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EXISTENCE OF SOLUTIONS TO NONLOCAL AND SINGULAR VARIATIONAL ELLIPTIC INEQUALITY VIA GALERKIN METHOD

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Abstract. In this article, we study the existence of solutions for nonlocal variational elliptic inequality

$$-M(\|u\|^2)\Delta u \geq f(x,u)$$

Making use of the penalized method and Galerkin approximations, we establish existence theorems for both cases when M is continuous and when M is discontinuous.

1. Introduction

Let $\Omega \subset \mathbb{R}^{\mathbb{N}}$ be a bounded smooth domain with boundary $\partial\Omega$. Let us consider the Sobolev spaces $L^2(\Omega)$, $H_0^1(\Omega)$ whose inner products and norms will be denoted by $(,), |,|, ((,)), \|,\|$, respectively. We have $H_0^1(\Omega) \subset L^2(\Omega)$ dense and continuous. Then the duals $(L^2(\Omega))' \subset (H_0^1(\Omega))'$ also dense and continuous. If we identify $L^2(\Omega) = (L^2(\Omega))'$ we have

$$H_0^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega),$$

where $H^{-1}(\Omega)$ is the dual of $H_0^1(\Omega)$. Throughout this paper, let us represent by K a closed convex set of $L^2(\Omega)$, with $0 \in K$, which has the following property:

(H1) There exists a contraction $\rho : \mathbb{R} \longrightarrow \mathbb{R}$ (e.g., $\mid \rho(\lambda_1) - \rho(\lambda_2) \mid \leq \mid \lambda_1 - \lambda_2 \mid$) with $\rho(0) = 0$ such that $(P_K v)(x) = \rho(v(x)), \forall\ v \in L^2(\Omega)$, where P_K is the projection operator from $L^2(\Omega)$ into K.

For example, let us consider $K = \{v \in L^2(\Omega);\ a \leq v(x) \leq b\ \ a.e.\ in\ \Omega\}$ where $-\infty \leq a \leq 0 \leq b \leq \infty$. We define $\rho(\lambda)$ as follows

$$\rho(\lambda) = \begin{cases} \lambda & for \quad a < \lambda < b \\ b & for \quad \lambda \geq b \qquad (if \quad b \quad is \quad finite) \\ a & for \quad \lambda \leq a \qquad (if \quad a \quad is \quad finite) \end{cases}$$

In this paper we study some questions related to the existence of solutions for the nonlocal elliptic variational inequality:

$$u \in K : (-M(\|u\|^2)\Delta u, v) \geq (f, v)\ , for\ all\ v \in K \cap (H_0^1(\Omega) \cap H^2(\Omega)) \tag{1.1}$$

where $f \in H_0^1(\Omega)$ and $M : \mathbb{R} \to \mathbb{R}$ is a function whose behavior will be stated later. The main purpose of this work is establishing properties on M under which problem (1.1), and its nonlinear counterpart, possesses a solution. This inequality has called our attention because the operator

$$Lu := M(\|u\|^2)\Delta u$$

contains the nonlocal term $M(\|u\|^2)$ which poses some interesting mathematical questions. Also the operator L appears in the Kirchhoff equation, which arises in nonlinear vibrations, namely

$$\begin{gathered} u_{tt} - M\Big(\int_\Omega |\nabla u|^2 dx\Big)\Delta u = f(x,u) \quad \text{in } \Omega \times (0,T), \\ u = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x). \end{gathered}$$

The mathematical aspects of this model were largely investigated. See, for example, Arosio-Spagnolo [2], Hazoya-Yamada [11], Lions [16], Pohozhaev [18]. A survey of the results about the mathematical aspects of Kirchhoff model can be found in Medeiros-Limaco-Menezes [14]. Unilateral problems for nonlinear operators of the Kirchhoff type were initially studied by Kludnev [12], Larkin [13], Medeiros-Milla Miranda [17] among others.
Recently, Alves-Corrêa [1] focused their attention on problem related with (1.1) in case $M(t) \geq m_0 > 0$, for all $t \geq 0$, where m_0 is a constant. Among other things they studied the above M-linear problem (1.1) where M, besides the strict positivity mentioned before, satisfies the following assumption:

The function $H : \mathbb{R} \to \mathbb{R}$ with

$$H(t) = M(t^2)t$$

is monotone and $H(\mathbb{R}) = \mathbb{R}$.

In a previous paper, Menezes-Corrêa [8] proved a similar results by allowing M to attain negative values and $M(t) \geq m_0 > 0$ only for t large enough. This is possible thanks to a device explored by Alves-de Figueiredo [3], who use Galerkin method to attack a non-variational elliptic system. The technique can be conveniently adapted to problems such as (1.1). In this way we improve substantially the existence result on the above problem mainly because our assumptions on M are weakened. Indeed, we may also consider the case in which M possesses a singularity. The methodology used in our proof consists in transforming, by penalty, the inequality (1.1) into a family of equations depending of a parameter $\epsilon > 0$ and apply Galerkin's method. In the application of Galerkin's methods we use the sharp angle lemma(see Lions [15, p.53]). This paper is organized as follows: Section 2 is devoted to the study of (1.1) in the continuous case. In Section 3 the inequality (1.1) is studied in case M possesses a discontinuity. In Section 4 we analyze another type of variational inequality.

2. The M-linear Problem: Continuous Case

In this section we are concerned with the M-linear problem (1.1) where $f \in H_0^1(\Omega)$ and $M : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying

(M1) There exists a positive number m_0 such that $M(t) \geq m_0$, for all $t \geq 0$.

We have

Theorem 2.1. *Assume that and (M1) and (H1) hold. Then for any choice of $0 \neq f \in H_0^1(\Omega)$ the problem* (1.1) *admits at least one solution.*

The proof of Theorem 2.1 is given by the penalty method. In fact, let us represent by β the operator from $L^2(\Omega)$ into $L^2(\Omega)$ defined by $\beta = I - P_K$, e.g., $(\beta v)(x) = v(x) - \rho(v(x))$ for all $v \in L^2(\Omega)$. The operator β is monotone and Lipschitzian. The next result can be found in Haraux [10] p. 58.

Lemma 2.2. *Let $g : \mathbb{R} \to \mathbb{R}$ be a Lipschitzian and increasing function with $g(0) = 0$. Then $(g(u), -\Delta u) \geq 0$ for all $u \in H_0^1(\Omega) \cap H^2(\Omega)$.*

We have that $g(s) = s - \rho(s)$ is under the condition of the Lemma 2.2, then

$$(\beta(v), -\Delta v) \geq 0 \; for\ all \;\; v \in H_0^1(\Omega) \cap H^2(\Omega). \tag{2.1}$$

The penalized problem associated to the problem (1.1), consists in given $\epsilon > 0$, find u_ϵ solution in Ω of the problem

$$\begin{gathered} -M(\|u_\epsilon\|^2)\Delta u_\epsilon + \frac{1}{\epsilon}\beta(u_\epsilon) = f \quad \text{in } \Omega, \\ u_\epsilon = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{2.2}$$

where f is given in $H_0^1(\Omega)$.
The existence of one solution of the penalized problem (2.2) is give by the

Theorem 2.3. *Assume that* $(M1)$ *hold. Then, for any $0 \neq f \in H_0^1(\Omega)$ there exists at least one $u_\epsilon \in H_0^1(\Omega) \cap H^2(\Omega)$, solution of the problem* (2.2) .

Proof. We employ the Galerkin Method by using the sharp angle lemma. Let us consider the Hilbertian basis of spectral objects $(e_j)_{j\in\mathbb{N}}$ and $(\lambda_j)_{j\in\mathbb{N}}$ for the operator $-\Delta$ in $H_0^1(\Omega)$, cf. Brezis [5]. We know that the eigenvectors $(e_j)_{j\in\mathbb{N}}$ are orthonormal complete in $L^2(\Omega)$ and complete in $H_0^1(\Omega) \cap H^2(\Omega)$. For each $m \in \mathbb{N}$ consider

$$\mathbb{V}_m = \operatorname{span}\{e_1, \ldots, e_m\},$$

the subspace of $H_0^1(\Omega) \cap H^2(\Omega)$ generated by m eigenvectors $e_1, e_2, ..., e_m$. If $u_{\epsilon_m} \in V_m$, then

$$u_{\epsilon_m} = \sum_{j=1}^{m} \xi_j e_j \;\; with\ real \;\; \xi_j, \;\; 1 \leq j \leq m.$$

We will consider u_m instead of u_{ϵ_m} . The approximate problem consists in finding a solution $u_m \in V_m$ of the system of algebraic equations

$$M(\|u_m\|^2)((u_m, e_i)) + \frac{1}{\epsilon}(\beta(u_\epsilon), e_i) = (f, e_i), \quad i = 1, \ldots, m. \tag{2.3}$$

We need to prove that (2.3) has a solution $u_m \in V_m$. To this end, we will consider the vector $\eta = (\eta_i)_{1\leq i\leq m}$ of $\mathbb{R}^m$ defined by

$$\eta_i = M(\|u_m\|^2)((u_m, e_i)) + \frac{1}{\epsilon}(\beta(u_m), e_i) - (f, e_i), i = 1, \ldots, m.$$

Let $\xi = (\xi_i)_{1\le i\le m}$ be the components of the vector u_m of V_m. The mapping $P : \mathbb{R}^m \to \mathbb{R}^m$ defined by $P\xi = \eta$ is continuous. If we prove that $\langle P\xi, \xi\rangle \ge 0$ for $\|\xi\|_{\mathbb{R}^n} = r$, with an appropriate r, it will follow by the sharp angle lemma that there exists a ξ in the ball $B_r(0) \subset \mathbb{R}^m$ such that $P\xi = 0$. This implies the existence of a solution to (2.3). In fact, we have

$$\langle P\xi, \xi\rangle = \sum_1^m \xi_i\eta_i = M(\|u_m\|)((u_m, u_m)) + \frac{1}{\epsilon}(\beta(u_m), u_m) - (f, u_m).$$

Using (M1), Hölder and Poincaré inequalities and observing that $(\beta(u_m), u_m) \ge 0$, we get

$$\langle P\xi, \xi\rangle \ge m_0\|u_m\|^2 - C\|f\|\|u_m\|$$

We can consider r large enough that $(P\xi, \xi) \ge 0$ for $\|\xi\|_{\mathbb{R}^m} = r$. Then $P\xi = 0$ for some $\xi \in B_r(0)$, which implies that system (2.3) has a solution $u_m \in V_m$ corresponding to this ξ. Thus, there is $u_m \in \mathbb{V}_m$,

$$\|u_m\| \le r, \tag{2.4}$$

r does not depend on m and ϵ, such that

$$M(\|u_m\|^2)((u_m, e_i)) + \frac{1}{\epsilon}(\beta(u_m), e_i) = (f, e_i), \quad i = 1, \ldots, m,$$

which implies that

$$M(\|u_m\|^2)((u_m, \omega)) + \frac{1}{\epsilon}(\beta(u_m), \omega) = (f, \omega), \quad \text{for all } \omega \in \mathbb{V}_m. \tag{2.5}$$

Because $(\|u_m\|^2)$ is a bounded real sequence and M is continuous one has

$$\|u_m\|^2 \to \tilde{t}_0, \tag{2.6}$$

for some $\tilde{t}_0 \ge 0$, and

$$u_m \rightharpoonup u \text{ in } H_0^1(\Omega), \quad u_m \to u \text{ in } L^2(\Omega), \quad M(\|u_m\|^2) \to M(\tilde{t}_0), \tag{2.7}$$

perhaps for a subsequence.

Take $k \le m$, $\mathbb{V}_k \subset \mathbb{V}_m$. Fix k and let $m \to \infty$ in equation (2.5) to obtain

$$M(\tilde{t}_0)((u, \omega)) + \frac{1}{\epsilon}(\beta(u), \omega) = (f, \omega), \quad \text{for all } \omega \in (H_0^1 \cap H^2)(\Omega). \tag{2.8}$$

Since $-\Delta e_j = \lambda_j e_j$ we take $\omega = -\Delta u_m$ in (2.8). We obtain

$$M(\widetilde{t_0})(\Delta u_m, \Delta u_m) + \frac{1}{\epsilon}(\beta(u_m), -\Delta u_m) = (\nabla f, \nabla u_m) \tag{2.9}$$

By (2.1), we obtain

$$|\Delta u_m|^2 \le C(|\nabla f|^2 + |\nabla u_m|^2). \tag{2.10}$$

Since $f \in H_0^1(\Omega)$ and by (2.6), we have that

$$|\Delta u_m|^2 \le C \tag{2.11}$$

where C does not depend on m and ϵ. By (2.7) and (2.11) and by compact imbedding of $(H_0^1(\Omega) \cap H^2(\Omega)) \subset H_0^1(\Omega)$ we deduce that

$$u_m \to u \;\; in \;\; H_0^1(\Omega) \tag{2.12}$$

The continuity of M and convergence (2.7) and (2.12) permit to pass to the limit in (2.5). We obtain

$$M(\|u\|^2)((u,v)) + \frac{1}{\epsilon}(\beta(u),v) = (f,v), \quad \text{for all } v \in H_0^1(\Omega) \cap H^2(\Omega). \tag{2.13}$$

Thus u_ϵ is a weak solution of problem (2.2) and the proof of Theorem 2.3 is complete. □

Proof. of Theorem 2.1
Let $(\epsilon_n)_{n\in\mathbb{N}}$ be a sequence of real numbers such that

$$0 < \epsilon_n < 1 \; for \; all \; n \in \mathbb{N} \;\; and \;\; \lim_{n\to\infty} \epsilon_n = 0.$$

For each $n \in \mathbb{N}$, we get function u_{ϵ_n} which satisfies Theorem 2.3. Since the estimates were uniform on ϵ *and* n, we can see that there exists a subsequence of u_{ϵ_n}, again called u_{ϵ_n}, and a function $u \in H_0^1(\Omega) \cap H^2(\Omega)$ such that

$$u_{\epsilon_m} \to u \text{ in } H_0^1(\Omega). \tag{2.14}$$

Consider v in $H_0^1(\Omega) \cap H^2(\Omega)$ with v belongs to K. By (2.5), we have

$$(-M(\|u_{\epsilon_n}\|^2)\Delta u_{\epsilon_n} - f, v) = (-\frac{1}{\epsilon_n}\beta(u_{\epsilon_n}), v) \tag{2.15}$$

and

$$(-M(\|u_{\epsilon_n}\|^2)\Delta u_{\epsilon_n} - f, -u_{\epsilon_n}) = (-\frac{1}{\epsilon_n}\beta(u_{\epsilon_n}), -u_{\epsilon_n}). \tag{2.16}$$

Follows of (2.15) and (2.16) that

$$\begin{aligned}(-M(\|u_{\epsilon_n}\|^2)\Delta u_{\epsilon_n} - f, v - u_{\epsilon_n}) &= \frac{1}{\epsilon}(-\beta(u_{\epsilon_n}), v - u_{\epsilon_n}) \\ &= \frac{1}{\epsilon}(\beta(v) - \beta(u_{\epsilon_n}), v - u_{\epsilon_n}) \geq 0,\end{aligned} \tag{2.17}$$

because $v \in K$ and monotonicity of β. Hence

$$(-M(\|u_{\epsilon_n}\|^2)\Delta u_{\epsilon_n} - f, v) + (f, u_{\epsilon_n}) \geq (-M(\|u_{\epsilon_n}\|^2)\Delta u_{\epsilon_n}, u_{\epsilon_n}), \tag{2.18}$$

But

$$\begin{gathered}(-M(\|u_{\epsilon_n}\|^2)\Delta u_{\epsilon_n}, u_{\epsilon_n}) = M(\|u_{\epsilon_n}\|^2)(\nabla u_{\epsilon_n}, \nabla u_{\epsilon_n}) = \\ M(\|u_{\epsilon_n}\|^2)(\nabla(u_{\epsilon_n} - u), \nabla(u_{\epsilon_n} - u)) + M(\|u_{\epsilon_n}\|^2)(\nabla u_{\epsilon_n}, \nabla u) + \\ M(\|u_{\epsilon_n}\|^2)(\nabla u, \nabla(u_{\epsilon_n} - u)) \geq M(\|u_{\epsilon_n}\|^2)(\nabla u_{\epsilon_n}, \nabla u) + \\ M(\|u_{\epsilon_n}\|^2)(\nabla u, \nabla(u_{\epsilon_n} - u)),\end{gathered} \tag{2.19}$$

because $M(\|u_{\epsilon_n}\|^2)(\nabla(u_{\epsilon_n} - u), \nabla(u_{\epsilon_n} - u)) \geq 0$. Then,

$$\liminf[-M(\|u_{\epsilon_n}\|^2)(\Delta u_{\epsilon_n}, u_{\epsilon_n})] \geq -M(\|u\|^2)(\nabla u, u). \tag{2.20}$$

By (2.18) and (2.20), we obtain (1.1).
In order to prove $u \in K$ we observe that from (2.5), with $\omega = u_{\epsilon_m}$, that

$$0 \leq (\beta(u_{\epsilon_n}), u_{\epsilon_n}) \leq \epsilon C \tag{2.21}$$

therefore

$$(\beta(u_{\epsilon_n}), u_{\epsilon_n}) \;\; converges \; to \; zero. \tag{2.22}$$

Let $v \in H_0^1(\Omega)$ be arbitrarily fixed; then the inequality

$$(\beta(u_{\epsilon_n}) - \beta(v), u_{\epsilon_n} - v) \geq 0$$

yields $(\beta(v), u - v) \leq 0$, hence $(\beta(u - \lambda\omega)), \omega) \leq 0$ with the choice of $v = u - \lambda\omega$ with $\lambda > 0$ and $\omega \in H_0^1(\Omega)$. By hemicontinuity of β we can let $\lambda \to 0^+$ and obtain $(\beta(u), \omega) \geq 0$, hence $\beta(u) = 0$ by the arbitrariness of ω. Therefore $u \in K$.

□

3. The M-linear Problem: A Discontinuous Case

In this section we concentrate our atention on problem (1.1) when M possesses a discontinuity. More precisely, we study problem (1.1) with $M : \mathbb{R}/\{\theta\} \to \mathbb{R}$ continuous such that

(M2) $\lim_{t\to\theta^+} M(t) = \lim_{t\to\theta^-} M(t) = +\infty$
(M3) $\limsup_{t\to+\infty} M(t^2)t = +\infty$.

Theorem 3.1. *Assume that (M1)–(M3) and (H1) hold. Then for any choice of $f \in H_0^1(\Omega)$ the problem* (1.1) *admits at least one solution solution.*

The proof of Theorem 3.1 is given as in the proof of Theorem 2.1. We will formulate the penalized problem, associated with the variational inequality (1.1), as follows. Given $\epsilon > 0$, find a function $u_\epsilon \in H_0^1(\Omega)$ solution of the problem

$$\begin{aligned} -M(\|u_\epsilon\|^2)\Delta u_\epsilon + \frac{1}{\epsilon}\beta(u_\epsilon) = f \quad \text{in } \Omega, \\ u_\epsilon = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{3.1}$$

where f is given in $H_0^1(\Omega)$.
The existence of one solution of the penalized problem (3.1) is given by the

Theorem 3.2. *Assume that* $(M1) - (M3)$ *hold. Then, for any* $0 \neq f \in H_0^1(\Omega)$ *there exists at least one* $u_\epsilon \in L^2(\Omega)$*, solution of the problem* (3.1) .

Proof. We first consider the sequence of functions $M_n : \mathbb{R} \to \mathbb{R}$ given by

$$M_n(t) = \begin{cases} n, & \theta - \delta_n' \leq t \leq \theta + \delta_n'', \\ M(t), & t \leq \theta - \delta_n' \text{ or } t \geq \theta + \delta_n'', \end{cases}$$

for $n > m_0$, where $\theta - \delta_n'$ and $\theta + \delta_n''$, δ_n', $\delta_n'' > 0$, are, respectively, the points closest to θ, at left and at right, so that

$$M(\theta - \delta_n') = M(\theta + \delta_n'') = n.$$

We point out that, in this case, $\delta_n', \delta_n'' \to 0$ as $n \to \infty$.

Take $n > m_0$ and observe that the horizontal lines $y = n$ cross the graph of M. Hence M_n is continuous and satisfies (M1), for each $n > m_0$. In view of this, for each n like above, there is $u_n \in H_0^1(\Omega) \bigcap H^2(\Omega)$ satisfying

$$\begin{aligned} M_n(\|u_n\|^2)((u_n, \omega)) + \frac{1}{\epsilon}(\beta(u_n), \omega) = (f, \omega), \\ \text{for all } \omega \in H_0^1(\Omega) \cap H^2(\Omega). \end{aligned} \tag{3.2}$$

Taking $\omega = u_n$ in the above equation one has

$$M_n(\|u_n\|^2)\|u_n\|^2 + \frac{1}{\epsilon}(\beta(u_n), u_n) = (f, u_n),$$

and so

$$M_n(\|u_n\|^2)\|u_n\| \leq \|f\|,$$

since $0 \leq (\beta(u_n), u_n)$. Because of (M3) the sequence $(\|u_n\|)$ must be bounded. Hence

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{in } H_0^1(\Omega), \\ u_n &\to u \quad \text{in } L^2(\Omega), \\ \|u_n\|^2 &\to \theta_0, \quad \text{for some } \theta_0, \end{aligned} \tag{3.3}$$

perhaps for subsequences. We note that if $(M_n(\|u_n\|^2)$ converges its limit is different of zero. Suppose that $\|u_n\|^2 \to \theta$. If $\|u_n\|^2 > \theta + \delta_n''$ or $\|u_n\|^2 < \theta - \delta_n'$, for infinitely many n, we would get $M_n(\|u_n\|^2) = M(\|u_n\|^2)$, for such n, and so

$$M(\|u_n\|^2)\|u_n\|^2 \leq (f, u_n) \;\Rightarrow\; +\infty \leq (f, u)$$

which is a contradiction. On the other hand, if there are infinitely many n so that $\theta - \delta_n' \leq \|u_n\|^2 \leq \theta + \delta_n'' \;\Rightarrow\; M_n(\|u_n\|^2) = n$ and so $n\|u_n\|^2 \leq (f, u_n) \;\Rightarrow\; +\infty \leq (f, u)$ and we arrive again in a contradiction.

Consequently $\|u_n\|^2 \to \theta_0 \neq \theta$ which implies that for n large enough

$$\|u_n\|^2 < \theta - \delta_n' \quad \text{or} \quad \|u_n\|^2 > \theta + \delta_n''$$

and so $M_n(\|u_n\|^2) = M(\|u_n\|^2)$ which yields

$$M(\|u_n\|^2)((u_n, \omega)) + \frac{1}{\epsilon}(\beta(u_n), \omega) = (f, \omega), \quad \forall \omega \in H_0^1(\Omega) \cap H^2(\Omega). \tag{3.4}$$

Taking $\omega = -\Delta u_n$ in (3.4), using(2.1) and arguments of compactness as in the proof of theorem 2.1, we obtain that

$$u_n \to u \ \in H_0^1(\Omega) \tag{3.5}$$

The estimates (3.3) and (3.5) are sufficient to pass to the limit in the approximate equation and to obtain

$$M(\|u\|^2)((u, \omega)) + \frac{1}{\epsilon}(\beta(u), \omega) = (f, \omega), \quad \forall \omega \in H_0^1(\Omega) \cap H^2(\Omega). \tag{3.6}$$

□

Proof. of Theorem 3.1
As in the proof of Theorem 2.1, let $(\epsilon_n)_{n\in\mathbb{N}}$ be a sequence of real numbers such, that

$$0 < \epsilon_n < 1 \; for \; all \; n \in \mathbb{N} \;\; and \;\; \lim_{n\to\infty} \epsilon_n = 0.$$

Applying Theorem 3.1, for each $n \in \mathbb{N}$, we get a function $u_{\epsilon_n} \in H_0^1(\Omega) \cap H^2(\Omega)$ which satisfies

$$M(\|u_{\epsilon_n}\|^2)((u_{\epsilon_n}, \omega)) + \frac{1}{\epsilon_n}(\beta(u_{\epsilon_n}), \omega) = (f, \omega), \quad \forall \omega \in H_0^1(\Omega) \cap H^2(\Omega). \tag{3.7}$$

Since the estimates were uniform on ϵ *and* n, we can see that there exists a subsequence of (u_{ϵ_n}), again called (u_{ϵ_n}), and a function $u \in H_0^1(\Omega) \cap H^2(\Omega)$ such that

$$u_{\epsilon_n} \to u \text{ in } H_0^1(\Omega). \tag{3.8}$$

We note that $\|u_{\epsilon_n}\|$ does not converges to θ. In fact, we assume by contradiction that $\|u_{\epsilon_n}\| \to \theta$. By (3.7) and (3.8), with $\omega = u_{\epsilon_n}$, we get

$$M(\|u_{\epsilon_n}\|^2)\|u_{\epsilon_n}\| \le C$$

By hyphotese (M2), letting $n \to \infty$, we have a contradiction. Thus

$$M(\|u_{\epsilon_n}\|^2) \to M(\|u\|^2),$$

since $\|u_{\epsilon_n}\| \to \|u\|$ and M is continuous in an neighboard of $\|u\| \neq \theta$.

Consider v in $H_0^1(\Omega)$ with v belonging to K. By (3.7), we have

$$(-M(\|u_{\epsilon_n}\|^2)\Delta u_{\epsilon_n} - f, v) = (-\frac{1}{\epsilon}\beta(u_{\epsilon_n}), v) \tag{3.9}$$

and

$$(-M(\|u_{\epsilon_n}\|^2)\Delta u_{\epsilon_n} - f, -u_{\epsilon_n}) = (-\frac{1}{\epsilon}\beta(u_{\epsilon_n}), -u_{\epsilon_n}). \tag{3.10}$$

Follows of (3.9) and (3.10) that

$$\begin{aligned}(-M(\|u_{\epsilon_n}\|^2)\Delta u_{\epsilon_n} - f, v - u_{\epsilon_n}) &= \frac{1}{\epsilon}(-\beta(u_{\epsilon_n}), v - u_{\epsilon_n}) \\ &= \frac{1}{\epsilon}(\beta(v) - \beta(u_{\epsilon_n}), v - u_{\epsilon_n}) \ge 0,\end{aligned} \tag{3.11}$$

because $v \in K$ and monotonicity of β. Hence

$$(-M(\|u_{\epsilon_n}\|^2)\Delta u_{\epsilon_n} - f, v) + (f, u_{\epsilon_n}) \ge (-M(\|u_{\epsilon_n}\|^2)\Delta u_{\epsilon_n}, u_{\epsilon_n}). \tag{3.12}$$

As in the proof of Theorem 2.1, letting $n \to \infty$, we obtain that u satisfies the conditions of Theorem 3.2.

□

4. Another Nonlocal Problem

Next, we make some remarks on a nonlocal problem which is a slight generalization of one studied by Chipot-Lovat [6] and Chipot-Rodrigues [7]. More precisely, the above authors studied the problem

$$\begin{aligned}-a\Big(\int_\Omega u\Big)\Delta u = f \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega,\end{aligned} \tag{4.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $N \ge 1$, and $a : \mathbb{R} \to (0, +\infty)$ is a given function. Equation (4.1) is the stationary version of the parabolic problem

$$\begin{aligned}u_t - a\Big(\int_\Omega u(x,t)dx\Big)\Delta u = f \quad \text{in } \Omega \times (0,T), \\ u = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x).\end{aligned}$$

Here T is some arbitrary time and u represents, for instance, the density of a population subject to spreading. See [6, 7] for more details. In particular, [6] studies problem (4.1), with $f \in H^{-1}(\Omega)$, and proves the following result.

Proposition 4.1. *Let $a : \mathbb{R} \to (0, +\infty)$ be a positive function, $f \in H^{-1}(\Omega)$. Then problem (4.1) has as many solutions μ as the equation*

$$a(\mu)\mu = \langle\langle f, \varphi \rangle\rangle,$$

where φ is the function(unique) satisfying

$$-\Delta\varphi = 1 \quad in\ \Omega,$$
$$\varphi = 0 \quad on\ \partial\Omega.$$

In the present work, we consider the variational inequality

$$u \in K : (-a(\int_\Omega |u|^q)\Delta u, \omega) \geq (f, \omega) \ \ for\ all\ v \in K \cap (H_0^1(\Omega) \cap H^2(\Omega)) \tag{4.2}$$

where K and f are as before and $1 < q < 2N/(N-2), N \geq 3$. When $q = 2$ we have the well known Carrier model.

Theorem 4.2. *If $t \mapsto a(t)$ is a decreasing and continuous function, for $t \geq 0$, $\lim_{t\to+\infty} a(t^q)t = +\infty$ and $t \mapsto a(t^q)t$ is injective, for $t \geq 0$, then, for each $0 \neq f \in H^{-1}(\Omega)$, problem (4.2) possesses a weak solution.*

The proof of Theorem 4.2 is given as in the proof of Theorem 2.1. We will formulate the penalized problem, associated with the variational inequality (4.2), as follows: Given $\epsilon > 0$, find a function $u_\epsilon \in H_0^1(\Omega)$ solution of the problem

$$\begin{aligned} -a(\int_\Omega |u_\epsilon|^q)\Delta u_\epsilon + \frac{1}{\epsilon}\beta(u_\epsilon) &= f \quad \text{in } \Omega, \\ u_\epsilon &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{4.3}$$

where f is given in $H_0^1(\Omega)$.
The existence of one solution of the penalized problem (4.3) is given by the

Theorem 4.3. *Under the hypotheses of Theorem 4.2, then for each $0 \neq f \in H^{-1}(\Omega)$ there exists at least one $u_\epsilon \in H_0^1(\Omega)$ solution of the problem* (4.3) .

Proof. As in the proof of Theorem 2.1, let $\sum = \{e_1, \ldots, e_m, \ldots\}$ be an orthonormal basis of the Hilbert space $H_0^1(\Omega)$. For each $m \in \mathbb{N}$ consider the finite dimensional Hilbert space

$$\mathbb{V}_m = \text{span}\{e_1, \ldots, e_m\},$$

$P : \mathbb{R}^m \to \mathbb{R}^m$ be the function $P(\xi) = (P_1(\xi), \ldots, P_m(\xi))$, where

$$P_i(\xi) = a(\|u\|_q^q)((u, e_i)) + \frac{1}{\epsilon}(\beta(u), e_i) - \langle\langle f, e_i \rangle\rangle, \quad i = 1, \ldots, m$$

with $u = \sum_{j=1}^m \xi_j e_j$ and the identifications of $\mathbb{R}^m$ and $\mathbb{V}_m$ mentioned before. So

$$P_i(\xi) = a(\|u\|_q^q)\xi_i + \frac{1}{\epsilon}(\beta(u), e_i) - \langle\langle f, e_i \rangle\rangle, \quad i = 1, \ldots, m$$

and then

$$< P(\xi), \xi >= a(\|u\|_q^q)\|u\|^2 + \frac{1}{\epsilon}((\beta(u), u)) - \langle\langle f, u \rangle\rangle$$

We have to show that there is $r > 0$ so that $\langle P(\xi), \xi\rangle \geq 0$, for all $\|\xi\|_{\mathbb{R}^m} = r$ in $\mathbb{V}_m$. Suppose, on the contrary, that for each $r > 0$ there is $u_r \in \mathbb{V}_m$ such that $\|u_r\| = r$ and

$$\langle P(\xi_r), \xi_r\rangle < 0, \quad \xi_r \leftrightarrow u_r.$$

Taking $r = n \in \mathbb{N}$ we obtain a sequence (u_n), $\|u_n\| = n$, $u_n \in \mathbb{V}_m$ and

$$\langle P(u_n), u_n\rangle = a(\|u_n\|_q^q)\|u_n\|^2 + \frac{1}{\epsilon}(\beta(u_n), u_n) - \langle\langle f, u_n\rangle\rangle < 0$$

and so

$$a(\|u_n\|_q^q)\|u_n\| < C\|f\|, \quad \forall n = 1, 2, \ldots,$$

since$(\beta(u_n), u_n) > 0$. Because of the continuous immersion $H_0^1(\Omega) \subset L^q(\Omega)$ one gets $\|u\|_q \leq C\|u\|$ and the monotonicity of a yields $a(\|u\|_q^q) \geq a(C\|u\|^q)$ and so

$$a(C\|u_n\|^q)\|u_n\| < C\|f\|.$$

In view of $\lim_{t\to+\infty} a(t^q)t = +\infty$ one has that $\|u_n\| \leq C, \forall n \in \mathbb{N}$, which contradicts $\|u_n\| = n$. So, there is $r_m > 0$ such that $\langle F(\xi), \xi\rangle \geq 0$, for all $|\xi| = r_m$. In view of the sharp angle lemma there is $u_m \in \mathbb{V}_m$, $\|u_m\| \leq r_m$ such that $P_i(u_m) = 0, i = 1, \ldots, m$, that is,

$$a\left(\|u_m\|_q^q\right)((u_m, \omega)) + \frac{1}{\epsilon}(\beta(u_m), \omega) = \langle\langle f, \omega\rangle\rangle, \quad \forall\omega \in \mathbb{V}_m. \tag{4.4}$$

Reasoning as before, by using the facts that $t \to a(t)$ is decreasing for $t \geq 0$, $\lim_{t\to+\infty} a(t^q)t = +\infty$ and $\int_\Omega \beta(u)u > 0$ we conclude that $\|u_m\| \leq C, \forall m = 1, 2\ldots$ for some constant C that does not depend on m. Hence, $u_m \rightharpoonup u$ in $H_0^1(\Omega)$, $u_m \to u$ in $L^q(\Omega)$, $1 < q < \frac{2N}{N-2}$, and so $\|u_m\|_q \to \|u\|_q$. Taking limits on both sides of equation (4.4) we conclude that the function u is a solution of problem (4.3). $\square$

Remark 4.4. The function

$$a(t) = \frac{1}{t^{2\beta} + 1},$$

where β and q are related by $2\beta q < 1$, satisfies the assumptions of Theorem 4.2

Proof. of Theorem 4.2
Let $(\epsilon_n)_{n\in\mathbb{N}}$ be a sequence of real numbers such, that

$$0 < \epsilon_n < 1 \; for\ all\ n \in \mathbb{N} \;\; and \;\; \lim_{n\to\infty} \epsilon_n = 0.$$

For each $n \in \mathbb{N}$, we get function u_{ϵ_n} which satisfies Theorem 4.3. Since the estimates were uniform on ϵ *and* n, we can see that there exists a subsequence of u_{ϵ_n}, again called u_{ϵ_n}, and a function $u \in H_0^1(\Omega) \cap L^q(\Omega)$, $1 \leq q \leq \frac{2N}{N-2}$ such that

$$u_{\epsilon_n} \rightharpoonup u \text{ in } H_0^1(\Omega) \cap H^2(\Omega) \tag{4.5}$$

$$u_{\epsilon_n} \to u \text{ in } L^q(\Omega), \quad 1 \leq q \leq \frac{2N}{N-2} \tag{4.6}$$

$$\|u_{\epsilon_n}\|_q \to \|u\|_q \tag{4.7}$$

Consider v in $H_0^1(\Omega)$ with v belongs to K. By (4.3), we have

$$
\begin{aligned}
(-a(\|u_{\epsilon_n}\|_q^p)\Delta u_{\epsilon_n} - f, v - u_{\epsilon_n}) = (-\frac{1}{\epsilon}\beta(u_{\epsilon_n}), v - u_{\epsilon_n}) = \\
\frac{1}{\epsilon}(\beta(v) - \beta(u_{\epsilon_n}), v - u_{\epsilon_n}) \geq 0
\end{aligned}
\tag{4.8}
$$

because $v \in K$ and monotonicity of β.
As in the proof of Theorem 2.1, when $\epsilon_n \to 0$ we obtain that u satisfies the conditions of Theorem 4.2.

□

References

[1] C. O. Alves & F. J. S. A. Corrêa, *On existence of solutions for a class of problem involving a nonlinear operator*, Communications on applied nonlinear analysis, 8(2001), N. 2, 43-56.

[2] A. Arosio & S. Spagnolo, *Global solution to the Cauchy problem for a nonlinear hyperbolic equation*, Nonlinear Partial Differential Equations and Their Applications, College de France Seminar, Vol. 16, H. Brezis and J. L. Lions edts., Pitman, London, 1984.

[3] C. O. Alves & D. G. de Figueiredo, *Nonvariational elliptic systems via Galerkin methods*, Function Spaces, Differential Operators and Nonlinear Analysis - The Hans Triebel Anniversary Volume, Ed. Birkhauser, Switzerland, 47-57, 2003.

[4] A. Ambrosetti, H. Brezis & G. Cerami, *Combined effects of concave and convex nonlinearities in some elliptic problems*, J. Funct. Anal. 122(1994), 519-543.

[5] H. Brezis *Analyse Fonctionnelle, Théorie et Applications*, Masson, Paris (1983).

[6] M. Chipot & B. Lovat, *Some remarks on nonlocal elliptic and parabolic problems*, Nonlinear Analysis, T.M.A., Vol. 30, No. 7,(1997), 4619-4627.

[7] M. Chipot & J. F. Rodrigues, *On a class of nonlocal nonlinear elliptic problems*, Mathematical Modelling and Numerical Analysis, Vol. 26, No. 3, 1992, 447-468.

[8] F. J. S. A. Corrêa & S. B. de Menezes, *Existence of solutions to nonlocal and singular elliptic problems via Galerkin Method*, Eletronic Journal of Differential Equations, Vol. 2004(2004), No. 19, 2004, 1-10.

[9] J. G. Eisley, *Nonlinear vibrations of beams and rectangular plates*, Z. Anger. Math. Phys. 15,(1964)167-175.

[10] A. Haraux *Nonlinear Evolution Equations- Global Behaviour of Solutions*, Lectures Notes in Mathematics 841, Springer Verlag, (1981).

[11] A. Hazoya & Y. Yamada *On some nonlinear wave equations II, Global existence and decay of solutions*, J. Fac. Sci. (Univ. Tokio) Vol.38, No. 2 (1991) 230-250.

[12] A. N. Kludnev *One-side problem connected with a weakly nonlinear parabolic operator*, Siberian Math. J. 19 (1978) 412-417.

[13] N. A. Larkin *The unilateral problem for nonlocal quasilinear hyperbolic equation of the theory of elasticity* , Doklady Acad. URSS, 274(6) (1984) 341-344 (in Russian).

[14] J. Limaco, L. A. Medeiros & S. B. de Menezes, *Vibrations of elastic strings: Mathematical aspects, Part one*, Journal of Computational Analysis and Applications, Vol. 4. N. 2,(2002)91-127.

[15] J. L. Lions, *Quelques Méthodes de résolution des problémes aux limites non linéaires*, Dunod, Gauthier-Villars, Paris, 1969.

[16] J. L. Lions, *On some questions in boundary value problems of Mathematical Physics*, Contemporary Development in Continuum Mechanics and Partial Differential Equations, Edited by G. M. de La Penha and L. A. Medeiros, North-Holland, Amsterdan (1978) 285-346.

[17] L. A. Medeiros & M. Milla Miranda *Local solutions for nonlinear unilateral problems* , Revue Romaine de Math. Pures et Appliquées, 31(1986) 371-382.

[18] S. I. Pohozhaev, *The Kirchhoff quasilinear hyperbolic equation*, Differential Equations, Vol. 21, No. 1 (1985) 101-108 (in Russian)

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