*Electronic Journal of Qualitative Theory of Differential Equations*
2013, No. **22**, 1-11; **http://www.math.u-szeged.hu/ejqtde/**

# COMPARISON THEOREM FOR OSCILLATION OF FOURTH-ORDER NONLINEAR RETARDED DYNAMIC EQUATIONS

CHENGHUI ZHANG, RAVI P. AGARWAL, AND TONGXING LI*

Abstract. This work is concerned with oscillation of a class of fourth-order nonlinear delay dynamic equations on a time scale. A new comparison theorem is established that improves related results reported in the literature.

## 1. Introduction

Fourth-order differential equations naturally appear in models concerning physical, biological, and chemical phenomena, for instance, problems of elasticity, deformation of structures, or soil settlement; see [5]. In this work, we study oscillation of a fourth-order nonlinear delay dynamic equation

$$x^{\Delta^4}(t) + p(t)x^{\gamma}(\tau(t)) = 0 \tag{1.1}$$

on an arbitrary time scale $\mathbb{T}$, where $\gamma > 0$ is the quotient of odd positive integers, $p$ is a real-valued positive rd-continuous function defined on $\mathbb{T}$, $\tau \in \mathrm{C}_{\mathrm{rd}}(\mathbb{T}, \mathbb{T})$, $\tau(t) \leq t$, and $\tau(t) \to \infty$ as $t \to \infty$.

Since we are interested in oscillatory behavior, we assume throughout this paper that the given time scale $\mathbb{T}$ is unbounded above, i.e., it is a time scale interval of the form $[t_0, \infty)_{\mathbb{T}} := [t_0, \infty) \cap \mathbb{T}$ with $t_0 \in \mathbb{T}$.

By a solution of (1.1) we mean a nontrivial real-valued function $x \in \mathrm{C}^4_{\mathrm{rd}}[T_x, \infty)_{\mathbb{T}}$, $T_x \in [t_0, \infty)_{\mathbb{T}}$ which satisfies (1.1) on $[T_x, \infty)_{\mathbb{T}}$. The solutions vanishing in some neighbourhood of infinity will be excluded from our consideration. A solution $x$ of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is nonoscillatory. Equation (1.1) is called oscillatory if all its solutions oscillate.

The theory of time scales, which has recently received a lot of attention, was introduced by Stefan Hilger [16] in his PhD thesis in order to

---

1991 *Mathematics Subject Classification.* 34K11, 34N05, 39A10.
*Key words and phrases.* Oscillation, comparison theorem, fourth-order nonlinear retarded dynamic equation, time scale.
*Corresponding author.

unify continuous and discrete analysis. The study of the oscillation of dynamic equations on time scales is a new area of applied mathematics, and work in this topic is rapidly growing. In recent years, there has been increasing interest in obtaining sufficient conditions for oscillatory and nonoscillatory of solutions of different classes of dynamic equations on time scales. We refer the reader to [1–4, 6–24] and the references cited therein. Agarwal et al. [1], Erbe et al. [8], Şahiner [21], Zhang and Zhu [23] considered a second-order delay dynamic equation

$$x^{\Delta^2}(t) + p(t)x(\tau(t)) = 0.$$

Akın-Bohner et al. [4] investigated a second-order Emden–Fowler dynamic equation

$$x^{\Delta^2}(t) + p(t)x^{\gamma}(\sigma(t)) = 0.$$

Han et al. [15] studied a second-order Emden–Fowler delay dynamic equation

$$x^{\Delta^2}(t) + p(t)x^{\gamma}(\tau(t)) = 0.$$

For the oscillation of higher-order dynamic equations on time scales, Erbe et al. [9] considered a third-order dynamic equation

$$x^{\Delta^3}(t) + p(t)x(t) = 0.$$

Grace et al. [11] studied a fourth-order dynamic equation

$$x^{\Delta^4}(t) + p(t)(x^{\sigma})^{\gamma}(t) = 0.$$

Monotone and oscillatory behavior of solutions to a fourth-order dynamic equation

$$(a(x^{\Delta^2})^{\alpha})^{\Delta^2}(t) + p(t)(x^{\sigma})^{\beta}(t) = 0$$

with the property that

$$\frac{x(t)}{\int_{t_0}^{t}\int_{t_0}^{s} a^{-1/\alpha}(\tau)\Delta\tau\Delta s} \to 0 \quad \text{as} \quad t \to \infty$$

were established by Grace et al. [12]. Grace et al. [14] studied a fourth-order dynamic equation

$$x^{\Delta^4}(t) + p(t)x^{\gamma}(t) = 0. \tag{1.2}$$

They obtained some oscillation criteria for (1.2), one of which we present below for the convenience of the reader.

**Theorem 1.1** (See [14, Theorem 2.2]). *Assume* $\int_{t_0}^{\infty} p(s)\sigma(s)\Delta s < \infty$ *and define* $Q_1(t) := \int_t^{\infty}\int_s^{\infty} p(\tau)\Delta\tau\Delta s$ *and* $Q_2(t) := [(\alpha - t_0)h_2(t,\alpha)/(t - t_0)]^{\gamma} p(t)$ *for* $\alpha \in \mathbb{T}^k$, $t \in \mathbb{T}$, *and* $t \geq \alpha > t_0$. *If both second-order dynamic equations*

$$y^{\Delta^2}(t) + Q_1(t)y^{\gamma}(t) = 0$$

*and*

$$z^{\Delta^2}(t) + Q_2(t)z^{\gamma}(t) = 0$$

*are oscillatory, then* (1.2) *is oscillatory.*

The purpose of this paper is to improve those results obtained in [14]. This paper is organized as follows: In the next section, we present the basic definitions and the theory of calculus on time scales. In Section 3, we establish some new oscillation results for (1.1).

In what follows, all functional inequalities are assumed to hold eventually, that is, for all sufficiently large $t$.

## 2. Some preliminaries

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. Since we are interested in oscillatory behavior, we suppose that the time scale under consideration is not bounded above and is a time scale interval of the form $[t_0, \infty)_{\mathbb{T}}$. On any time scale we define the forward and backward jump operators by

$$\sigma(t) := \inf\{s \in \mathbb{T} | s > t\} \quad \text{and} \quad \rho(t) := \sup\{s \in \mathbb{T} | s < t\},$$

where $\inf \emptyset := \sup \mathbb{T}$ and $\sup \emptyset := \inf \mathbb{T}$, $\emptyset$ denotes the empty set.

A point $t \in \mathbb{T}$ is said to be left-dense if $\rho(t) = t$ and $t > \inf \mathbb{T}$, right-dense if $\sigma(t) = t$ and $t < \sup \mathbb{T}$, left-scattered if $\rho(t) < t$, and right-scattered if $\sigma(t) > t$. The graininess $\mu$ of the time scale is defined by $\mu(t) := \sigma(t) - t$.

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous if it is continuous at each right-dense point and if there exists a finite left limit in all left-dense points. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$.

Fix $t \in \mathbb{T}$ and let $f : \mathbb{T} \to \mathbb{R}$. Define $f^{\Delta}(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood $U$ of $t$ (i.e., $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$|[f(\sigma(t)) - f(s)] - f^{\Delta}(t)[\sigma(t) - s]| \leq \varepsilon|\sigma(t) - s| \quad \text{for all} \quad s \in U.$$

In this case, $f^\Delta(t)$ is called the (delta) derivative of $f$ at $t$. $f$ is said to be differentiable if its derivative exists. The set of functions $f : \mathbb{T} \to \mathbb{R}$ that are differentiable and whose derivative is rd-continuous function is denoted by $\mathrm{C}^1_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$. If $f$ is differentiable at $t$, then $f$ is continuous at $t$. If $f$ is continuous at $t$ and $t$ is right-scattered, then $f$ is differentiable at $t$ with

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

If $t$ is right-dense, then $f$ is differentiable at $t$ iff the limit

$$f^\Delta(t) = \lim_{s\to t} \frac{f(t) - f(s)}{t - s}$$

exists as a finite number. In this case

$$f^\Delta(t) = \lim_{s\to t} \frac{f(t) - f(s)}{t - s}.$$

If $f$ is differentiable at $t$, then

$$f^\sigma(t) = f(\sigma(t)) = f(t) + \mu(t)f^\Delta(t).$$

Let $f$ be a real-valued function defined on an interval $[a, b]_\mathbb{T}$. We say that $f$ is increasing, decreasing, nondecreasing, and nonincreasing on $[a, b]_\mathbb{T}$ if $t_1,\ t_2 \in [a, b]_\mathbb{T}$ and $t_2 > t_1$ imply $f(t_2) > f(t_1)$, $f(t_2) < f(t_1)$, $f(t_2) \geq f(t_1)$, and $f(t_2) \leq f(t_1)$, respectively. Let $f$ be a differentiable function on $[a, b]_\mathbb{T}$. Then $f$ is increasing, decreasing, nondecreasing, and nonincreasing on $[a, b]_\mathbb{T}$ if $f^\Delta(t) > 0$, $f^\Delta(t) < 0$, $f^\Delta(t) \geq 0$, and $f^\Delta(t) \leq 0$ for all $t \in [a, b)_\mathbb{T}$, respectively.

We will make use of the following product and quotient rules for the derivative of the product $fg$ and the quotient $f/g$ (where $g(t)g(\sigma(t)) \neq 0$) of two differentiable functions $f$ and $g$

$$(fg)^\Delta(t) = f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g(\sigma(t)),$$

$$\left(\frac{f}{g}\right)^\Delta(t) = \frac{f^\Delta(t)g(t) - f(t)g^\Delta(t)}{g(t)g(\sigma(t))}.$$

For $a,\ b \in \mathbb{T}$ and a differentiable function $f$, the Cauchy integral of $f^\Delta$ is defined by

$$\int_a^b f^\Delta(t)\Delta t = f(b) - f(a).$$

The integration by parts formula reads

$$\int_a^b f^\Delta(t)g(t)\Delta t = f(b)g(b) - f(a)g(a) - \int_a^b f^\sigma(t)g^\Delta(t)\Delta t,$$

and infinite integrals are defined as

$$\int_a^\infty f(s)\Delta s = \lim_{t\to\infty}\int_a^t f(s)\Delta s.$$

## 3. Main results

In this section, we present some sufficient conditions which ensure that every solution of (1.1) is oscillatory. We begin with the following lemma.

*Lemma* 3.1. Assume there exists $T \in [t_0,\infty)_{\mathbb{T}}$ such that

$$y(t) > 0, \quad y^{\Delta}(t) > 0, \quad y^{\Delta^2}(t) < 0 \quad \text{for} \quad t \in [T,\infty)_{\mathbb{T}}.$$

Then, there exists a constant $T_k \in [T,\infty)_{\mathbb{T}}$ such that

$$\frac{y(\tau(t))}{y(\sigma(t))} \geq \frac{\tau(t)-T}{\sigma(t)-T} \geq k\frac{\tau(t)}{\sigma(t)} \quad \text{and} \quad \frac{y(\tau(t))}{y(t)} \geq \frac{\tau(t)-T}{t-T} \geq k\frac{\tau(t)}{t}$$

for each $k \in (0,1)$ and for $t \in [T_k,\infty)_{\mathbb{T}}$.

*Proof.* The proof is similar to that of [21, Lemma 1], and hence is omitted. □

*Lemma* 3.2 (See [9, Lemma 4]). Assume $y$ satisfies

$$y(t) > 0, \quad y^{\Delta}(t) > 0, \quad y^{\Delta^2}(t) > 0, \quad y^{\Delta^3}(t) \leq 0$$

for $t \in [t_1,\infty)_{\mathbb{T}}$. Then

$$\liminf_{t\to\infty} \frac{ty(t)}{h_2(t,t_0)y^{\Delta}(t)} \geq 1,$$

where $h_2(t,s)$ is the Taylor monomial of degree 2 (see Bohner and Peterson [6, Section 1.6]).

*Lemma* 3.3. Assume $x$ is an eventually positive solution of (1.1). Then there are only the following two possible cases for $t \in [t_1,\infty)_{\mathbb{T}} \subseteq [t_0,\infty)_{\mathbb{T}}$ sufficiently large:

(1) $x > 0$, $x^{\Delta} > 0$, $x^{\Delta^2} > 0$, $x^{\Delta^3} > 0$, $x^{\Delta^4} < 0$;
(2) $x > 0$, $x^{\Delta} > 0$, $x^{\Delta^2} < 0$, $x^{\Delta^3} > 0$, $x^{\Delta^4} < 0$.

*Proof.* The proof is simple, and so is omitted. □

**Theorem 3.4.** *Assume there exists a positive function $m \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$ such that*

$$\frac{tm(t)}{lh_2(t,t_0)} - m^\Delta(t) \le 0 \tag{3.1}$$

*for some $l \in (0,1)$. Suppose further that there exists a positive function $v \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$ such that*

$$\frac{m(t)v(t)}{\int_{t_l}^t m(s)\Delta s} - v^\Delta(t) \le 0 \tag{3.2}$$

*for all $t \in [t_*,\infty)_\mathbb{T} \subset (t_l,\infty)_\mathbb{T}$, sufficiently large. If both second-order dynamic equations*

$$z^{\Delta^2}(t) + l^\gamma p(t)\left[\frac{v(\tau(t))}{v(t)}\frac{h_2(t,t_0)}{tm(t)}\int_{t_l}^t m(s)\Delta s\right]^\gamma z^\gamma(t) = 0 \tag{3.3}$$

*and*

$$u^{\Delta^2}(t) + k^\gamma\left[\int_t^\infty \int_s^\infty p(\varsigma)\left(\frac{\tau(\varsigma)}{\varsigma}\right)^\gamma \Delta\varsigma\Delta s\right] u^\gamma(t) = 0 \tag{3.4}$$

*are oscillatory for all sufficiently large $t_l$ and for some $k \in (0,1)$, then (1.1) is oscillatory.*

*Proof.* Suppose that (1.1) has a nonoscillatory solution $x$ on $[t_0,\infty)_\mathbb{T}$. We may assume without loss of generality that there exists a $t_1 \in [t_0,\infty)_\mathbb{T}$ such that $x(t) > 0$ and $x(\tau(t)) > 0$ for $t \in [t_1,\infty)_\mathbb{T}$. It follows from Lemma 3.3 that $x$ satisfies either case (1) or case (2).

Assume case (1). Set $y := x^\Delta$. It follows from Lemma 3.2 that

$$x^\Delta(t) \ge l\frac{h_2(t,t_0)}{t}x^{\Delta^2}(t) \tag{3.5}$$

for $t \in [t_l,\infty)_\mathbb{T}$ and for given $l \in (0,1)$. Since

$$\begin{aligned}\left(\frac{x^\Delta}{m}\right)^\Delta(t) &= \frac{x^{\Delta^2}(t)m(t) - x^\Delta(t)m^\Delta(t)}{m(t)m^\sigma(t)}\\ &\le \frac{x^\Delta(t)}{m(t)m^\sigma(t)}\left[\frac{tm(t)}{lh_2(t,t_0)} - m^\Delta(t)\right] \le 0,\end{aligned}$$

we see that $x^\Delta/m$ is nonincreasing. Then, we obtain

$$x(t) = x(t_l) + \int_{t_l}^t \frac{x^\Delta(s)}{m(s)}m(s)\Delta s \ge \frac{x^\Delta(t)}{m(t)}\int_{t_l}^t m(s)\Delta s. \tag{3.6}$$

Using (3.5) and (3.6), we have

$$x(t) \geq l\left[\frac{h_2(t,t_0)}{tm(t)}\int_{t_l}^{t} m(s)\Delta s\right] x^{\Delta^2}(t). \tag{3.7}$$

On the other hand, we find by (3.6) that

$$\begin{aligned}\left(\frac{x}{v}\right)^{\Delta}(t) &= \frac{x^{\Delta}(t)v(t)-x(t)v^{\Delta}(t)}{v(t)v^{\sigma}(t)}\\ &\leq \frac{x(t)}{v(t)v^{\sigma}(t)}\left[\frac{m(t)v(t)}{\int_{t_l}^{t} m(s)\Delta s}-v^{\Delta}(t)\right] \leq 0.\end{aligned}$$

Hence $x/v$ is nonincreasing. Thus, we get

$$\frac{x(\tau(t))}{v(\tau(t))} \geq \frac{x(t)}{v(t)}, \quad \text{since} \quad \tau(t) \leq t. \tag{3.8}$$

Using (3.7) and (3.8), we obtain

$$\begin{aligned}x(\tau(t)) &\geq \frac{v(\tau(t))}{v(t)}x(t)\\ &\geq l\left[\frac{v(\tau(t))}{v(t)}\frac{h_2(t,t_0)}{tm(t)}\int_{t_l}^{t} m(s)\Delta s\right] x^{\Delta^2}(t).\end{aligned}$$

Substituting the latter inequality into (1.1), we have

$$x^{\Delta^4}(t)+l^{\gamma}p(t)\left[\frac{v(\tau(t))}{v(t)}\frac{h_2(t,t_0)}{tm(t)}\int_{t_l}^{t} m(s)\Delta s\right]^{\gamma}(x^{\Delta^2})^{\gamma}(t) \leq 0.$$

Let $z := x^{\Delta^2}$. Then we see that $z$ is a positive solution of

$$z^{\Delta^2}(t)+l^{\gamma}p(t)\left[\frac{v(\tau(t))}{v(t)}\frac{h_2(t,t_0)}{tm(t)}\int_{t_l}^{t} m(s)\Delta s\right]^{\gamma} z^{\gamma}(t) \leq 0.$$

It follows from [14, Lemma 2.1] that equation (3.3) has positive solutions, which is a contradiction.

Assume now case (2). Using (1.1), we calculate

$$x^{\Delta^3}(z)-x^{\Delta^3}(t)+\int_{t}^{z} p(s)x^{\gamma}(\tau(s))\Delta s = 0.$$

Set $y := x$. By Lemma 3.1, we find

$$\frac{x(\tau(t))}{x(t)} \geq k\frac{\tau(t)}{t}$$

for given $k \in (0,1)$. Thus, we get by $x^{\Delta} > 0$ that

$$x^{\Delta^3}(z) - x^{\Delta^3}(t) + k^{\gamma} x^{\gamma}(t) \int_t^z p(s) \left(\frac{\tau(s)}{s}\right)^{\gamma} \Delta s \le 0.$$

Letting $z \to \infty$ in the above inequality, we obtain

$$-x^{\Delta^3}(t) + k^{\gamma} x^{\gamma}(t) \int_t^{\infty} p(s) \left(\frac{\tau(s)}{s}\right)^{\gamma} \Delta s \le 0$$

due to $\lim_{z\to\infty} x^{\Delta^3}(z) \ge l_1 \ge 0$. Therefore,

$$-x^{\Delta^2}(z) + x^{\Delta^2}(t) + k^{\gamma} x^{\gamma}(t) \int_t^z \int_s^{\infty} p(\varsigma) \left(\frac{\tau(\varsigma)}{\varsigma}\right)^{\gamma} \Delta\varsigma \Delta s \le 0.$$

Letting $z \to \infty$ in the last inequality, from $\lim_{z\to\infty}(-x^{\Delta^2}(z)) \ge l_2 \ge 0$, we have

$$x^{\Delta^2}(t) + k^{\gamma} x^{\gamma}(t) \int_t^{\infty} \int_s^{\infty} p(\varsigma) \left(\frac{\tau(\varsigma)}{\varsigma}\right)^{\gamma} \Delta\varsigma \Delta s \le 0.$$

It follows from [14, Lemma 2.1] that equation (3.4) has positive solutions, which is a contradiction. The proof is complete. □

*Remark* 3.5. From Theorem 3.4, one can obtain some corollaries for the oscillation of (1.1). For example, if we use some related results in [15], then we get the following results.

*Corollary* 3.6. Let $\gamma > 1$ and assume there exists a positive function $m \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$ such that (3.1) holds for some $l \in (0,1)$. Suppose also that there exists a positive function $v \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$ such that (3.2) holds for all $t \in [t_*, \infty)_{\mathbb{T}} \subset (t_l, \infty)_{\mathbb{T}}$, sufficiently large. If

$$\int^{\infty} \frac{p(t)}{\sigma^{\gamma-1}(t)} \left[\frac{v(\tau(t))}{v(t)} \frac{h_2(t,t_0)}{m(t)} \int_{t_l}^t m(s)\Delta s\right]^{\gamma} \Delta t = \infty \tag{3.9}$$

and

$$\int^{\infty} \left[\int_t^{\infty} \int_s^{\infty} p(\varsigma) \left(\frac{\tau(\varsigma)}{\varsigma}\right)^{\gamma} \Delta\varsigma \Delta s\right] \frac{t^{\gamma}}{\sigma^{\gamma-1}(t)} \Delta t = \infty, \tag{3.10}$$

then (1.1) is oscillatory.

*Corollary* 3.7. Let $\gamma < 1$ and assume there exists a positive function $m \in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$ such that (3.1) holds for some $l \in (0,1)$. Suppose further that there exists a positive function $v \in C^1_{rd}(\mathbb{T}, \mathbb{R})$ such that (3.2) holds for all $t \in [t_*, \infty)_{\mathbb{T}} \subset (t_l, \infty)_{\mathbb{T}}$, sufficiently large. If

$$\int^{\infty} p(t) \left[ \frac{v(\tau(t))}{v(t)} \frac{h_2(t, t_0)}{m(t)} \int_{t_l}^{t} m(s) \Delta s \right]^{\gamma} \Delta t = \infty \tag{3.11}$$

and

$$\int^{\infty} \left[ \int_t^{\infty} \int_s^{\infty} p(\varsigma) \left( \frac{\tau(\varsigma)}{\varsigma} \right)^{\gamma} \Delta\varsigma \Delta s \right] t^{\gamma} \Delta t = \infty, \tag{3.12}$$

then (1.1) is oscillatory.

Now, we give an example to illustrate the main results.

It is well known that the second-order sublinear Emden–Fowler differential equation

$$x''(t) + q(t) x^{\gamma}(t) = 0, \quad \gamma < 1$$

is oscillatory if

$$\int^{\infty} q(t) t^{\gamma} \mathrm{d}t = \infty.$$

Using this result, we consider the following equation

$$x^{(4)}(t) + \frac{k_0}{t^{1+3\gamma}} x^{\gamma}(t) = 0, \quad \frac{2}{3} < \gamma < 1, \tag{3.13}$$

where $k_0 > 0$ is a constant. Let $m(t) = t^3$ and $v(t) = t^5$. Applying Corollary 3.7, we see that equation (3.13) is oscillatory. At the same time, we find that [14, Theorem 2.2] cannot be applied in equation (3.13). Hence our result improves those of [14].

## 4. Acknowledgements

This research is supported by NNSF of P. R. China (Grant Nos. 61034007, 51277116, and 50977054).

## References

[1] Ravi P. Agarwal, Martin Bohner and Samir H. Saker, Oscillation of second order delay dynamic equations. *Can. Appl. Math. Q.*, 13 (2005) 1–17.

[2] Ravi P. Agarwal, Martin Bohner, Shuhong Tang, Tongxing Li and Chenghui Zhang, Oscillation and asymptotic behavior of third-order nonlinear retarded dynamic equations. *Appl. Math. Comput.*, 219 (2012) 3600–3609.

[3] Ravi P. Agarwal, Said R. Grace and Donal O'Regan, *Oscillation Theory for Second Order Dynamic Equations, volume 5 of Series in Mathematical Analysis and Applications.* Taylor and Francis Ltd., London, 2003.

[4] Elvan Akın-Bohner, Martin Bohner and Samir H. Saker, Oscillation criteria for a certain class of second order Emden–Fowler dynamic equations. *Electron. Trans. Numer. Anal.*, 27 (2007) 1–12.

[5] M. Bartušek, M. Cecchi, Z. Došlá and M. Marini, Fourth-order differential equation with deviating argument. *Abstr. Appl. Anal.*, 2012 (2012) 1–17.

[6] Martin Bohner and Allan Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications*. Birkhäuser, Boston, 2001.

[7] Martin Bohner and Allan Peterson, *Advances in Dynamic Equations on Time Scales*. Birkhäuser, Boston, 2003.

[8] Lynn Erbe, Allan Peterson and Samir H. Saker, Oscillation criteria for second-order nonlinear delay dynamic equations. *J. Math. Anal. Appl.*, 333 (2007) 505–522.

[9] Lynn Erbe, Allan Peterson and Samir H. Saker, Hille and Nehari type criteria for third-order dynamic equations. *J. Math. Anal. Appl.*, 329 (2007) 112–131.

[10] Said R. Grace, Ravi P. Agarwal, Martin Bohner and Donal O'Regan, Oscillation of second-order strongly superlinear and strongly sublinear dynamic equations. *Commun. Nonlinear Sci. Numer. Simulat.*, 14 (2009) 3463–3471.

[11] Said R. Grace, Ravi P. Agarwal and Sandra Pinelas, On the oscillation of fourth order superlinear dynamic equations on time scales. *Dynam. Syst. Appl.*, 20 (2011) 45–54.

[12] Said R. Grace, Ravi P. Agarwal and Wichuta Sae-jie, Monotone and oscillatory behavior of certain fourth order nonlinear dynamic equations. *Dynam. Syst. Appl.*, 19 (2010) 25–32.

[13] Said R. Grace, Martin Bohner and Ailian Liu, On Kneser solutions of third-order delay dynamic equations. *Carpathian J. Math.*, 26 (2010) 184–192.

[14] Said R. Grace, Martin Bohner and Shurong Sun, Oscillation of fourth-order dynamic equations. *Hacet. J. Math. Stat.*, 39 (2010) 545–553.

[15] Zhenlai Han, Shurong Sun and Bao Shi, Oscillation criteria for a class of second order Emden–Fowler delay dynamic equations on time scales. *J. Math. Anal. Appl.*, 334 (2007) 847–858.

[16] Stefan Hilger, Analysis on measure chains–a unified approach to continuous and discrete calculus. *Results Math.*, 18 (1990) 18–56.

[17] Başak Karpuz, Unbounded oscillation of higher-order nonlinear delay dynamic equations of neutral type with oscillating coefficients. *Electron. J. Qual. Theory Differ. Equ.*, 34 (2009) 1–14.

[18] Başak Karpuz, Asymptotic behaviour of bounded solutions of a class of higher-order neutral dynamic equations. *Appl. Math. Comput.*, 215 (2009) 2174–2183.

[19] Tongxing Li, Zhenlai Han, Chenghui Zhang, and Ying Sun, Oscillation criteria for third-order nonlinear delay dynamic equations on time scales. *Bull. Math. Anal. Appl.*, 3 (2011) 52–60.

[20] Tongxing Li, Ethiraju Thandapani and Shuhong Tang, Oscillation theorems for fourth-order delay dynamic equations on time scales. *Bull. Math. Anal. Appl.*, 3 (2011) 190–199.

[21] Yeter Şahiner, Oscillation of second-order delay differential equations on time scales. *Nonlinear Anal. TMA*, 63 (2005) 1073–1080.

[22] Samir H. Saker, *Oscillation Theory of Dynamic Equations on Time Scales, Second and Third Orders.* Lambert Academic Publisher, 2010.

[23] Binggen Zhang and Shanliang Zhu, Oscillation of second order nonlinear delay dynamic equations on time scales. *Comput. Math. Appl.*, 49 (2005) 599–609.

[24] Chenghui Zhang, Tongxing Li, Ravi P. Agarwal and Martin Bohner, Oscillation results for fourth-order nonlinear dynamic equations. *Appl. Math. Lett.*, 25 (2012) 2058–2065.

(Received January 23, 2013)

Shandong University, School of Control Science and Engineering, Jinan, Shandong 250061, P. R. China
*E-mail address*: `zchui@sdu.edu.cn`

Texas A&M University–Kingsville, Department of Mathematics, 700 University Blvd., Kingsville, TX 78363-8202, USA
*E-mail address*: `agarwal@tamuk.edu`

Shandong University, School of Control Science and Engineering, Jinan, Shandong 250061, P. R. China
*E-mail address*: `litongx2007@163.com`