Gx(t), & t \in I_0. \end{cases} \tag{2.3}$$

Obviously T defines the operator $T : X \to X$. We show that T is completely continuous on X. Using the standard arguments as in Granas et al. [3], it is shown that T is a continuous operator on X, with respect to the norm $\|\cdot\|$. Let Y be a bounded set in X. Then there is real number $r > 0$ such that $\|x\| \leq r$ for all $x \in Y$. We shall show that $T(Y)$ is a uniformly bounded and equi-continuous set in X. Since Y is bounded, then there exists a constant $r > 0$ such that $\|x\| \leq r$ for all $x \in Y$. Now by (A_1),

$$\begin{aligned} |Tx(t)| &\leq N + \int_0^t |f(s,x(s),Sx)|\,ds \\ &\leq N + \int_0^t h_r(s)\,ds \\ &\leq N + \|h_r\|_{L^1}, \end{aligned}$$

i.e. $\|Tx\| \leq M$ for all $x \in Y$, where $M = N + \|h_r\|_{L^1}$. This shows that $T(Y)$ is a uniformly bounded set in X. Now we show that $T(Y)$ is an equi-continuous set. Let

$t, \tau \in I$. Then for any $x \in Y$ we have by (2.3),

$$\begin{aligned}
|Tx(t) - Tx(\tau)| &\leq \left| \int_0^t f(s, x(s), Sx)\, ds - \int_0^\tau f(s, x(s), Sx)\, ds \right| \\
&\leq \left| \int_\tau^t |f(s, x(s), Sx)|\, ds \right| \\
&\leq \left| \int_\tau^t h_r(s)\, ds \right| \\
&\leq |p(t) - p(\tau)|
\end{aligned}$$

where $p(t) = \displaystyle\int_0^t h_k(s)\, ds$.

Similarly if $\tau, t \in I_0$, then we obtain

$$|Tx(t) - Tx(\tau)| = |Gx(t) - Gx(\tau)|.$$

Since G is compact and continuous on X, $G(Y)$ is a relatively compact set in $C(I_0, \mathbb{R}^n)$. Consequently it is a equi-continuous set in $C(I_0, \mathbb{R}^n)$ and hence we have

$$|Gx(t) - Gx(\tau)| \to 0.$$

for all $x \in Y$. If $\tau \in I_0$ and $t \in I$, then we obtain

$$\begin{aligned}
|Tx(t) - Tx(\tau)| &\leq |Gx(\tau) - Gx(0)| + \left| \int_0^t g(s, x(s), Sx)\, ds \right| \\
&\leq |Gx(\tau) - Gx(0)| + \left| \int_0^t |g(s, x(s), Sx)|\, ds \right| \\
&\leq |Gx(\tau) - Gx(0)| + \int_0^t h_r\, ds.
\end{aligned}$$

Note that if $|t - \tau| \to 0$ implies that $t \to 0$ and $\tau \to 0$. Therefore in all above three cases,

$$|Tx(t) - Tx(\tau)| \to 0 \quad \text{as} \quad t \to \tau.$$

Hence $T(Y)$ is an equi-continuous set and consequently $T(Y)$ is relatively compact by Arzelá-Ascoli theorem. Consequently T is a completely continuous operator on X. Thus all the conditions of Theorem 3.1 are satisfied and a direct application of it yields that either conclusion (i) or conclusion (ii) holds. We show that the conclusion (ii) is not possible. Let $u \in X$ be any solution to FDE (1.1). Then we have, for any $\lambda \in (0, 1)$,

$$\begin{aligned}
u(t) &= \lambda Tu(t) \\
&= \lambda \left[Gx(0) + \int_0^t f(s, u(s), Su)\, ds \right]
\end{aligned}$$

for $t \in I$, and

$$u(t) = \lambda T u(t) = \lambda G u(t)$$

for all $t \in I_0$. Then we have

$$\begin{aligned} |u(t)| &\leq N + \left| \int_0^t f(s, u(s), Su)\, ds \right| \\ &\leq N + \int_0^t |f(s, u(s), Su)|\, ds \\ &\leq N + \int_0^t \gamma(s)\phi(|u(s)|)\, ds \\ &\leq N + \int_0^t \gamma(s)\phi(|u(s)|)\, ds. \end{aligned} \tag{2.4}$$

Let $w(t) = N + \displaystyle\int_0^t \gamma(s)\phi(u(s))\, ds$ for $t \in I$. Then we have $|u(t)| \leq w(t)$ for all $t \in I$. Since ϕ is nondecreasing, a direct differentiation of $w(t)$ yields

$$\left.\begin{aligned} w'(t) &\leq \gamma(s)\phi(w(t)) \\ w(0) &= N, \end{aligned}\right\} \tag{2.5}$$

that is,

$$\int_0^t \frac{w'(s)}{\phi(w(s))}\, ds \leq \int_0^t \gamma(s)\, ds.$$

By the change of variables in the above integral gives that

$$\begin{aligned} \int_N^{w(t)} \frac{ds}{\phi(s)} &\leq \int_0^t \gamma(s)\, ds \\ &\leq \|\gamma\|_{L^1} \\ &< \int_N^\infty \frac{ds}{\phi(s)}. \end{aligned}$$

Now an application of mean value theorem yields that there is a constant $M > 0$ such that $w(t) \leq M$ for all $t \in J$. This further implies that

$$|u(t)| \leq w(t) \leq M,$$

for all $t \in I$. Again if $t \in I_0$, then we have

$$|u(t)| \leq \lambda |Gu(t)| \leq \|Gu\| \leq N.$$

Hence we have $|u(t)| \leq M$ for all $t \in J$. Thus the conclusion (ii) of Theorem 2.1 does not hold. Therefore the operator equation $Tx = x$ and consequently the FDE (1.1) has a solution on J. This completes the proof. $\square$

Example 2.1 Let $I_0 = [-1, 0]$ and $I = [0, 1]$ be two closed and bounded intervals in $\mathbb{R}$. For a given function $x \in C(J, \mathbb{R})$, consider the functional differential equation (FDE)

$$\left.\begin{aligned} x'(t) &= p(t)\frac{|x(t)|}{1 + x_t^2} \text{ a.e. } \ t \in I \\ x(t) &= \cos t, \quad t \in I_0 \end{aligned}\right\} \tag{2.6}$$

where $p \in L^1(I, \mathbb{R}^+)$ and $x_t \in C(I_0, \mathbb{R})$ with $x_t(\theta) = x(t + \theta), \theta \in I_0$.

Define functional operator S and operator G on $BM(J, \mathbb{R})$ by $Sx = x_t \in C(I_0, \mathbb{R})$ for $t \in I$ and $Gx(t) = \cos t$ for all $t \in I_0$. Obviously S is continuous and G is completely continuous with $N = \max\{\|Gx\| : x \in BM(J, \mathbb{R})\} = 1$.

Define a function $f : I \times \mathbb{R} \times BM(J, \mathbb{R}) \to \mathbb{R}$ by

$$f(t, x, y) = p(t)\frac{|x|}{1 + y^2}.$$

It is very easy to prove that the function $f(t, x, y)$ is L^1-Carathéodory. Again we have

$$\begin{aligned} |f(t, x, y)| &= \left| p(t)\frac{|x|}{1 + y^2} \right| \\ &\leq p(t)[1 + |x|] \end{aligned}$$

and so hypothesis (A_5) is satisfied with $\phi(r) = 1 + r$. Now by the definition of ϕ we obtain

$$\|p\|_{L^1} = \int_2^\infty \frac{ds}{\phi(s)} = \int_2^\infty \frac{ds}{1 + s} = +\infty.$$

Now we apply Theorem 3.1 to yields that the FDE (1.1) has a solution on $J = I_0 \bigcup I$.

3 Uniqueness Theorem

Let X be a Banach space with norm $\|\cdot\|$. A mapping $T : X \to X$ is called **$\mathcal{D}$-Lipschitzian** if there exists a continuous nondecreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$\|Tx - Ty\| \leq \psi(\|x - y\|) \tag{3.7}$$

for all $x, y \in X$ with $\psi(0) = 0$. Sometimes we call the function ψ a **$\mathcal{D}$-function** of T. In the special case when $\psi(r) = \alpha r$, $\alpha > 0$, T is called a Lipchitzian with a Lipschitz constant α. In particular if $\alpha < 1, T$ is called a contraction with a contraction constant α. Further if $\psi(r) < r$ for $r > 0$, then T is called a nonlinear contraction on X. Finally if $\psi(r) = r$, then T is called a nonexpansive operator on X.

The following fixed point theorem for the nonlinear contraction is well-known and useful for proving the existence and the uniqueness theorems for the nonlinear differential and integral equations.

Theorem 3.1 (Browder [1]) *Let X be a Banach space and let $T : X \to X$ be a nonlinear contraction. Then T has a unique fixed point.*

We will need the following hypotheses:

(B_1) The function $f : I \times \mathbb{R}^n \times BM(J, \mathbb{R}^n) \to \mathbb{R}^n$ is continuous and satisfies

$$|f(t, x_1, y_1) - f(t, x_2, y_2)| \leq \max\left\{\frac{|x_1 - x_2|}{a + |x_1 - x_2|}, \frac{\|y_1 - y_2\|}{a + \|y_1 - y_2\|}\right\}, \quad \text{a.e.} \quad t \in I$$

for all $x_1, x_2 \in \mathbb{R}^n$ and $y_1, y_2 \in BM(J, \mathbb{R}^n)$.

(B_2) The operator $S : BM(J, \mathbb{R}^n) \to BM(J, \mathbb{R}^n)$ is nonexpansive.

(B_3) The operator $G : BM(J, \mathbb{R}^n) \to C(I_0, \mathbb{R}^n)$ satisfies

$$|Gx(t) - Gy(t)| \leq \frac{|x(t) - x(t)|}{a + |x(t) - x(t)|}, \text{ a.e. } t \in I_0$$

for all $x, y \in BM(J, \mathbb{R})$.

Theorem 3.2 *Assume that the hypotheses $(B_1) - (B_3)$ hold. Then the FDE (1.1) has a unique solution on J.*

Proof : Let $X = C(J, \mathbb{R}^n)$ and define an operator T on X by (2.2). We show that T is a nonlinear contraction on X. Let $x, y \in X$. By hypothesis (B_1),

$$\begin{aligned} |Tx(t) - Ty(t)| &\leq \int_0^t |f(s, x(s), Sx) - f(s, y(s), Sy)|\, ds \\ &\leq \int_0^t \left(\max\left\{\frac{|x(s) - y(s)|}{a + |x(s) - y(s)|}, \frac{\|Sx - Sy\|}{a + \|Sx - Sy\|}\right\}\right) ds \\ &\leq \frac{a\|x - y\|}{a + \|x - y\|}. \end{aligned}$$

for all $t \in I$. Again

$$\begin{aligned} |Tx(t) - Ty(t)| &\leq |Gx(t) - Gy(t)| \\ &\leq \frac{|x(t) - x(t)|}{a + |x(t) - x(t)|} \\ &\leq \frac{a\|x - y\|}{a + \|x - y\|} \end{aligned}$$

for all $t \in J$. Taking supremum over t we obtain

$$\|Ax - Ay\| \le \psi(\|x-y\|),$$

for all $x, y \in X$ where $\psi(r) = \dfrac{ar}{a+r} < r$, which shows that T is a nonlinear contraction on X. We now apply Theorem 3.1 to yield that the operator T has a unique fixed point. This further implies that the FDE (1.1) has a unique solution on J. This completes the proof. $\square$

Example 3.1 Let $I_0 = [-\pi/2, 0]$ and $I = [0,1]$ be two closed and bounded intervals in $\mathbb{R}$. For a given function $x \in C(J, \mathbb{R})$, consider the functional differential equation (FDE)

$$\left.\begin{aligned} x'(t) &= \frac{1}{2}\left[\frac{|x(t)|}{1+|x(t)|} + \frac{\|x_t\|_C}{1+\|x_t\|_C}\right] \ \text{ a.e. } \ t \in I \\ x(t) &= \cos t, \quad t \in I_0 \end{aligned}\right\} \tag{3.8}$$

where $x_t \in C(I_0, \mathbb{R})$ with $x_t(\theta) = x(t+\theta), \theta \in I_0$.

Define the functional operator S and the operator G on $BM(J, \mathbb{R})$ by $Sx = x_t \in C(I_0, \mathbb{R})$ for $t \in I$ and $Gx(t) = \cos t$ for all $t \in I_0$. Obviously S is continuous and G is bounded with $C = \max\{\|Gx\| : x \in BM(J, \mathbb{R})\} = 1$. Clearly S is nonexpansive on $BM(J, \mathbb{R})$.

Define a function $f : I \times \mathbb{R} \times BM(J, \mathbb{R}) \to \mathbb{R}$ by

$$f(t, x, y) = \frac{1}{2}\left[\frac{|x(t)|}{1+|x(t)|} + \frac{\|y\|}{1+\|y\|}\right].$$

It is very easy to prove that the function f is continuous on $I \times \mathbb{R} \times BM(J, \mathbb{R})$. Finally we show that the function f satisfies the inequality given in (B_1). Let $x_1, x_2 \in \mathbb{R}$ and $y_1, y_2 \in BM(J, \mathbb{R})$ be arbitrary. Then we have

$$\begin{aligned} |f(t,x_1,y_1) \ &- \ f(t,x_2,y_2)| \\ &\le \ \frac{1}{2}\left|\frac{|x_1|}{1+|x_1|} - \frac{|x_2|}{1+|x_2|}\right| + \frac{1}{2}\left|\frac{\|y_1\|}{1+\|y_1\|} - \frac{\|y_2\|}{1+\|y_2\|}\right| \\ &\le \ \frac{1}{2}\frac{|\,|x_1|-|x_2|\,|}{(1+|x_1|)(1+|x_2|)} + \frac{1}{2}\frac{|\|y_1\|-\|y_2\||}{(1+\|y_1\|)(1+\|y_2\|)} \\ &\le \ \frac{1}{2}\frac{|x_1-x_2|}{1+|x_1-x_2|} + \frac{1}{2}\frac{\|y_1-y_2\|}{1+\|y_1-y_2\|} \\ &\le \ \max\left\{\frac{|x_1-x_2|}{1+|x_1-x_2|}, \frac{\|y_1-y_2\||}{1+\|y_1-y_2\|}\right\} \end{aligned}$$

for all $t \in J$. Hence the hypothesis (B_1) of Theorem 3.1 is satisfied. Therefore an application of Theorem 3.1 yields that the FDE (3.8 has a unique solution on $[-\pi/2, 1]$.

4 Existence of Extremal Solutions

Let $x, y \in \mathbb{R}^n$ be such that $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. We define the co-ordinate wise order relation in $\mathbb{R}^n$, that is, $x \leq y \Leftrightarrow x_i \leq y_i \; \forall \, i = 1, \ldots, n$. We equip the Banach space $C(J, \mathbb{R}^n)$ with the order relation $\leq$ by $\xi_1 \leq \xi_2$ i and only if $\xi_1(t) \leq \xi_2(t) \; \forall \, t \in J$. By the order interval $[a, b]$ in a subset $AC(J, \mathbb{R}^n)$ of the Banach space $C(J, \mathbb{R}^n)$ we mean

$$[a, b] = \{x \in AC(J, \mathbb{R}^n) \mid a \leq x \leq b\}.$$

We use the following fixed point theorem of Heikkila and Lakshmikantham [5] in the sequel.

Theorem 4.1 *Let $[a, b]$ be an order interval in a subset Y of an ordered Banach space X and let $Q : [a, b] \to [a, b]$ be a nondecreasing mapping. If each sequence $\{Qx_n\} \subseteq Q([a, b])$ converges, whenever $\{x_n\}$ is a monotone sequence in $[a, b]$, then the sequence of Q-iteration of a converges to the least foxed point x_* of Q and the sequence of Q-iteration of b converges to the greatest fixed point x^* of Q. Moreover*

$$x_* = \min\{y \in [a, b] \mid y \geq Qy\} \quad \textit{and} \quad x^* = \max\{y \in [a, b] \mid y \leq Qy\}.$$

We need the following definitions in the sequel.

Definition 4.1 *A mapping $\beta : J \times \mathbb{R}^n \times C \to \mathbb{R}^n$ is said to satisfy* **Chandrabhan***'s conditions or simply is called L^1-***Chandrabhan** *if*

(i) $t \to \beta(t, x, y)$ is measurable for each $x \in \mathbb{R}^n$ and $y \in BM(J, \mathbb{R}^n)$,

(ii) The function $\beta(t, x, y)$ is nondecreasing in x and y almost everywhere for $t \in J$, and

(iii) for each real number $k > 0$, there exists a function $h_k \in L^1(J, \mathbb{R})$ such that

$$|\beta(t, x, y)| \leq h_k(t), \quad \textit{a.e.} \;\; t \in J$$

for all $x \in \mathbb{R}^n$ and $y \in BM(J, \mathbb{R}^n)$ with $|x| \leq k$, $\|y\| \leq k$.

Definition 4.2 *A function $u \in AC(J, \mathbb{R}^n)$ is called a lower solution of the FDE (1.1) on J if*

$$u'(t) \leq f(t, u(t), Su) \quad \textit{a.e} \quad t \in I$$

and

$$u(t) \leq Gu(t) \ \textit{for all}\ t \in I_0.$$

Again a function $v \in AC(J, \mathbb{R}^n)$ *is called an upper solution of the BVP (1.1) on* J *if*

$$v'(t) \geq f(t, v(t), Sv) \quad a.e \quad t \in I$$

and

$$v(t) \geq Gv(t) \quad \textit{for all}\ t \in I_0.$$

Definition 4.3 *A solution* x_M *of the FDE (1.1) is said to be maximal if for any other solution* x *to FDE(1.1) one has* $x(t) \leq x_M(t), \forall t \in J$. *Again a solution* x_m *of the FDE (1.1) is said to be minimal if* $x_m(t) \leq x(t), \forall t \in J$, *where* x *is any solution of the FDE (1.1) on* J.

We consider the following set of assumptions:

(C_1) The operator $S : BM(J, \mathbb{R}^n) \to BM(J, \mathbb{R}^n)$ is nondecreasing.

(C_2) The functions $f(t, x, y)$ is Chandrabhan.

(C_3) The operator $G : BM(J, \mathbb{R}^n) \to C(I_0, \mathbb{R}^n)$ is nondecreasing.

(C_4) The FDE (1.1) has a lower solution u and an upper solution v on J with $u \leq v$.

Remark 4.1 *Assume that hypotheses* $(C_1) - (C_4)$ *hold. Define a function* $h : J \to \mathbb{R}^+$ *by*

$$h(t) = |f(t, u(t), Su)| + |f(t, v(t), Sv)|, \forall\, t \in I.$$

Then h *is Lebesgue integrable and*

$$|f(t, x(t), Sx)| \leq h(t), \quad a.e.\ t \in I,\ \forall\, x(t) \in [u, v].$$

Theorem 4.2 *Suppose that the assumptions* (A_2), (C_1)-(C_4) *hold. Then FDE (1.1) has a minimal and a maximal solution on* J.

Proof. Now FDE (1.1) is equivalent to FIE (2.2) on J. Let $X = AC(J, \mathbb{R}^n)$. Define the operators T on $[a, b]$ by (2.3). Then FIE (1.1) is transformed into an operator equation $Tx(t) = x(t)$ in a Banach space X. Now the hypotheses (C_2) implies that T

is nondecreasing on $[u, v]$. To see this, let $x, y \in [u, v]$ be such that $x \leq y$. Then by (C_2),

$$\begin{aligned} Tx(t) &= Gx(0) + \int_0^t f(s, x(s), Sx)\, ds \\ &\leq Gy(0) + \int_0^t f(s, y(s), Sy)\, ds \\ &= Ty(t),\ \forall t \in I, \end{aligned}$$

and

$$Tx(t) = Gx(t) \leq Gy(t) = Ty(t) \quad \text{for all} \quad t \in I_0.$$

So T is nondecreasing operator on $[u, v]$. Finally we show that A defines a mapping $T : [u, v] \to [u, v]$. Let $x \in [u, v]$ be an arbitrary element. Then for any $t \in I$, we have

$$\begin{aligned} u(t) &\leq Gu(0) + \int_0^t f(s, u(s), Su)\, ds \\ &\leq Gx(0) + \int_0^t f(s, x(s), Sx)\, ds \\ &\leq Gv(0) + \int_0^t f(s, v(s), Sv)\, ds \\ &\leq v(t), \end{aligned}$$

for all $t \in I$. Again from (C_2) it follows that

$$u(t) \leq Tu(t) = Gu(t) \leq Gx(t) \leq Tx(t) \leq Gv(t) = Tu(t) \leq v(t)$$

for all $t \in I_0$. As a result $u(t) \leq Tx(t) \leq v(t), \forall\, t \in J$. Hence $Ax \in [u, v], \forall\, x \in [u, v]$.

Finally let $\{x_n\}$ be a monotone sequence in $[u, v]$. We shall show that the sequence $\{Tx_n\}$ converges in $T([u, v])$. Obviously the sequence $\{Tx_n\}$ is monotone in $T([u, v])$. Now it can be shown as in the proof of Theorem 2.2 that the sequence $\{Tx_n\}$ is uniformly bounded and equi-continuous in $T([u, v])$ with the function h playing the role of h_k. Hence an application of Arzela-Ascoli theorem yields that the sequence $\{Tx_n\}$ converges in $T([u, v])$. Thus all the conditions of Theorem 4.1 are satisfied and hence the operator T has a least and a greatest fixed point in $[u, v]$. This further implies that the the FDE (1.1) has maximal and minimal solutions on J. This completes the proof. $\square$

Remark 4.2 The main existence result proved in Liz and Pouso [8] seems to be not correct wherein the authors assume the function $f(t, x, y)$ to be nondecreasing only in y, whereas we need both in x and y. In consequence the example 4.1 quoted in Liz and Pouso [8] also goes wrong. Therefore our existence theorem proved in this section is an improvement of the result of Liz and Pouso [8] with correct proof.

Example 4.1 Given two closed and bounded intervals $I_0 = [-r, 0]$ and $I = [0, 1]$ in $\mathbb{R}$ for some $0 < r < 1$, Consider the functional differential equation

$$\begin{aligned} x'(t) &= \frac{\tanh\left(\left[\max\limits_{s\in[-r,t]} x(s)\right]\right)}{\sqrt{t}} + \operatorname{sgn}(x(t)) \text{ a.e. } t \in I \\ x_0 &= \sin t \text{ for } t \in I_0. \end{aligned} \tag{4.9}$$

where tanh is the hyperbolic tangent, square bracket means the integer part and

$$\operatorname{sgn}(x) = \begin{cases} \dfrac{x}{|x|} \text{ if } x \neq 0 \\ 0, \text{ if } x = 0 \end{cases}$$

Define the operators $S, G : BM(J, \mathbb{R}) \to BM(J, \mathbb{R})$ by

$$Sx(t) = \begin{cases} \left[\max\limits_{s\in[-r,t]} x(s)\right] \text{ if } t \in I \\ 0, \text{otherwise.} \end{cases}$$

and

$$Gx(t) = \begin{cases} \sin t \text{ if } t \in I_0 \\ 0, \text{ otherwise.} \end{cases}$$

Consider the mapping $f : I \times \mathbb{R} \times BM(J, \mathbb{R}) \to \mathbb{R}$ defined by

$$f(t, x, y) = \frac{\tanh y}{\sqrt{t}} + \operatorname{sgn}(x)$$

for $t \neq 0$. Obviously the operators S and G are nondecreasing on $BM(J, \mathbb{R})$. It is not difficult to verify that the function $f(t, x, y)$ is L^1-Chandrabhan. Again note that

$$-1 - \frac{1}{\sqrt{t}} < f(t, x, y) < 1 + \frac{1}{\sqrt{t}}$$

for all $t \in J, x \in \mathbb{R}$ and $y \in BM(J, \mathbb{R})$. Therefore if we define the functions α and β by

$$\alpha'(t) = -1 - \frac{1}{\sqrt{t}} \ , \ \alpha(0) = 0$$

and

$$\beta'(t) = 1 + \frac{1}{\sqrt{t}} \ , \ \beta(0) = 0$$

for all $t \in I$ with

$$\alpha(t) = \sin t = \beta(t) \ t \in I_0,$$

then α and β are respectively the lower and upper solutions of FDE (4.9) on J with $\alpha \leq \beta$. Thus all the conditions of Theorem 4.1 are satisfied and hence the FDE (4.9) has maximal and minimal solutions on J.

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