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Bounds and optimization of the minimum eigenvalue for a vibrating system

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Abstract

We consider the problem of the oscillation of a string fixed at one end with a mass connected to a spring at the other end. The problem of minimizing the first eigenvalue of the system subject to a fixed total mass constraint is investigated. We discuss both a Sturm–Liouville and a Stieltjes integral formulation of the boundary value problem. For small spring constant, the minimum eigenvalue for both formulations is obtained by concentrating all the mass at the end with the spring. For large spring constants, the Stieltjes eigenvalue is minimized by a point mass at an interior point. We also formulate the problem with an α-norm constraint on the density ρ in which case the optimal eigenpair satisfies a nonlinear boundary value problem. Numerical evidence suggests that this case tends to the point-mass case at the end as $\alpha \to 1+$.

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1 Introduction

We consider the vibration problem of a string fixed at one end with a mass m at the other end which is connected to a spring. The mass is free to oscillate in the vertical direction. If the sum of the mass of the string and m is fixed, we pose the problem of finding the least eigenvalue of this system, and to determine also the eigenfunction which gives rise to this minimal eigenvalue.

Some motivation for this problem is provided by the classic paper of Krein [12] where a string of length L and fixed mass m is considered. The eigenvalue problem considered by Krein is

$$-y'' = \lambda\rho(x)y, \quad y(0) = y(L) = 0.$$

Krein solves the problem of determining the maximum and minimum of the eigenvalues $\lambda_n(\rho)$, $n = 1, 2, ...$, when the density ρ is subjected to the constraints

$$h \le \rho(x) \le H, \quad \int_0^L \rho(x)\,dx = m.$$

In the case $h = 0$, $H = \infty$, Krein points out that the only conclusion is $\lambda_n(\rho) \ge 4n^2/mL$, $n = 1, 2, ...$, and the inequality is an equality and attained when the string is divided into n equal parts, and at the center of each part is concentrated a mass of value m/n.

A related problem is that of a maximum or minimum of the lowest eigenvalue of the Schrodinger equation $-\Delta u + V(x)u = Eu$ on a domain Ω where V is subjected to an α-norm constraint, $\int_\Omega |V(x)|^\alpha\,dx < \infty$. For $\alpha > 1$, it is possible to derive a necessary condition for V to be an extremizer which is given by $u^2 = c|V|^{\alpha-1}$ where u is the associated eigenfunction. We refer to the paper of Ashbaugh and Harrell [2] for a discussion of this problem.

Our design problem is to see how to minimize the lowest frequency of vibration for a certain vibrational system subject to having a fixed amount of mass to distribute. In section 2 we define the minimization problem we are studying, discuss a self-adjoint formulation and a Stieltjes integral formulation. In order to achieve the minimum of the lowest eigenvalue under a total mass constraint, the Stieltjes extension of the problem is necessary. Section 3 gives two discrete examples which turn out to be minimizers of the Stieltjes design problem. These examples of concentrated mass are analogous to those of Krein's $n = 1$ case. In section 4, we establish new lower bounds for the eigenvalues of each of the two formulations. In section 5 we prove that the infimum of the Sturm–Liouville eigenvalues and the minimum Stieltjes eigenvalue are equal. Finally, in section 6, the problem is reformulated with an α-norm constraint on the density ρ with $\alpha > 1$ and contrasted with the first problem where $\alpha = 1$. First we prove that a mimimizer exists. Then we derive a necessary condition for a minimizer analogous to that of the Schrodinger equation, but with an additional complication that involves the terminal value of the eigenfunction in the associated nonlinear equation. We provide numerical evidence that this case reduces to our point-mass cases with all mass concentrated at the endpoint as $\alpha \to 1+$. There is a critical value of the length of our string so that for a length less than the critical value, the minimum eigenvalue is right continuous at $\alpha = 1$, while for larger lengths the limit as $\alpha \to 1+$ is greater than the minimum eigenvalue for $\alpha = 1$.

We use $\mathcal{L}^\alpha(0, L)$ to denote the Banach space of (equivalence classes) of complex valued functions satisfying $\int_0^L |f(x)|^\alpha \, dx < \infty$. The norm is given by $||f||_\alpha$.

2 The model and eigenvalue problem

A string of density $\rho(x)$ is distributed over the interval $[0, L]$ and is fixed at the end $x = 0$. A mass m is attached at the end $x = L$ and also to a spring with spring constant k. The mass is free to move vertically in a frictionless groove which is perpendicular to the string. The string tension is T. As we see below, this leads to an eigenvalue problem with the eigenvalue parameter in the boundary condition. Such problems occur frequently in vibration and heat flow problems. An extensive list of references where such problems occur has been given by Fulton [7]. Further discussion of the vibration model given here can be found in section 4.3 of [8]. If the deflection of the string is denoted by $u(x, t)$, then we have by the wave equation that

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = T\frac{\partial^2 u}{\partial x^2}, \qquad 0 \le x < L, \quad t > 0. \tag{2.1}$$

The boundary condition at $x = 0$ is

$$u(0, t) = 0. \tag{2.2}$$

It is assumed the equilibrium position of the string is $u(x, 0) = 0$. At the end $x = L$, we have by Newton's second law (assuming small vibrations as in the derivation of the wave equation) that

$$m\frac{d^2}{dt^2}u(L, t) = -ku(L, t) - T\frac{\partial u}{\partial x}(L, t). \tag{2.3}$$

Employing separation of variables, setting $u(x,t) = h(t)\phi(x)$, and substituting into (2.1)–(2.3) yields after some simplification that

$$\phi''(x) = -\lambda\rho(x)\phi(x), \qquad 0 \le x < L. \tag{2.4}$$

$$\phi(0) = 0. \tag{2.5}$$

$$\tilde{k}\phi(L) + \phi'(L) = \lambda m\phi(L), \quad \tilde{k} = \frac{k}{T}. \tag{2.6}$$

It is well known how to cast (2.4)–(2.6) as a self-adjoint problem. This shows the eigenvalues are real. The form of the Rayleigh quotient shows the least eigenvalue $\lambda_0(\rho, m)$ is also positive. Here we follow Fulton [7] and Hinton [9]. Let $\mathcal{L}_\rho(0, L)$ be the Hilbert space of (equivalence classes) of Lebesgue measurable functions f satisfying $\int_0^L \rho(s)|f(s)|^2\, ds < \infty$. Let H be the Hilbert space $\mathcal{L}^2_\rho(0, L) \oplus \mathbb{C}$, and define the inner product in H by

$$\left\langle \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}, \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} \right\rangle = \int_0^L \rho(s)F_1(s)\bar{G}_1(s)\, ds + \frac{1}{m}F_2\bar{G}_2.$$

We define the domain of an operator A by

$$\mathfrak{D}(A) = \left\{ \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} : (i), (ii), (iii) \text{ below hold} \right\}.$$

(i) F_1, F_1' are absolutely continuous on $[0, L]$.

(ii) $F_1(0) = 0$.

(iii) $F_2 = mF_1(L)$.

The operator A is defined on $\mathfrak{D}(A)$ by

$$A\left(\begin{bmatrix} F_1 \\ F_2 \end{bmatrix}\right) = \begin{bmatrix} -F_1''/\rho \\ \tilde{k}F_1(L) + F_1'(L) \end{bmatrix}.$$

It follows from [7, 9] that A is self-adjoint, A has compact resolvent, and the eigenvalues of A are the same as those of (2.4)–(2.6). Also the Rayleigh quotient is given by

$$\frac{\left\langle A\left(\begin{bmatrix} F_1 \\ F_2 \end{bmatrix}\right), \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \right\rangle}{\left\langle \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}, \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \right\rangle} = \frac{\tilde{k}F_1^2(L) + \int_0^L F_1'(s)^2\, ds}{mF_1^2(L) + \int_0^L \rho(s)F_1^2(s)\, ds}.$$

We will need the Rayleigh quotient for (2.4)–(2.6). By multiplying (2.4) by ϕ and integrating by parts, we have after applying (2.4)–(2.5) and solving for λ that

$$\lambda = \frac{\tilde{k}\phi^2(L) + \int_0^L \phi'(s)^2\, ds}{m\phi^2(L) + \int_0^L \rho(s)\phi^2(s)\, ds}. \tag{2.7}$$

The set of admissible or test functions in the Rayleigh quotient requires on F_1 that $F_1(0) = 0$, F_1 is absolutely continuous on $[0, L]$, and $\int_0^L F_1'(s)^2\, ds < \infty$. Since $F_1 = \phi$ in the self-adjoint formulation, these are also the conditions on ϕ in (2.7).

Let $\lambda_0(\rho, m)$ be the least eigenvalue of (2.4)–(2.6). Now subject (2.4)–(2.6) to a total mass constraint

$$K = m + \int_0^L \rho(x)\, dx, \tag{2.8}$$

where K is a given positive number, and consider the problem of minimizing $\lambda_0(\rho, m)$ subject to the constraint (2.8). We suppress the dependence of $\lambda_0(\rho, m)$ on $K, \tilde{k}$ to simplify the notation. In the last section we will also subject (2.4)–(2.6) to a constraint

$$K = m + \int_0^L \rho^\alpha(x)\, dx, \quad \alpha > 1,$$

and investigate the behavior as α tends to one.

The eigenvalue problem (2.4)–(2.6) will be defined over an admissible class $\mathcal{A}_1$ defined by

$$\mathcal{A}_1 = \{(\rho, m) : \rho(x) > 0 \text{ a.e., } m \geq 0,\ \rho \in \mathcal{L}(0, L),\ K = m + \int_0^L \rho(s)\, ds\}.$$

The minimization problem is then to find

$$\lambda^*(K) := \inf_{(\rho,m)\in\mathcal{A}_1} \lambda_0(\rho, m). \tag{2.9}$$

It turns out that the value $\lambda^*(K) < \lambda_0(\rho, m)$ for all $\lambda_0(\rho, m) \in \mathcal{A}_1$ so that the lowest eigenvalue of (2.4)–(2.6) has no minimum within the class $\mathcal{A}_1$. To achieve this minimum it is necessary to allow for point masses distributed along the string. To allow for point masses distributed between $x = 0$ and $x = L$ as in [12], we change to the Stieltjes integral formulation of (2.4). We change the eigenparameter to Λ to distinguish the two problems.

$$\phi'(x) = \phi'(0) - \int_0^x \Lambda\phi(s)dP(s), \qquad 0 \leq x < L. \tag{2.10}$$

and the boundary conditions become

$$\phi(0) = 0, \quad \tilde{k}\phi(L) + \phi'(L-) = \Lambda m\phi(L), \quad \tilde{k} = \frac{k}{T}. \tag{2.11}$$

Here $P(x)$ is the cumulative mass distribution function, i.e., $P(x)$ is the total mass on $[0, x]$ for $0 \le x \le L$. In the case where there are no point masses on $[0, L)$, then

$$P(x) = \int_0^x \rho(s)\, ds, \quad 0 \le x < L, \quad P(L) = m + \int_0^L \rho(s)\, ds = K.$$

By a solution of (2.10)–(2.11) we mean an absolutely continuous function ϕ on $[0, L]$ so that (2.10) holds except at the jumps of P, and the conditions (2.11) hold. The class $\mathcal{A}_2$ of admissible $P(x)$ is defined by

$$\mathcal{A}_2 = \{P : P \text{ is nondecreasing on } [0, L],\ P(0) = 0,\ P(L) = K\}. \tag{2.12}$$

The conditions on P ensure that ϕ' has one-sided limits at each point. Further ϕ' is bounded on $[0, L]$. We denote the smallest eigenvalue of (2.10)–(2.11) by $\Lambda_0(P)$. Thus our new minimization problem is to find

$$\Lambda^{**}(K) := \inf_{P \in \mathcal{A}_2} \Lambda_0(P). \tag{2.13}$$

In analogy with (2.4)–(2.6) for $P \in \mathcal{A}_2$, we will always let $m = P(L) - P(L-)$ so $m \ge 0$ is the mass concentrated at $x = L$.

In section 4 we prove that there is an element $P \in \mathcal{A}_2$ so that $\Lambda^{**}(K) = \Lambda_0(P)$. Following this in section 5 we prove $\Lambda^{**}(K) = \lambda^*(K)$.

We also need the Rayleigh quotient for (2.10)–(2.11). Multiplying (2.10) by ϕ', integrating over $[0, L]$, applying (2.11), and solving for Λ gives

$$\Lambda = \frac{\tilde{k}\phi^2(L) + \int_0^L \phi'(s)^2\, ds}{m\phi^2(L) + \int_0^{L-} \phi^2(s) dP(s)\, ds} \tag{2.14}$$

with the same conditions on ϕ in (2.14) as in (2.7). The L-limit in (2.14) indicates the possible jump in P at L in not included in the integral; this jump is the first term of the denominator. Alternatively, the denominator could be written as a single term $\int_0^L \phi^2(s) dP(s)\, ds$.

The existence–uniqueness theory of (2.10) can be found in Reid [14], see also Hinton–Lewis [10]. For $P \in \mathcal{A}_2$, and given initial conditions $\phi(0)$, $\phi'(0)$, there is a unique solution of (2.10).

3 Two degenerate cases

Example 3.1. Let $P_1(x) = 0$ for $0 \leq x < L$, and $P_1(L) = K$, i.e., all the mass is concentrated at $x = L$. We solve (2.10)–(2.11) by normalizing ϕ with $\phi'(0) = 1$. We compute that

$$\phi(x) = x, \quad 0 \leq x \leq L, \quad \phi'(x) = 1, \quad 0 \leq x < L.$$

A substitution of these expressions into the second boundary condition of (2.11) and use of the constraint $P(L) = K$ yields that

$$\Lambda_0(P_1) = \frac{L\tilde{k} + 1}{LK}. \tag{3.1}$$

In our next example we concentrate all the mass at a point $\bar{x}$, $0 < \bar{x} < L$.

Example 3.2. Let $m = 0$ and $\rho(x) = K\delta(x - \bar{x})$, where δ is the delta function, i.e., $P_2(x) = 0$ for $0 \leq x < \bar{x}$ and $P_2(x) = K$ for $\bar{x} \leq x \leq L$. We then compute

$$\phi'(x) = \begin{cases} 1, & \text{if } 0 \leq x < \bar{x}, \\ 1 - \Lambda_0(P_2)K\bar{x}, & \text{if } \bar{x} < x \leq L. \end{cases} \tag{3.2}$$

Computing ϕ yields

$$\phi(x) = \begin{cases} x, & \text{if } 0 \leq x \leq \bar{x}, \\ \bar{x} + [1 - \Lambda_0(P_2)K\bar{x}](x - \bar{x}), & \text{if } \bar{x} < x \leq L. \end{cases} \tag{3.3}$$

A substitution of these expressions into the second boundary condition of (2.11) yields that

$$\Lambda_0(P_2) = \frac{L\tilde{k} + 1}{K\bar{x}(1 + \tilde{k}(L - \bar{x}))}. \tag{3.4}$$

Note with this value of $\Lambda_0(P_2)$,

$$\phi(L) = \frac{\bar{x}}{1 + \tilde{k}(L - \bar{x})} \to 0 \text{ as } \tilde{k} \to \infty.$$

In Example 3.2 we now minimize $\Lambda_0(P_2)$ with respect to $\bar{x}$. A calculation gives

$$\begin{aligned} \frac{d\Lambda_0}{d\bar{x}} &= -\frac{(1 + \tilde{k}L)(1 + \tilde{k}(L - \bar{x}) - \tilde{k}\bar{x}))}{K[\bar{x}(1 + \tilde{k}(L - \bar{x}))]^2} \\ &= 0 \text{ at } \bar{x} = \frac{L}{2} + \frac{1}{2\tilde{k}}. \end{aligned} \tag{3.5}$$

For $\tilde{k} > 1/L$, we see that $0 < L/2 + 1/2\tilde{k} < L$. Hence $\Lambda_0(P_2)$ is minimized with respect to $\bar{x}$ at $\bar{x} = L/2 + 1/2\tilde{k}$ and the minimum value attained is

$$\Lambda_0(P_2) = 4\tilde{k}/K(1 + L\tilde{k}). \tag{3.6}$$

Note that for $\tilde{k} \to \infty$, the value of $\Lambda_0(P_2)$ tends tends to the Krein value of $4/KL$ (fixed endpoints). For $\tilde{k} \leq 1/L$, the critical point $\bar{x} = L/2 + 1/2\tilde{k} \geq L$. In this case $\Lambda_0(P)$ is minimized at $\bar{x} = L$ with minimum value of $\lambda_0(P_2) = (1 + L\tilde{k})/KL$. Thus for a single point mass of K, we see that as $\tilde{k}$ varies from 0 to ∞, the minimum value of $\Lambda_0(P)$ is achieved by locating all the mass at $x = L$ for $\tilde{k} \leq 1/L$, and is achieved by locating all the mass at $\bar{x} = L/2 + 1/2\tilde{k}$ for $\tilde{k} > 1/L$.

Some algebra shows that with $\bar{x} = L/2+1/2\tilde{k}$ and $\Lambda_0(P)$ given by (3.6), then (3.3) reduces to

$$\phi(x) = \begin{cases} x, & \text{if } 0 \leq x \leq \bar{x}, \\ 2\bar{x} - x, & \text{if } \bar{x} < x \leq L. \end{cases} \tag{3.7}$$

4 Lower bounds for λ_0 and Λ_0

We begin by establishing some inequalities which will be needed in the sequel. For these inequalities we suppose that ψ is a function such that $\psi : [0, L] \to \mathbb{R}$, $\psi(0) = 0$, ψ is absolutely continuous, and $\int_0^L \psi'(s)^2\,ds < \infty$.

By application of the Cauchy–Schwarz inequality we obtain

$$\psi(x) \leq x^{\frac{1}{2}} \left(\int_0^x |\psi'(s)|^2\,ds \right)^{\frac{1}{2}}, \tag{4.1}$$

and

$$\psi(x) - \psi(L) \leq (L - x)^{\frac{1}{2}} \left(\int_x^L |\psi'(s)|^2\,ds \right)^{\frac{1}{2}}, \tag{4.2}$$

Adding (4.1) and (4.2) gives

$$2\psi(x) - \psi(L) \leq x^{\frac{1}{2}} \left(\int_0^x |\psi'(s)|^2\,ds \right)^{\frac{1}{2}} + (L - x)^{\frac{1}{2}} \left(\int_x^L |\psi'(s)|^2\,ds \right)^{\frac{1}{2}},$$

which by another application of Cauchy–Schwarz gives

$$2\psi(x) - \psi(L) \leq L^{\frac{1}{2}} \left(\int_0^L |\psi'(s)|^2\,ds \right)^{\frac{1}{2}}. \tag{4.3}$$

Note that we have equality in (4.1) and (4.2) only for ψ linear on each subinterval. Further equality implies that if $\psi' = c$ on $[0, x]$, then $\psi' = \pm c$ on $[x, L]$.

From $(\psi^2)' = 2\psi\psi'$, and Cauchy–Schwarz we obtain

$$\psi^2(x) \leq 2 \int_0^x |\psi(s)\psi'(s)|\, ds, \quad \psi^2(x) - \psi^2(L) \leq 2 \int_x^L |\psi(s)\psi'(s)|\, ds,$$

and by adding these two inequalities we have

$$2\psi^2(x) - \psi^2(L) \leq 2 \int_0^L |\psi(s)\psi'(s)|\, ds. \tag{4.4}$$

We will also need Opial's inequality [13], see also [1],

$$\int_0^L |\psi(s)\psi'(s)|\, ds \leq \frac{L}{2} \int_0^L |\psi'(s)|^2\, ds. \tag{4.5}$$

4.1 Lower bounds for $\Lambda_0(P)$

Our first theorem is a lower bound for eigenvalues of (2.10)–(2.11) for small $\tilde{k}$.

Theorem 4.1. *If $\Lambda_0(P)$ is the least eigenvalue for* (2.10)–(2.11) *for some $P \in \mathcal{A}_2$ and $\tilde{k} \leq 1/L$, then*

$$\Lambda_0(P) \geq \frac{1 + \tilde{k}L}{KL}. \tag{4.6}$$

Proof. Let ϕ be the eigenfunction corresponding to $\Lambda_0(P)$. By the Rayleigh quotient (2.14), (4.4), and (4.5), for $x_0 \in [0, L]$,

$$\begin{aligned}\Lambda_0(P) &= \frac{\tilde{k}\phi^2(L) + \int_0^L \phi'(s)^2\, ds}{m\phi^2(L) + \int_0^{L-} \phi^2(s) dP(s)\, ds} \\ &\geq \frac{\tilde{k}[2\phi^2(x_0) - L\int_0^L \phi'(s)^2\, ds] + \int_0^L \phi'(s)^2\, ds}{m\phi^2(L) + \int_0^{L-} \phi^2(s) dP(s)\, ds}.\end{aligned}$$

Now choose x_0 so that $\phi^2(x_0)$ is the maximum of $\phi^2(x)$ for $x \in [0, L]$. Then

$$m\phi^2(L) + \int_0^{L-} \phi^2(s) dP(s) \leq m\phi^2(x_0) + \phi^2(x_0) \int_0^{L-} dP(s) = \phi^2(x_0)K. \tag{4.7}$$

Thus with this substitution in the above we have

$$\begin{aligned}\Lambda_0(P) &\geq \frac{\tilde{k}[2\phi^2(x_0) - L\int_0^L \phi'(s)^2\,ds] + \int_0^L \phi'(s)^2\,ds}{K\phi^2(x_0)} \\ &= \frac{2\tilde{k}}{K} + (1-\tilde{k}L)\frac{\int_0^L \phi'(s)^2\,ds}{K\phi^2(x_0)}.\end{aligned} \tag{4.8}$$

However (4.1) gives $\phi^2(x_0) \leq L\int_0^L \phi'(s)^2\,ds$, and when this is substituted into (4.8) we have

$$\Lambda_0(P) \geq \frac{2\tilde{k}}{K} + (1-\tilde{k}L)\frac{1}{KL} = \frac{1+\tilde{k}L}{KL}.$$

□

We now treat the case $\tilde{k} > 1/L$.

Theorem 4.2. *If $\Lambda_0(P)$ is the least eigenvalue for* (2.10)–(2.11) *for some $P \in \mathcal{A}_2$ and $\tilde{k} > 1/L$, then*

$$\Lambda_0(P) \geq \frac{4\tilde{k}}{K(1+\tilde{k}L)}. \tag{4.9}$$

Proof. Let ϕ be the eigenfunction corresponding to $\Lambda_0(P)$ which we normalize by making $\phi'(0) = 1$. Choose x_0 so that $\phi(x_0)$ is the maximum value of $\phi(x)$ on $[0, L]$. Since $\phi'(0) = 1$, we have $\phi(x_0) > 0$, and the left hand side of (4.3) is positive with $x = x_0$. Square both sides of (4.3) to obtain

$$[2\phi(x_0) - \phi(L)]^2 \leq L\int_0^L \phi'(s)^2\,ds. \tag{4.10}$$

For the remainder of the proof we need $\phi(x) \geq 0$ on $[0, L]$ so that we have that $\phi^2(x_0)$ is the maximum value of $\phi^2(x)$ on $[0, L]$. In the case that P is given by $(\rho, m) \in \mathcal{A}_1$, it is known that ϕ has no zeros in $(0, L)$, see Linden [11] or Binding et al. [5]. Thus $\phi(x_0) \geq \phi(x)$ on $[0, L]$ so that (4.7) holds. We assume this case first and indicate below how to do the general case.

Using (4.10) and (4.7) in the Rayleigh quotient (2.14) gives

$$\begin{aligned}\Lambda_0(P) &\geq \frac{\tilde{k}\phi^2(L) + L^{-1}[2\phi(x_0) - \phi(L)]^2}{m\phi^2(L) + \int_0^L \phi^2(s)dP(s)\,ds} \\ &= \frac{1}{KL\phi^2(x_0)}[(\tilde{k}L+1)\phi^2(L) + 4\phi^2(x_0) - 4\phi(x_0)\phi(L)] \\ &= \frac{1}{KL} f\left(\frac{\phi(L)}{\phi(x_0)}\right),\end{aligned} \tag{4.11}$$

where $f(y) = (\tilde{k}L+1)y^2 - 4y + 4$. It follows that $f'(y) = 0$ at $y_0 = 2/(\tilde{k}L+1)$, so that the minimum value of $f(y)$ is given by $f(y_0)$. Hence

$$\Lambda_0(P) \geq \frac{f(y_0)}{KL} = \frac{4\tilde{k}}{(1+\tilde{k}L)K}.$$

For the general case of $P \in \mathcal{A}_2$, we can choose a sequence of absolutely continuous, increasing P_n such that $P_n \in \mathcal{A}_2$ and such that $P_n(x) \to P(x)$ as $n \to \infty$ for each $x \in [0, L]$. Then $(\rho_n, m_n) \in \mathcal{A}_1$ with $\rho_n = P_n'$, $m_n = 0$. By choosing a fixed test function, we can bound the sequence $\lambda_0(P_n', 0)$ above. Since the $\lambda_0(P_n', 0)$ are also bounded below, we can assume without loss of generality that the sequence $\lambda_0(P_n', 0)$ converges with limit μ. Let ϕ_n be the eigenfunction corresponding to $\lambda_0(P_n', 0)$ with $\phi_n(0) = 0, \phi_n'(0) = 1$. The theory of Battle [3, 4] now applies to give that if ϕ is the solution of (2.10) with λ replaced by μ, and initial conditions $\phi(0) = 0, \phi'(0) = 1$, then

$$\phi_n(x) \to \phi(x) \text{ uniformly on } [0, L], \quad \phi_n'(x) \to \phi'(x) \text{ pointwise on } [0, L],$$

and further the sequence ϕ_n' is uniformly bounded on $[0, L]$. Substituting ϕ_n, P_n into (2.14) and letting $n \to \infty$ shows that $\mu = \Lambda_0(P)$. Since $\Lambda_0(P_n', 0)$ satisfies (4.9), μ will as well.

□

Theorems 4.1 and 4.2 combined with the examples of Section 3 solve the second minimization problem yielding

$$\Lambda^{**}(K) = \inf_{P \in \mathcal{A}_2} \Lambda_0(P) = \begin{cases} \frac{1+\tilde{k}L}{KL} \text{ if } \tilde{k} \leq \frac{1}{L}, \\ \frac{4\tilde{k}}{(1+\tilde{k}L)K} \text{ if } \tilde{k} > \frac{1}{L}. \end{cases} \tag{4.12}$$

The minima are realized by point masses of mass K located at $x = L$ in case $\tilde{k} \leq 1/L$, and located at $x = L/2 + 1/2\tilde{k}$ in case $\tilde{k} > 1/L$. Note that for $\tilde{k} \to \infty$, the value of $\Lambda^{**}(K)$ tends tends to the Krein value of $4/KL$ (fixed endpoints).

We now show that the functions P_1 and P_2 from examples 3.1 and 3.2, respectively, which realize the minima in (4.12) are unique. We consider the case of P_1 as the other case is similar. If $\tilde{k} \leq 1/L$, and P is such that $\Lambda_0(P) = (1+\tilde{k}L)/KL$, then in the proof of Theorem 4.1 we have equality in all inequalities of the proof. In particular with the first use of (4.4) in the Rayleigh quotient the function ϕ must be linear on $[0, L]$ and $\phi'(0) = 1$ implies $\phi(x) = x$ on $[0, L]$. We can substitute this function into (2.10) to conclude $P(x) = 0$ for $0 \leq x < L$; hence $P = P_1$ since $P(L) = K$.

4.2 Lower bounds for $\lambda_0(\rho, m)$

Here we give a general lower bound for eigenvalues of (2.4)–(2.6). Upper bounds can be obtained by the use of test functions.

Theorem 4.3. *If* $\lambda_0(\rho, m)$ *is the least eigenvalue of* (2.4)–(2.6) *, then*

$$m \geq \tilde{k}L \int_0^L \rho(s)\,ds \Rightarrow \lambda_0(\rho, m) \geq \frac{1+\tilde{k}L}{L(m+\int_0^L \rho(s)\,ds)}. \tag{4.13}$$

and

$$m < \tilde{k}L \int_0^L \rho(s)\,ds \Rightarrow \lambda_0(\rho, m) \geq \frac{1}{L\int_0^L \rho(s)\,ds}. \tag{4.14}$$

Proof. Let ϕ be the eigenfunction for $\lambda_0(\rho, m)$ with $\phi'(0) = 1$. Set $Q(x) = \int_x^L \rho(s)\,ds$. Then integrating by parts and applying Opial's inequality (4.5) yields

$$\begin{aligned} \int_0^L \rho(s)\phi^2(s)\,ds &= -\int_0^L Q'(s)\phi^2(s)\,ds \\ &= \int_0^L 2Q(s)\phi(s)\phi'(s)\,ds \\ &\leq 2Q(0)\int_0^L |\phi(s)\phi'(s)|\,ds \\ &\leq Q(0)L\int_0^L \phi'(s)^2\,ds. \end{aligned} \tag{4.15}$$

Using (4.15) in the Rayleigh quotient (2.7), we obtain

$$\begin{aligned} \lambda_0(\rho, m) &\geq \frac{\tilde{k}\phi^2(L)+\int_0^L \phi'(s)^2\,ds}{m\phi^2(L)+Q(0)L\int_0^L \phi^2(s)\,ds} \\ &= \frac{\tilde{k}+y}{m+Q(0)Ly}, \quad y = \frac{\int_0^L \phi'(s)^2\,ds}{\phi^2(L)}. \end{aligned} \tag{4.16}$$

From (4.1) we see that $y \geq 1/L$. For the function f defined by

$$f(y) = \frac{\tilde{k}+y}{m+Q(0)Ly},$$

we compute that

$$f'(y) = \frac{m - \tilde{k}Q(0)L}{(m + Q(0)Ly)^2}.$$

We can now draw the following conclusions. If $m \geq \tilde{k}L \int_0^L \rho(s)\,ds = \tilde{k}LQ(0)$, then f is increasing which implies

$$\lambda_0(\rho, m) \geq \inf_{y \geq 1/L} f(y) = f(1/L) = \frac{1 + \tilde{k}L}{L(M + \int_0^L \rho(s)\,ds)}.$$

If $m \leq \tilde{k}L \int_0^L \rho(s)\,ds$, then f is decreasing which implies

$$\lambda_0(\rho, m) \geq \inf_{y \geq 1/L} f(y) = f(\infty) = \frac{1}{Q(0)L} = \frac{1}{L\int_0^L \rho(s)\,ds}.$$

□

5 The relationship between $\lambda^*(K)$ and $\Lambda^{**}(K)$

We come now to the question as to whether $\lambda^*(K) = \Lambda^{**}(K)$. Since $\Lambda^{**}(K)$ is obtained as a minimum over a set which contains the set of which $\lambda^*(K)$ is an infimum, it is clear that

$$\Lambda^{**}(K) \leq \lambda^*(K).$$

To show that equality holds when $\tilde{k} \leq 1/L$, we consider

$$\rho_n(x) = \begin{cases} \epsilon & \text{if } 0 \leq x < L - \frac{1}{n} \\ n(K - \eta),\ \eta = \epsilon(L - \frac{1}{n}) & \text{if } L - \frac{1}{n} \leq x \leq L. \end{cases} \tag{5.1}$$

Let $\tilde{\phi}(x) = x$ be a test function in the Rayleigh quotient expression for $\lambda_0(\rho_n, 0)$. Then

$$\begin{aligned} \Lambda^{**}(K) \leq \lambda^*(K) \leq \lambda(\rho_n, 0) &\leq \frac{\tilde{k}\tilde{\phi}^2(L) + \int_0^L \tilde{\phi}'(s)^2\,ds}{\int_0^L \rho_n(s)\tilde{\phi}^2(s)\,ds} \\ &= \frac{\tilde{k}L^2 + L}{\int_0^{L-1/n} \epsilon s^2\,ds + \int_{L-1/n}^L n(K - \eta)s^2\,ds}. \end{aligned} \tag{5.2}$$

Letting $\epsilon \to 0$ and then $n \to \infty$ in (5.2) yields

$$\lambda^*(K) \leq (1 + \tilde{k}L)/KL \tag{5.3}$$

which demonstrates that

$$\Lambda^{**}(K) = \lambda^*(K) = (1 + \tilde{k}L)/KL, \quad \tilde{k} \leq 1/L \tag{5.4}$$

For the case $\tilde{k} > 1/L$, we use as test function (3.7) and take ρ_n as

$$\rho_n(x) = \begin{cases} \epsilon & \text{if } x \notin [\bar{x} - \frac{1}{n}, \bar{x}] \\ n(K - \eta) & \text{if } x \in [\bar{x} - \frac{1}{n}, \bar{x}]. \end{cases} \tag{5.5}$$

where $\bar{x}$ is an is (3.7). The calculation proceeds as in the case $\tilde{k} \leq 1/L$.

6 The constraint $K = m + \int_0^L \rho^\alpha(x)\, dx,\ \alpha > 1$

We now consider the problem (2.4)–(2.6) subject to the constraint, for some $\alpha > 1$,

$$K = m + \int_0^L \rho^\alpha(x)\, dx. \tag{6.1}$$

Letting $\lambda_0(\rho, m)$ be the least eigenvalue of (2.4)–(2.6) as before we consider the minimization problem,

$$\lambda^*_\alpha(K) = \inf_{(\rho,m)\in\mathcal{A}_3} \lambda_0(\rho, m),$$

where

$$\mathcal{A}_3 = \{(\rho, m) : \rho(x) > 0 \text{ a.e.},\ m \geq 0,\ \rho \in \mathcal{L}^\alpha(0, L),\ K = m + \int_0^L \rho^\alpha(s)\, ds\}.$$

We first establish existence of an optimal design pair (ρ, m) that minimizes the least eigenvalue subject to our new constraint (6.1). We will use Calculus of Variations to characterize our design pair and investigate the optimality conditions numerically.

Theorem 6.1. *There exists $(\rho_0, m_0) \in \mathcal{A}_3$ such that*

$$\lambda^*_\alpha(K) = \lambda_0(\rho_0, m_0). \tag{6.2}$$

Proof. Let $(\rho_n, m_n) \in \mathcal{A}_3$ be such that $\lambda_0(\rho_n, m_n) \to \lambda^*_\alpha(K)$ as $n \to \infty$. Since m_n, $||\rho_n||_\alpha$ are bounded by the constraint in $\mathcal{A}_3$, the sequence $\{m_n\}$ contains a convergent subsequence, and the sequence $\{\rho_n\}$ contains a weakly convergent subsequence. Without loss of generality, we assume

$$m_n \to m_0, \quad \rho_n \to \rho_0 \text{ (weakly) as } n \to \infty.$$

Let ϕ_n be the eigenfunction of (2.4)–(2.6) corresponding to (ρ_n, m_n) normalized by $\phi'(0) = 1$ i.e.,

$$\phi_n''(x) = -\lambda_0(\rho_n, m_n)\rho_n(x)\phi_n(x),\ \phi(0) = 0,\ \phi'(0) = 1, \quad 0 \le x < L.$$

Define the functions $Q_n(x),\ Q(x)$ by

$$Q_n(x) = \begin{cases} \lambda_0(\rho_n, m_n) \int_0^x \rho_n(s)\, ds, & \text{if } 0 \le x < L, \\ \lambda_0(\rho_n, m_n)K, & \text{if } x = L, \end{cases}$$

$$Q(x) = \begin{cases} \lambda_0(\rho_0, m_0) \int_0^x \rho_0(s)\, ds, & \text{if } 0 \le x < L, \\ \lambda_0(\rho_0, m_0)K, & \text{if } x = L. \end{cases}$$

Now with $1/\alpha + 1/\beta = 1$,

$$||\rho_n|| = \int_0^L \rho_n(x)\, dx \le \left(\int_0^L \rho_n^\alpha(x)\, dx \right)^{1/\alpha} L^{1/\beta} \le K^{1/\alpha} L^{1/\beta}. \tag{6.3}$$

Applying (6.3) and the weak convergence of the ρ_n to

$$\begin{aligned} Q_n(x) - Q(x) =& [\lambda_0(\rho_n, m_n) - \lambda_0(\rho_0, m_0)] \int_0^x \rho_n(s)\, ds \\ &+ \lambda_0(\rho_0, m_0) \int_0^x [\rho_n(s) - \rho_0(s)]\, ds, \end{aligned}$$

we conclude that $Q_n(x) \to Q(x)$ as $n \to \infty$ for all x in $[0, L]$. The results of Battle [3, 4] now give that $\phi_n(x) \to \phi_0(x)$ uniformly on $[0, L]$, and $\phi_n'(x) \to \phi_0'(x)$ pointwise on $[0, L]$ as $n \to \infty$. The convergence of ϕ_n' also follows from

$$\phi_n'(x) = 1 - \int_0^x \lambda_0(\rho_n, m_n)\rho_n(s)\phi_n(s)\, ds, \quad 0 \le x < L,$$

which also implies $\{|\phi_n'(x)|\}$ is uniformly bounded on $[0, L]$. Now we let $n \to \infty$ in

$$\lambda_0(\rho_n, m_n) = \frac{\tilde{k}\phi_n^2(L) + \int_0^L \phi_n'(s)^2\, ds}{m_n\phi_n^2(L) + \int_0^L \rho_n(s)\phi_n^2(s)\, ds}. \tag{6.4}$$

The only problematic term is

$$\int_0^L \rho_n(s)\phi_n^2(s)\, ds = \int_0^L \rho_n(s)\phi_0^2(s)\, ds + \int_0^L \rho_n(s)[\phi_n^2(s) - \phi_0^2(s)]\, ds.$$

The weak convergence of the ρ_n, the uniform convergence of the ϕ_n, and the uniform boundedness on $||\rho_n||$ allows to take the limit inside the integral for this term as well. Thus

$$\lim_{n\to\infty} \lambda_0(\rho_n, m_n) = \lambda_0(\rho_0, m_0)$$

which completes the proof. □

Having established existence of the optimal design over the class $\mathcal{A}_3$, we now obtain necessary conditions for optimality using the Calculus of Variations techniques.

Recall that the first eigenvalue of (2.4)–(2.6) is given by the Rayleigh quotient in (2.7)

$$\lambda_0(\rho, m) = \frac{\tilde{k}\phi^2(L) + \int_0^L \phi'(s)^2\, ds}{m\phi^2(L) + \int_0^L \rho(s)\phi^2(s)\, ds}.$$

where ϕ is the eigenfunction corresponding to $\lambda_0(\rho, m)$. When $(\rho, m) \in \mathcal{A}_3$ the first eigenvalue $\lambda_0(\rho, m)$ exists, is real and isolated because we have a discrete set of eigenvalues. Consider the functional

$$\mathcal{F}(\rho, m) = \lambda_0(\rho, m) + \mu \left[m + \int_0^L \rho^\alpha(x)\, dx - K \right]. \tag{6.5}$$

Minimizing $\mathcal{F}$ with respect to ρ and m is equivalent to solving our constrained problem. Note that

$$
\begin{aligned}
\frac{\partial \mathcal{F}}{\partial m} &= \frac{\partial}{\partial m}\left(\lambda_0(\rho, m)\right) + \mu \\
&= \frac{\partial}{\partial m}\left(\frac{\tilde{k}\phi^2(L) + \int_0^L \phi'(s)^2\, ds}{m\phi^2(L) + \int_0^L \rho(s)\phi^2(s)\, ds}\right) + \mu \\
&= \frac{-\left(\tilde{k}\phi^2(L) + \int_0^L \phi'(s)^2\, ds\right)\phi^2(L)}{\left(m\phi^2(L) + \int_0^L \rho(s)\phi^2(s)\, ds\right)^2} + \mu \\
&= \frac{-\lambda_0(\rho, m)\phi^2(L)}{m\phi^2(L) + \int_0^L \rho(s)\phi^2(s)\, ds} + \mu
\end{aligned}
\tag{6.6}
$$

and

$$
\begin{aligned}
\frac{\partial \mathcal{F}}{\partial \rho} &= \frac{\partial}{\partial \rho}\left(\lambda_0(\rho, m)\right) + \mu\left(\int_0^L \alpha\rho^{\alpha-1}(x)\, dx\right) \\
&= \frac{-\lambda_0(\rho, m)\int_0^L \phi^2(s)\, ds}{m\phi^2(L) + \int_0^L \rho(s)\phi^2(s)\, ds} + \mu\left(\int_0^L \alpha\rho^{\alpha-1}(x)\, dx\right).
\end{aligned}
\tag{6.7}
$$

The necessary conditions for optimality are $\frac{\partial \mathcal{F}}{\partial \rho} = \frac{\partial \mathcal{F}}{\partial m} = 0$. These are satisfied by

$$
\mu = \frac{\lambda_0(\rho, m)\phi^2(L)}{m\phi^2(L) + \int_0^L \rho(s)\phi^2(s)\, ds}
$$

and

$$
\rho(x) = \left(\frac{\phi^2(x)}{\alpha\phi^2(L)}\right)^{1/(\alpha-1)}
\tag{6.8}
$$

Using (6.8) in the original Sturm–Liouville problem (2.4)–(2.6), we find that our optimal design (ρ_0, m_0) satisfies the nonlinear system

$$
\begin{cases}
\phi''(x) = -\lambda\left(\dfrac{\phi^2(x)}{\alpha\phi^2(L)}\right)^{1/(\alpha-1)}\phi(x), \qquad 0 \le x < L. \\
\phi(0) = 0 \\
\tilde{k}\phi(L) + \phi'(L) = \lambda m\phi(L), \qquad \tilde{k} = k/T \\
m = K - \displaystyle\int_0^L \left(\frac{\phi^2(x)}{\alpha\phi^2(L)}\right)^{\alpha/(\alpha-1)} dx.
\end{cases}
\tag{6.9}
$$

Without loss of generality, we can assume that $\phi'(0) = 1$. We find the least eigenvalue by using a shooting method to find the first zero of the function

$$f(\lambda) = \tilde{k}\phi(L;\lambda) + \phi'(L;\lambda) - m\lambda\phi(L;\lambda).$$

We computed several numerical examples and include just a few here for illustrative purposes. We expect that as $\alpha \to 1+$ and $\tilde{k} \le 1/L$, our optimal density will approach the point-mass case discussed in Examples 3.1. In particular, we expect that all of the mass will be located at $x = L$, the string will have no mass and the eigenvalue will satisfy $\lambda = (1+\tilde{k}L)/KL$. For $K = L = 1$ and $\tilde{k} = 0.1$, we expect $m = 1$ and $\lambda = 1.1$. Figures 1–2 support this conclusion. For $K = L = 1$ and $\tilde{k} = 10$, however, the eigenvalue λ and mass m seem to approach $m = 1$, $\lambda = 11$ which is not the value of $\lambda^*(1) = 4\tilde{k}/K(1 + \tilde{k}L) = 40/11$. Figures 3–4 support this conclusion. Furthermore, we see in Figure 5 that the density of the string $\rho(x)$ approaches zero on the interval $[0, L)$ as $\alpha \to 1+$, demonstrating that all of the mass will be concentrated as a point mass at $x = L$.

For $0 < \alpha < 1$, it turns out that $\lambda^*(K) = \Lambda^*(K) = 0$, and there is no minimizer as $\lambda_0(\rho, m) > 0$ for all $(\rho, m) \in \mathcal{A}_1$. To see that $\lambda^*(K) = 0$, let $0 < \delta < 1$, $L = 1$, and set

$$\rho(x) = \begin{cases} 0 & \text{if } 0 \le x < 1-\delta, \\ (K/\delta)^{1/\alpha}, & \text{if } 1-\delta \le x \le 1. \end{cases}$$

With $m = 0$, and using $\phi(x) = x$ as a test function, it follows that

$$\lambda_0(\rho, 0) \le \frac{\tilde{k}\phi^2(1) + \int_0^1 \phi'(s)^2\,ds}{m\phi^2(1) + \int_0^1 \rho(s)\phi^2(s)\,ds} = \frac{\tilde{k}+1}{K\delta^{(\alpha-1)/\alpha}[1 - \delta + \delta^2/3]}. \tag{6.10}$$

The right hand side of (6.10) tends to 0 as $\delta \to 0+$ implying $\lambda^*(K) = 0$. The situation here is analogous to the Dirichlet problem $\phi'' = -\lambda\rho\phi$, $\phi(0) = \phi(1) = 0$ with $\int_0^1 \rho(s)^\alpha\,ds = K$, $0 < \alpha < 1$. This problem is discussed in section 5.2 of [6].

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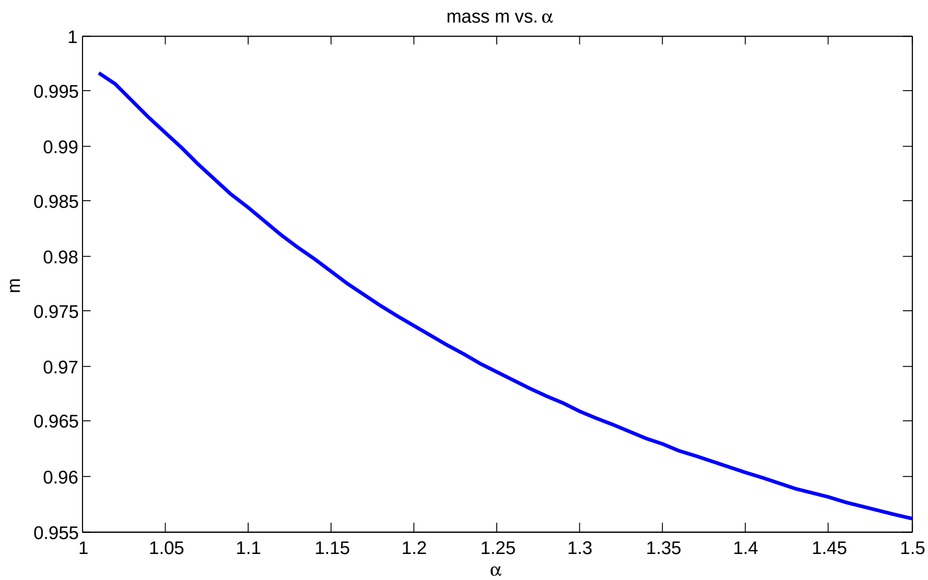

Figure 1: Mass m as $\alpha \to 1$ for $K = L = 1$ and $\tilde{k} = 0.1$

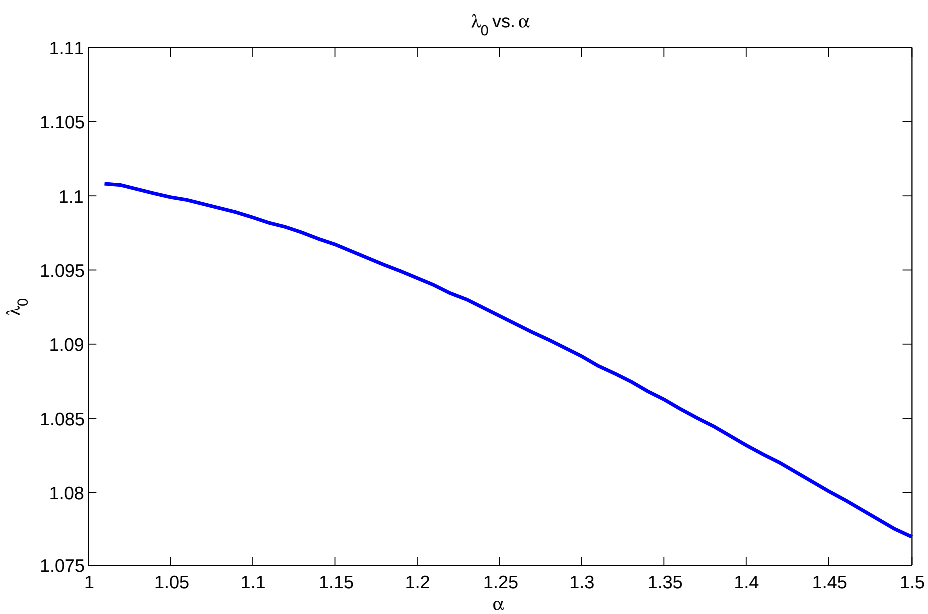

Figure 2: Least eigenvalue λ_0 as $\alpha \to 1$ for $K = L = 1$ and $\tilde{k} = 0.1$

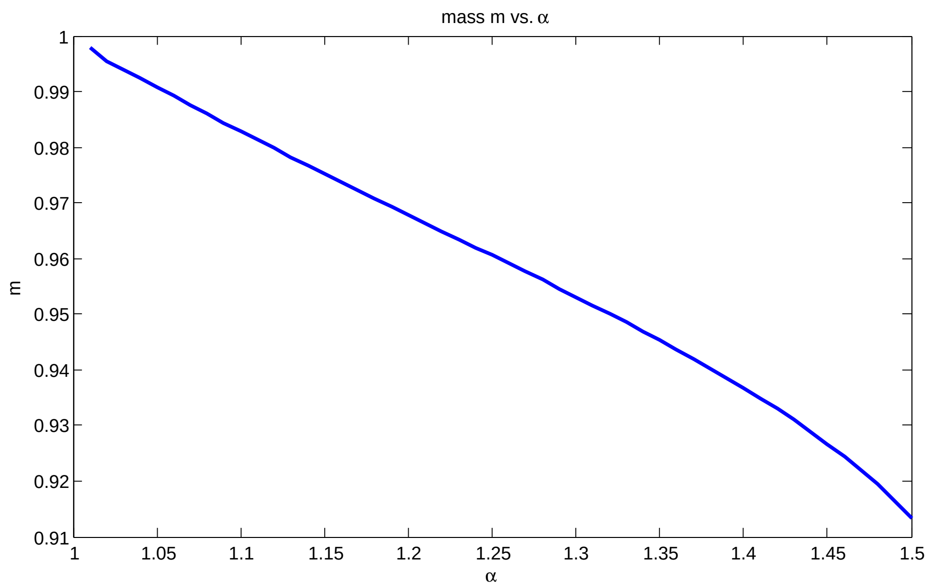

Figure 3: Mass m as $\alpha \to 1$ for $K = L = 1$ and $\tilde{k} = 10$.

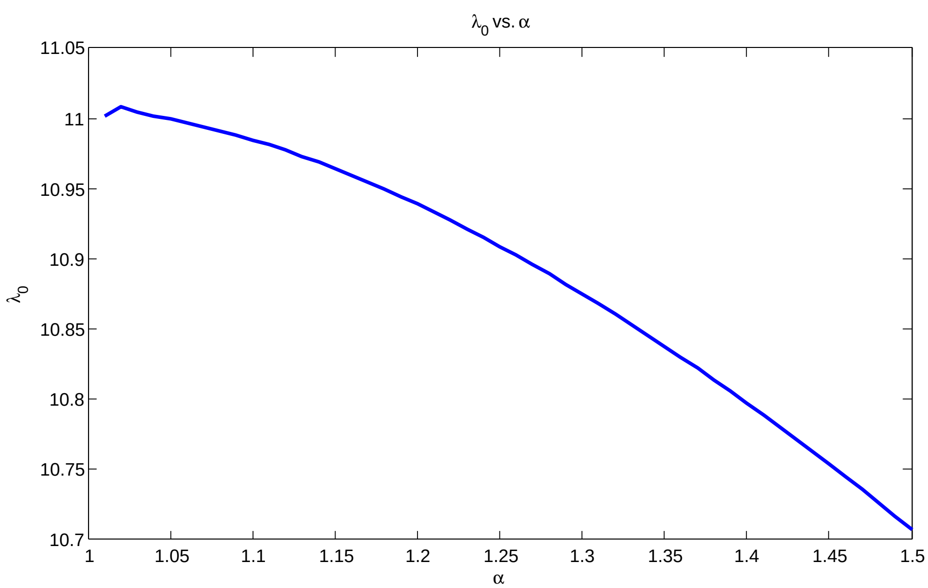

Figure 4: Least eigenvalue λ_0 as $\alpha \to 1$ for $K = L = 1$ and $\tilde{k} = 10$.

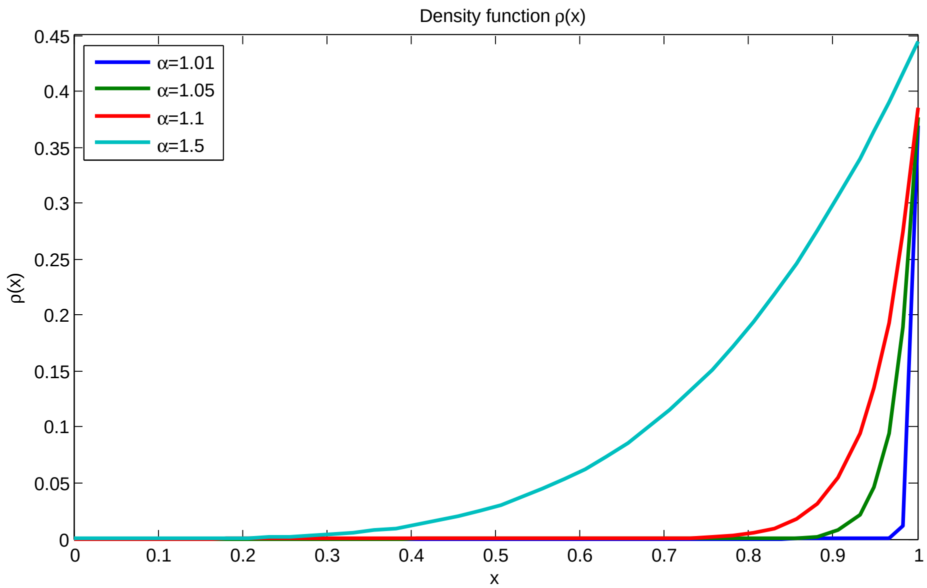

Figure 5: Comparison of densities for $\alpha = 1.01, 1.05, 1.1, 1.5$ for $k = 0.1$ show that the point mass case is approached when α is close to 1.

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