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Mild solutions for perturbed evolution equations with infinite state-dependent delay

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Abstract

In this paper, we give sufficient conditions to get the existence of mild solutions for two classes of first order partial and neutral of perturbed evolution equations by using the nonlinear alternative of Avramescu for contractions operators in Fréchet spaces, combined with semigroup theory. The solution here is depending on an infinite delay and is giving on the real half-line.

Keywords: Perturbed semilinear functional equations, neutral problem, mild solution, state-dependent delay, fixed point, nonlinear alternative, semigroup theory, Fréchet spaces, infinite delay.

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1 Introduction

In this paper, we give the existence of mild solutions defined on a semi-infinite positive real interval $J := [0, +\infty)$ for two classes of first order of semilinear functional and neutral functional perturbed evolution equations with state-dependent delay in a real separable Banach space $(E, |\cdot|)$ when the delay is infinite.

Firstly, we present some preliminary concepts and results in Section 2 and then in Section 3 we study the following semilinear functional perturbed evolution equations with state-dependent delay

$$y'(t) = A(t)y(t) + f(t, y_{\rho(t,y_t)}) + h(t, y_{\rho(t,y_t)}), \quad \text{a.e. } t \in J, \tag{1}$$

$$y_0 = \phi \in \mathcal{B}, \tag{2}$$

where $\mathcal{B}$ is an abstract *phase space* to be specified later, $f, h : J \times \mathcal{B} \to E$, $\rho : J \times \mathcal{B} \to \mathrm{I\!R}$ and $\phi \in \mathcal{B}$ are given functions and $\{A(t)\}_{0 \le t < +\infty}$ is a family of linear closed (not necessarily bounded) operators from E into E that generate an evolution system of operators $\{U(t, s)\}_{(t,s) \in J \times J}$ for $0 \le s \le t < +\infty$.

For any continuous function y and any $t \le 0$, we denote by y_t the element of $\mathcal{B}$ defined by $y_t(\theta) = y(t + \theta)$ for $\theta \in (-\infty, 0]$. Here $y_t(\cdot)$ represents the history of the

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state from time $t \leq 0$ up to the present time t. We assume that the histories y_t belong to $\mathcal{B}$.

Then, in Section 4, we consider the following neutral functional differential perturbed evolution equation with infinite delay

$$\frac{d}{dt}[y(t) - g(t, y_{\rho(t,y_t)})] = A(t)y(t) + f(t, y_{\rho(t,y_t)}) + h(t, y_{\rho(t,y_t)}), \quad \text{a.e.} \quad t \in J, \tag{3}$$

$$y_0 = \phi \in \mathcal{B}, \tag{4}$$

where $A(\cdot)$, f and ϕ are as in problem $(1)-(2)$ and $g : J \times \mathcal{B} \to E$ is a given function. Finally in Section 5, two examples are given to illustrate the abstract theory.

Differential delay equations, or functional differential equations, have been used in modeling scientific phenomena for many years. Often, it has been assumed that the delay is either a fixed constant or is given as an integral in which case is called distributed delay; see for instance the books [20, 23, 30], and the papers [15, 19].

An extensive theory is developed for evolution equations [3, 4, 17]. Uniqueness and existence results have been established for different evolution problems in the papers by Baghli and Benchohra in [2], [8]–[14]. Recently, Wang *et al.* look for nonlinear fractional impulsive evolution equations in [26]–[29].

However, complicated situations in which the delay depends on the unknown functions have been proposed in modeling in recent years. These equations are frequently called equations with state-dependent delay. Existence results and among other things were derived recently for functional differential equations when the solution is depending on the delay on a bounded interval for impulsive problems. We refer the reader to the papers by Abada *et al.* [1], Ait Dads and Ezzinbi [5], Anguraj *et al.* [6], Hernández *et al.* [21] and Li *et al.* [24].

Our main purpose in this paper is to extend some results from the cite literature devoted to state-dependent delay and those considered on a bounded interval for the evolution problems studied in [14]. We provide sufficient conditions for the existence of mild solutions on a semiinfinite interval $J = [0, +\infty)$ for the two classes of first order semilinear functional and neutral functional perturbed evolution equations with state-dependent delay $(1)-(2)$ and $(3)-(4)$ with *state-dependent delay* when the delay is infinite using the nonlinear alternative of Avramescu for contractions maps in Fréchet spaces [7], combined with semigroup theory [4, 25].

2 Preliminaries

We introduce notations, definitions and theorems which are used throughout this paper.

Let $C([0, +\infty); E)$ be the space of continuous functions from $[0, +\infty)$ into E and $B(E)$ be the space of all bounded linear operators from E into E, with the usual supremum norm

$$\|N\|_{B(E)} = \sup \{ \ |N(y)| \ : \ |y| = 1 \ \}, \quad N \in B(E).$$

A measurable function $y : [0, +\infty) \to E$ is Bochner integrable if and only if $|y|$ is Lebesgue integrable. (For the Bochner integral properties, see the classical monograph of Yosida [31]).

Let $L^1([0, +\infty), E)$ denotes the Banach space of measurable functions $y : [0, +\infty) \to E$ which are Bochner integrable normed by

$$\|y\|_{L^1} = \int_0^{+\infty} |y(t)| \, dt.$$

In this paper, we will employ an axiomatic definition of the phase space $\mathcal{B}$ introduced by Hale and Kato in [19] and follow the terminology used in [22]. Thus, $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ will be a seminormed linear space of functions mapping $(-\infty, 0]$ into E, and satisfying the following axioms.

(A_1) If $y : (-\infty, b) \to E, b > 0$, is continuous on $[0, b]$ and $y_0 \in \mathcal{B}$, then for every $t \in [0, b)$ the following conditions hold

(i) $y_t \in \mathcal{B}$;

(ii) There exists a positive constant H such that $|y(t)| \leq H\|y_t\|_{\mathcal{B}}$;

(iii) There exist two functions $K(\cdot), M(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ independent of y with K continuous and M locally bounded such that

$$\|y_t\|_{\mathcal{B}} \leq K(t) \sup\{\ |y(s)| : 0 \leq s \leq t\} + M(t)\|y_0\|_{\mathcal{B}}.$$

(A_2) For the function y in (A_1), y_t is a $\mathcal{B}-$valued continuous function on $[0, b]$.

(A_3) The space $\mathcal{B}$ is complete.

Denote $K_b = \sup\{K(t) : t \in [0, b]\}$ and $M_b = \sup\{M(t) : t \in [0, b]\}$.

Remark 2.1 1. *(ii) is equivalent to* $|\phi(0)| \leq H\|\phi\|_{\mathcal{B}}$ *for every* $\phi \in \mathcal{B}$.

2. *Since* $\|\cdot\|_{\mathcal{B}}$ *is a seminorm, two elements* $\phi, \psi \in \mathcal{B}$ *can verify* $\|\phi - \psi\|_{\mathcal{B}} = 0$ *without necessarily* $\phi(\theta) = \psi(\theta)$ *for all* $\theta \leq 0$.

3. *From the equivalence of in the first remark, we can see that for all* $\phi, \psi \in \mathcal{B}$ *such that* $\|\phi - \psi\|_{\mathcal{B}} = 0$ *: We necessarily have that* $\phi(0) = \psi(0)$.

We now indicate some examples of phase spaces. For other details we refer, for instance to the book by Hino *et al.* [22].

Example 2.2 *Let*

BC *denote the space of bounded continuous functions defined from* $(-\infty, 0]$ *to* E;

BUC *denote the space of bounded uniformly continuous functions defined from* $(-\infty, 0]$ *to* E;

$$C^\infty := \left\{ \phi \in BC : \lim_{\theta \to -\infty} \phi(\theta) \text{ exist in } E \right\};$$

$$C^0 := \left\{ \phi \in BC : \lim_{\theta \to -\infty} \phi(\theta) = 0 \right\}, \text{ endowed with the uniform norm}$$

$$\|\phi\| = \sup\{|\phi(\theta)| : \ \theta \leq 0\}.$$

We have that the spaces BUC, C^∞ *and* C^0 *satisfy conditions* $(A_1) - (A_3)$. *However,* BC *satisfies* $(A_1), (A_3)$ *but* (A_2) *is not satisfied.*

Example 2.3 *The spaces* C_g, UC_g, C_g^∞ *and* C_g^0.

Let g *be a positive continuous function on* $(-\infty, 0]$. *We define*

$$C_g := \left\{ \phi \in C((-\infty, 0], E) : \frac{\phi(\theta)}{g(\theta)} \text{ is bounded on } (-\infty, 0] \right\};$$

$$C_g^0 := \left\{ \phi \in C_g : \lim_{\theta \to -\infty} \frac{\phi(\theta)}{g(\theta)} = 0 \right\}, \text{ endowed with the uniform norm}$$

$$\|\phi\| = \sup \left\{ \frac{|\phi(\theta)|}{g(\theta)} : \ \theta \leq 0 \right\}.$$

Then we have that the spaces C_g *and* C_g^0 *satisfy conditions* (A_3). *We consider the following condition on the function* g.

(g_1) *For all* $a > 0$, $\displaystyle\sup_{0 \leq t \leq a} \sup \left\{ \frac{g(t+\theta)}{g(\theta)} : -\infty < \theta \leq -t \right\} < \infty.$

They satisfy conditions (A_1) *and* (A_2) *if* (g_1) *holds.*

Example 2.4 *The space* C_γ.

For any real constant γ, *we define the functional space* C_γ *by*

$$C_\gamma := \left\{ \phi \in C((-\infty, 0], E) : \lim_{\theta \to -\infty} e^{\gamma\theta} \phi(\theta) \text{ exists in } E \right\}$$

endowed with the following norm

$$\|\phi\| = \sup\{e^{\gamma\theta} |\phi(\theta)| : \ \theta \leq 0\}.$$

Then in the space C_γ *the axioms* $(A_1) - (A_3)$ *are satisfied.*

Definition 2.5 *A function $f : J \times \mathcal{B} \to E$ is said to be an L^1 -Carathéodory function if it satisfies*

(i) for each $t \in J$ the function $f(t, .) : \mathcal{B} \to E$ is continuous ;

(ii) for each $y \in \mathcal{B}$ the function $f(., y) : J \to E$ is measurable ;

(iii) for every positive integer k there exists $h_k \in L^1(J; \mathbb{R}^+)$ such that

$$|f(t, y)| \leq h_k(t) \quad \text{for all } \|y\|_{\mathcal{B}} \leq k \quad \text{and almost every } t \in J.$$

In what follows, we assume that $\{A(t)\}_{t\geq 0}$ is a family of closed densely defined linear unbounded operators on the Banach space E and with domain $D(A(t))$ independent of t.

Definition 2.6 *A family $\{U(t, s)\}_{(t,s)\in\Delta}$ of bounded linear operators $U(t, s) : E \to E$ where $(t, s) \in \Delta := \{(t, s) \in J \times J : 0 \leq s \leq t < +\infty\}$ is called an evolution system if the following properties are satisfied:*

1. *$U(t, t) = I$ where I is the identity operator in E,*
2. *$U(t, s)\ U(s, \tau) = U(t, \tau)$ for $0 \leq \tau \leq s \leq t < +\infty$,*
3. *$U(t, s) \in B(E)$ the space of bounded linear operators on E, where for every $(t, s) \in \Delta$ and for each $y \in E$, the mapping $(t, s) \to U(t, s)\ y$ is continuous.*

More details on evolution systems and their properties could be found on the books of Ahmed [3], Engel and Nagel [16] and Pazy [25].

Let X be a Fréchet space with a family of semi-norms $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$. We assume that the family of semi-norms $\{\|\cdot\|_n\}$ verifies

$$\|x\|_1 \leq \|x\|_2 \leq \ \|x\|_3 \leq ... \ \text{ for every } x \in X.$$

Let $Y \subset X$, we say that Y is bounded if for every $n \in \mathbb{N}$, there exists $\overline{M}_n > 0$ such that

$$\|y\|_n \leq \overline{M}_n \ \text{ for all } \ y \in Y.$$

To X we associate a sequence of Banach spaces $\{(X^n, \|\cdot\|_n)\}$ as follows : For every $n \in \mathbb{N}$, we consider the equivalence relation $\sim_n$ defined by : $x \sim_n y$ if and only if $\|x - y\|_n = 0$ for $x, y \in X$. We denote $X^n = (X|_{\sim_n}, \|\cdot\|_n)$ the quotient space, the completion of X^n with respect to $\|\cdot\|_n$. To every $Y \subset X$, we associate a sequence $\{Y^n\}$ of subsets $Y^n \subset X^n$ as follows : For every $x \in X$, we denote $[x]_n$ the equivalence class of x of subset X^n and we defined $Y^n = \{[x]_n : x \in Y\}$. We denote $\overline{Y^n}$, $int_n(Y^n)$ and $\partial_n Y^n$, respectively, the closure, the interior and the boundary of Y^n with respect to $\|\cdot\|_n$ in X^n.

The following definition is the appropriate concept of contraction in X.

Definition 2.7 *[18] A function $f : X \to X$ is said to be a contraction if for each $n \in \mathbb{N}$ there exists $k_n \in (0,1)$ such that*

$$\|f(x) - f(y)\|_n \leq k_n \ \|x - y\|_n \quad \textit{for all } x, y \in X.$$

The corresponding nonlinear alternative result is as follows

Theorem 2.8 *(Nonlinear Alternative of Avramescu, [7]). Let X be a Fréchet space and let $A, B : X \longrightarrow X$ be two operators satisfying*

(1) *A is a compact operator.*

(2) *B is a contraction.*

Then one of the following statements holds

$(Av1)$ *The operator $A + B$ has a fixed point;*

$(Av2)$ *The set $\{x \in X, x = \lambda A(x) + \lambda B(\frac{x}{\lambda})\}$ is unbounded for $\lambda \in]0,1[$.*

3 Semilinear evolution equations

Before stating and proving the main result, we give first the definition of a mild solution of the semilinear perturbed evolution problem $(1) - (2)$.

Definition 3.1 *We say that the function $y : \mathbb{R} \to E$ is a mild solution of $(1) - (2)$ if $y(t) = \phi(t)$ for all $t \leq 0$ and y satisfies the following integral equation*

$$y(t) = U(t,0)\ \phi(0) + \int_0^t U(t,s)\ [f(s, y_{\rho(s,y_s)}) + h(s, y_{\rho(s,y_s)})]\ ds \qquad a.e.\ t \in J. \tag{5}$$

Set

$$\mathcal{R}(\rho^-) = \{\rho(s,\phi) : (s,\phi) \in J \times \mathcal{B}, \rho(s,\phi) \leq 0\}.$$

We always assume that $\rho : J \times \mathcal{B} \to \mathbb{R}$ is continuous. Additionally, we introduce the following hypothesis

(H_ϕ) The function $t \to \phi_t$ is continuous from $\mathcal{R}(\rho^-)$ into $\mathcal{B}$ and there exists a continuous and bounded function $\mathcal{L}^\phi : \mathcal{R}(\rho^-) \to (0, \infty)$ such that

$$\|\phi_t\|_{\mathcal{B}} \leq \mathcal{L}^\phi(t) \|\phi\|_{\mathcal{B}} \quad \text{for every } t \in \mathcal{R}(\rho^-).$$

Remark 3.2 *The condition (H_ϕ), is frequently verified by continuous and bounded functions. For more details, see for instance [22].*

We will need to introduce the following hypotheses which are assumed thereafter

$(H0)$ $U(t,s)$ is compact for $t-s>0$.

$(H1)$ There exists a constant $\widehat{M}\geq 1$ such that

$$\|U(t,s)\|_{B(E)} \leq \widehat{M} \quad \text{for every } (t,s)\in\Delta.$$

$(H2)$ There exists a function $p\in L^1_{loc}(J;\mathbb{R}_+)$ and a continuous nondecreasing function $\psi:\mathbb{R}_+\to(0,\infty)$ and such that

$$|f(t,u)|\leq p(t)\ \psi(\|u\|_{\mathcal{B}}) \text{ for a.e. } t\in J \text{ and each } u\in\mathcal{B}.$$

$(H3)$ For all $R>0$, there exists $l_R\in L^1_{loc}(J;\mathbb{R}_+)$ such that

$$|f(t,u)-f(t,v)|\leq l_R(t)\ \|u-v\|_{\mathcal{B}}$$

for all $u,v\in\mathcal{B}$ with $\|u\|_{\mathcal{B}}\leq R$ and $\|v\|_{\mathcal{B}}\leq R$.

$(H4)$ There exists a function $\eta\in L^1(J,\mathbb{R}_+)$ such that

$$|h(t,u)-h(t,v)|\leq \eta(t)\quad \|u-v\|_{\mathcal{B}} \quad \text{a.e. } t\in J \quad et \quad \forall u,v\in\mathcal{B}.$$

Consider the following space

$$B_{+\infty}=\left\{y:\mathbb{R}\to E : y|_{[0,T]} \text{ continuous for } T>0 \text{ and } y_0\in\mathcal{B}\right\},$$

where $y|_{[0,T]}$ is the restriction of y to the real compact interval $[0,T]$.

Let us fix $\tau>1$. For every $n\in\mathbb{N}$, we define in $B_{+\infty}$ the semi-norms by

$$\|y\|_n := \sup\ \{\ e^{-\tau\ L_n^*(t)}\ |y(t)| : t\in[0,n]\ \},$$

where $L_n^*(t)=\displaystyle\int_0^t \bar{l}_n(s)\ ds$, $\bar{l}_n(t)=K_n\widehat{M}l_n(t)$ and l_n is the function from $(H3)$.

Then $B_{+\infty}$ is a Fréchet space with those family of semi-norms $\|\cdot\|_{n\in\mathbb{N}}$.

Lemma 3.3 *([21], Lemma 2.4)*
If $y:(-\infty,b]\to E$ is a function such that $y_0=\phi$, then

$$\|y_s\|_{\mathcal{B}}\leq (M_b+\mathcal{L}^\phi)\|\phi\|_{\mathcal{B}}+K_b\sup\{|y(\theta)|;\theta\in[0,max\{0,s\}]\},\ \ s\in\mathcal{R}(\rho^-)\cup J,$$

where $\mathcal{L}^\phi=\sup\limits_{t\in\mathcal{R}(\rho^-)}\mathcal{L}^\phi(t)$.

Proposition 3.4 *By (H_ϕ), Lemma 3.3 and the property (A_1), we have for each $t\in[0,n]$ and $n\in\mathbb{N}$*

$$\|y_{\rho(t,y_t)}\|\leq K_n|y(t)|+(M_n+\mathcal{L}^\varphi)\ \|y_0\|_{\mathcal{B}}$$

Theorem 3.5 *Assume that* (H_ϕ), $(H0)-(H2)$ *and* $(H4)$ *hold and moreover for all* $n \in \mathbb{N}$, *we have*

$$\int_{\sigma_n}^{+\infty} \frac{ds}{s+\psi(s)} > K_n \widehat{M} \int_0^n \max(p(s);\eta(s))\ ds. \tag{6}$$

with $\sigma_n = (M_n + \mathcal{L}^\phi + K_n \widehat{M} H)\|\phi\|_{\mathcal{B}} + K_n \widehat{M} \int_0^n \ |h(s,0)|\ ds$. *Then the problem* $(1)-(2)$ *has a mild solution on* $(-\infty,+\infty)$.

Proof. We transform the problem $(1)-(2)$ into a fixed-point problem. Consider the operator $N : B_{+\infty} \to B_{+\infty}$ defined by

$$N(y)(t) = \begin{cases} \phi(t), & \text{if } t \in \mathbb{R}^-; \\ U(t,0)\ \phi(0) + \displaystyle\int_0^t U(t,s)\ f(s, y_{\rho(s,y_s)})\ ds, & \\ \quad + \displaystyle\int_0^t U(t,s)\ h(s, y_{\rho(s,y_s)})\ ds, & \text{if } t \in J. \end{cases}$$

Clearly, fixed points of the operator N are mild solutions of the problem $(1)-(2)$.

For $\phi \in \mathcal{B}$, we will define the function $x(.) : \mathbb{R} \to E$ by

$$x(t) = \begin{cases} \phi(t), & \text{if } \ t \le 0; \\ U(t,0)\ \phi(0), & \text{if } \ t \in J. \end{cases}$$

Then $x_0 = \phi$. For each function $z \in B_{+\infty}$, set

$$y(t) = z(t) + x(t)$$

It is obvious that y satisfies (5) if and only if z satisfies $z_0 = 0$ and

$$\begin{aligned} z(t) \ &= \ \int_0^t U(t,s)\ f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})\ ds \\ &\quad + \int_0^t U(t,s)\ h(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})\ ds. \end{aligned}$$

Let

$$B^0_{+\infty} = \{ z \in B_{+\infty} : z_0 = 0 \}.$$

Define the operators $F, G : B^0_{+\infty} \to B^0_{+\infty}$ by

$$F(z)(t) = \int_0^t U(t,s)\ f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})\ ds$$

and

$$G(z)(t)=\int_0^t U(t,s)\ h(s,z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)})\ ds.$$

Obviously the operator N having a fixed point is equivalent to $F+G$ having one, so it turns to prove that $F+G$ has a fixed point.

First, show that F is continuous and compact.

Step 1 : First, we show the continuity of F. Let $(z_n)_{n\in\mathbb{N}}$ be a sequence in $B^0_{+\infty}$ such that $z_n\to z$ in $B^0_{+\infty}$. By the hypothesis $(H1)$, we have

$$\begin{aligned}|F(z_n)(t)-F(z)(t)| &\leq \int_0^t \|U(t,s)\|_{B(E)}\times\\ &\quad\times|f(s,z_{n\rho(s,z_{ns}+x_s)}+x_{\rho(s,z_{ns}+x_s)})-f(s,z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)})|ds\\ &\leq \widehat{M}\int_0^t \left|f(s,z_{n\rho(s,z_{ns}+x_s)}+x_{\rho(s,z_{ns}+x_s)})-f(s,z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)})\right|\ ds.\end{aligned}$$

Since f is continuous, by dominated convergence theorem of Lebesgue, we get

$$|F(z_n)(t)-F(z)(t)|\longrightarrow 0\ \text{ if }\ n\longrightarrow+\infty$$

So F is continuous.

Step 2 : Show that F transforms any bounded of $B^0_{+\infty}$ in a bounded set. For each $d>0$, there exists a positive constant ξ such that for all $z\in B_d=\{z\in B^0_{+\infty}\ :\|z\|_n\leq d\}$ we get $\|F(z)\|_n\leq\xi$. Let $z\in B_d$, from assumption $(H1)$ and $(H2)$, we have for each $t\in[0,n]$

$$\begin{aligned}|F(z)(t)| &\leq \int_0^t \|U(t,s)\|_{B(E)}\ |f(s,z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)})|\ ds\\ &\leq \widehat{M}\int_0^t p(s)\ \psi(\|z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}})\ ds.\end{aligned}$$

From (H_ϕ), Lemma 3.3 and assumption (A_1), we have for each $t\in[0,n]$

$$\begin{aligned}\|z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}} &\leq \|z_{\rho(s,z_s+x_s)}\|_{\mathcal{B}}+\|x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}}\\ &\leq K_n|z(s)|+(M_n+\mathcal{L}^\phi)\|z_0\|_{\mathcal{B}}\\ &\qquad+K_n|x(s)|+(M_n+\mathcal{L}^\phi)\|x_0\|_{\mathcal{B}}\\ &\leq K_n|z(s)|+K_n\|U(s,0)\|_{B(E)}|\phi(0)|+(M_n+\mathcal{L}^\phi)\|\phi\|_{\mathcal{B}}\\ &\leq K_n|z(s)|+K_n\widehat{M}|\phi(0)|+(M_n+\mathcal{L}^\phi)\|\phi\|_{\mathcal{B}}\end{aligned}$$

Using (ii), we get

$$\begin{aligned}\|z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}} &\leq K_n|z(s)|+K_n\widehat{M}H\|\phi\|_{\mathcal{B}}+(M_n+\mathcal{L}^\phi)\|\phi\|_{\mathcal{B}}\\ &\leq K_n|z(s)|+(M_n+\mathcal{L}^\phi+K_n\widehat{M}H)\|\phi\|_{\mathcal{B}}\end{aligned}$$

Set $c_n := (M_n + \mathcal{L}^\phi + K_n\widehat{M}H)\|\phi\|_{\mathcal{B}}$ and $\delta_n := K_n d + c_n$. Then

$$\|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}} \leq K_n|z(s)| + c_n \leq \delta_n. \tag{7}$$

Using the nondecreasing character of ψ, we get for each $t \in [0, n]$

$$|F(z)(t)| \leq \widehat{M}\psi(\delta_n)\|p\|_{L^1} := \varrho.$$

So there is a positive constant ϱ such that $\|F(z)\|_n \leq \varrho$. Then $F(B_d) \subset B_\varrho$.

Step 3 : F maps bounded sets into equicontinuous sets of $B^0_{+\infty}$. We consider B_d as in Step 2 and we show that $F(B_d)$ is equicontinuous. Let $\tau_1, \tau_2 \in J$ with $\tau_1 < \tau_2$ and $z \in B_d$.

$$\begin{aligned} |F(z)(\tau_2) - F(z)(\tau_1)| \;\leq\; & \int_0^{\tau_1} \|U(\tau_2, s) - U(\tau_1, s)\|_{B(E)} \times \\ & \times|f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})|\, ds \\ & + \int_{\tau_1}^{\tau_2} \|U(\tau_2, s)\|_{B(E)}\, |f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})|\, ds. \end{aligned}$$

Then by (7) and the nondecreasing character of ψ, we get

$$\begin{aligned} |F(z)(\tau_2) - F(z)(\tau_1)| \;\leq\; & \psi(\delta_n)\int_0^{\tau_1} \|U(\tau_2, s) - U(\tau_1, s)\|_{B(E)}\, p(s)\, ds \\ & + \widehat{M}\psi(\delta_n)\int_{\tau_1}^{\tau_2} p(s)\, ds. \end{aligned}$$

Note that $|F(z)(\tau_2) - F(z)(\tau_1)| \longrightarrow 0$ as $\tau_2 - \tau_1 \longrightarrow 0$ independently of $z \in B_d$. The right-hand of the above inequality tends to zero as $\tau_2 - \tau_1 \longrightarrow 0$, since $U(t, s)$ is a strongly continuous operator and the compactness of $U(t, s)$ for $t > s$, implies the continuity in the uniform operator topology (see [4, 25]). As a consequence of Steps 1 to 3 together with the Arzelà-Ascoli theorem it suffices to show that the operator F maps B_d into a precompact set in E.

Let $t \in J$ be fixed and let ε be a real number satisfying $0 < \varepsilon < t$. For $z \in B_d$, we define

$$F_\varepsilon(z)(t) = U(t, t - \varepsilon)\int_0^{t-\varepsilon} U(t - \varepsilon, s)Cu_{z+x}(s)\, ds.$$

Since $U(t, s)$ is a compact operator, the set $Z_\varepsilon(t) = \{F_\varepsilon(z)(t) \; : z \in B_d\}$ is pre-compact in E for every ε,$0 < \varepsilon < t$. Moreover, using the definition of w, we get

$$|F(z)(t) - F_\varepsilon(z)(t)| \leq \int_{t-\varepsilon}^{t} \|U(t, s)\|_{B(E)}\, |f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})|\, ds$$

Therefore the set $Z(t) = \{F(z)(t) \; : z \in B_d\}$ is totally bounded. So we deduce from Steps 1, 2 and 3 that F is a compact operator.

Step 4 : G is a contraction. Let $z, \overline{z} \in B^0_{+\infty}$. By the hypotheses $(H1)$ and $(H4)$, we get for each $t \in [0, n]$ and $n \in \mathbb{N}$

$$\begin{aligned}
|G(z)(t) - G(\overline{z})(t)| &\leq \int_0^t \|U(t,s)\|_{B(E)} \times \\
&\quad \times \mid h(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}) - h(s, \overline{z}_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}) \mid ds \\
&\leq \widehat{M} \int_0^t \eta(s) \ \|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)} - \overline{z}_{\rho(s,z_s+x_s)} - x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}} \ ds \\
&\leq \widehat{M} \int_0^t \eta(s) \ \|z_{\rho(s,z_s+x_s)} - \overline{z}_{\rho(s,z_s+x_s)}\|_{\mathcal{B}} \ ds.
\end{aligned}$$

Use the inequality (7), to get

$$\begin{aligned}
|G(z)(t) - G(\overline{z})(t)| &\leq \int_0^t \widehat{M} \ K_n \ \eta(s) \ |z(s) - (\overline{z})(s)| \ ds \\
&\leq \int_0^t \left[\overline{L}_n(s) \ e^{\tau L^*_n(s)}\right] \left[e^{-\tau L^*_n(s)} \ |z(s) - \overline{z}(s)|\right] \ ds \\
&\leq \int_0^t \left[\frac{e^{\tau \ L^*_n(s)}}{\tau}\right]' \ ds \ \|z - \overline{z}\|_n \\
&\leq \frac{1}{\tau} \ e^{\tau \ L^*_n(t)} \ \|z - \overline{z}\|_n.
\end{aligned}$$

Therefore

$$\|G(z) - G(\overline{z})\|_n \leq \frac{1}{\tau} \ \|z - \overline{z}\|_n.$$

Then the operator G is a contraction for all $n \in \mathbb{N}$.

Step 5 : For applying Theorem (2.8), we must check $(Av2)$: i.e. it remains to show that the set

$$\Gamma = \left\{ z \in B^0_{+\infty} \ : z = \lambda F(z) + \lambda G\left(\frac{z}{\lambda}\right) \quad \text{for some } \lambda \in]0,1[\right\}.$$

is bounded.

Let $z \in \Gamma$. By $(H1) - (H2)$ and $(H4)$, we have for each $t \in [0, n]$

$$\begin{aligned}
\frac{1}{\lambda}|z(t)| &\leq \int_0^t \|U(t,s)\|_{B(E)} \ \left|f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})\right| \ ds \\
&\quad + \int_0^t \|U(t,s)\|_{B(E)} \ \left|h\left(s, \frac{z_{\rho\left(s, \frac{z_s}{\lambda}+x_s\right)}}{\lambda}\right) - h(s,0) + h(s,0)\right| \ ds \\
&\leq \widehat{M} \int_0^t p(s) \ \psi(\|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})\|_{\mathcal{B}} \ ds) \\
&\quad + \widehat{M} \int_0^t \eta(s) \ \left\|\frac{z_{\rho\left(s, \frac{z_s}{\lambda}+x_s\right)}}{\lambda}\right\|_{\mathcal{B}} \ ds + \widehat{M} \int_0^t |h(s,0)| \ ds.
\end{aligned}$$

Use Proposition (3.4) and inequality (7)

$$\begin{aligned}\frac{1}{\lambda}|z(t)| &\leq \widehat{M}\int_0^t p(s)\ \psi(K_n|z(s)|+c_n)\ ds\\ &\quad+\widehat{M}\int_0^t \eta(s)\ \left(\frac{K_n}{\lambda}|z(s)|+c_n\right)\ ds+\widehat{M}\int_0^t\ |h(s,0)|\ ds.\end{aligned}$$

We consider the function $u(t) := \sup\limits_{\theta\in[0,t]} |z(\theta)|$. The nondecreasing character of ψ gives with the fact that $0<\lambda<1$

$$\begin{aligned}\frac{K_n}{\lambda}u(t)+c_n &\leq c_n+K_n\widehat{M}\int_0^t p(s)\ \psi\left(\frac{K_n}{\lambda}u(s)+c_n\right)\ ds\\ &\quad+K_n\widehat{M}\int_0^t \eta(s)\ \left(\frac{K_n}{\lambda}u(s)+c_n\right)\ ds+\widehat{M}\int_0^t\ |h(s,0)|\ ds.\end{aligned}$$

Set $\sigma_n := c_n+K_n\widehat{M}\displaystyle\int_0^n\ |h(s,0)|\ ds$. Then, we have

$$\begin{aligned}\frac{K_n}{\lambda}u(t)+c_n &\leq \sigma_n+K_n\widehat{M}\int_0^t p(s)\ \psi\left(\frac{K_n}{\lambda}u(s)+c_n\right)\ ds\\ &\quad+K_n\widehat{M}\int_0^t \eta(s)\ \left(\frac{K_n}{\lambda}u(s)+c_n\right)\ ds.\end{aligned}$$

We consider the function μ defined by

$$\mu(t)=\sup\ \{K_nu(s)+c_n\ :\ 0\leq s\leq t\},\quad 0\leq t\leq+\infty.$$

Let $t^\star\in[0,t]$ be such that $\mu(t)=K_nu(t^\star)+c_n$. From the previous inequality, we have for all $t\in[0,n]$

$$\mu(t)\leq\sigma_n+K_n\widehat{M}\int_0^t p(s)\psi(\mu(s))ds+K_n\widehat{M}\int_0^t \eta(s)\mu(s)ds.$$

Let us take the right-hand side of the above inequality as $v(t)$. Then, we have

$$\mu(t)\leq v(t)\quad\forall t\in[0,n].$$

From the definition of v, we have

$$v(0)=\sigma_n\quad\text{and}\quad v'(t)=K_n\widehat{M}\left[p(t)\psi(\mu(t))+\eta(t)\mu(t)\right]\quad\text{a.e.}\quad t\in[0,n].$$

Using the nondecreasing character of ψ, we get

$$v'(t)\leq K_n\widehat{M}[p(t)\psi(v(t))+\eta(t)v(t)]\quad\text{a.e. } t\in[0,n]$$

So, using (6) for each $t \in [0,n]$, we get

$$\begin{aligned}\int_{\sigma_n}^{v(t)} \frac{ds}{s+\psi(s)} &\leq K_n \widehat{M} \int_0^t \max(p(s);\eta(s))\, ds \\ &\leq K_n \widehat{M} \int_0^n \max(p(s);\eta(s))\, ds \\ &< \int_{c_n}^{+\infty} \frac{ds}{s+\psi(s)}.\end{aligned}$$

Thus, for every $t \in [0,n]$, there exists a constant Λ_n such that $v(t) \leq \Lambda_n$ and hence $\mu(t) \leq \Lambda_n$. Since $\|z\|_n \leq \mu(t)$, we have $\|z\|_n \leq \Lambda_n$. This shows that the set Γ is bounded. Then the statement $(Av2)$ in Theorem 2.8 does not hold. The nonlinear alternative of Avramescu implies that $(Av1)$ is satisfied, we deduce that the operator $F+G$ has a fixed point $z^\star$. Then $y^\star(t) = z^\star(t) + x(t)$, $t \in]-\infty, +\infty[$ is the fixed point of the operator N which is a mild solution of the problem $(1)-(2)$.

4 Semilinear neutral evolution equations

In this section, we give an existence result for the problem $(3)-(4)$. Firstly we define the concept of the mild solution for that problem.

Definition 4.1 *We say that the function $y(\cdot): \mathbb{R} \to E$ is a mild solution of $(3)-(4)$ if $y(t) = \phi(t)$ for all $t \leq 0$ and y satisfies the following integral equation*

$$\begin{aligned} y(t) &= U(t,0)[\phi(0) - g(0,\phi)] + g(t, y_{\rho(t,y_t)}) + \int_0^t U(t,s)A(s)\ g(s, y_{\rho(s,y_s)})\, ds \\ &\quad + \int_0^t U(t,s)\ [f(s, y_{\rho(s,y_s)}) + h(s, y_{\rho(s,y_s)})]\, ds \qquad \forall t \in J. \end{aligned} \tag{8}$$

We consider the hypotheses (H_ϕ), $(H0)-(H2)$ and $(H4)$ and we will need the following assumptions

$(H5)$ There exists a constant $\overline{M}_0 > 0$ such that

$$\|A^{-1}(t)\|_{B(E)} \leq \overline{M}_0 \quad for\ all\ t \in J.$$

$(H6)$ There exists a constant $0 < L < \dfrac{1}{\overline{M}_0 K_n}$, such that

$$|A(t)\ g(t,\phi)| \leq L\ (\|\phi\|_{\mathcal{B}} + 1) \text{ for all } t \in J \text{ and } \phi \in \mathcal{B}.$$

$(H7)$ There exists a constant $L_* > 0$ such that

$$|A(s)\ g(s,\phi) - A(\overline{s})\ g(\overline{s}, \overline{\phi})| \leq L_*\ (|s - \overline{s}| + \|\phi - \overline{\phi}\|_{\mathcal{B}})$$

for all $s, \overline{s} \in J$ and $\phi, \overline{\phi} \in \mathcal{B}$.

$(H8)$ The function g is completely continuous and for any bounded set $Q \subset \mathcal{B}$ the set $\{t \longrightarrow g(t, x_{\rho(t,y_t)}) : x \in Q\}$ is equicontinuous in $C(J, E)$.

Theorem 4.2 *Suppose that hypotheses* (H_ϕ), $(H0)-(H2)$, $(H4)$ *and* $(H5)-(H8)$ *are satisfied and moreover for all* $n \in \mathbb{N}$, *we have*

$$\int_{\xi_n}^{+\infty} \frac{ds}{s+\psi(s)} > \frac{K_n \widehat{M}}{1-\overline{M}_0 L K_n} \int_0^n \max(L+\eta(s), p(s))\ ds \tag{9}$$

with $c_n = (M_n + \mathcal{L}^\phi + K_n \widehat{M} H)\|\phi\|_{\mathcal{B}}$ *and*

$$\xi_n = c_n + K_n \frac{\left(\widehat{M}+1\right)\overline{M}_0 L + \widehat{M} L n + \overline{M}_0 L \left(c_n + \widehat{M}\right)\|\phi\|_{\mathcal{B}} + \widehat{M}\int_0^n |h(s,0)|\ ds}{1-\overline{M}_0 L K_n}.$$

Then the problem $(3)-(4)$ *has a mild solution.*

Proof. Consider the operator $\widetilde{N} : B_{+\infty} \to B_{+\infty}$ defined by

$$\widetilde{N}(y)(t) = \begin{cases} \phi(t), & \text{if } t \in \mathbb{R}^-; \\ U(t,0)\ [\phi(0) - g(0,\phi)] + g(t, y_{\rho(t,y_t)}) & \\ \quad + \displaystyle\int_0^t U(t,s)A(s)\ g(s, y_{\rho(s,y_s)})\ ds, & \\ \quad + \displaystyle\int_0^t U(t,s)\ \left[f(s, y_{\rho(s,y_s)}) + h(s, y_{\rho(s,y_s)})\right]\ ds & \text{if } t \in J. \end{cases}$$

Then, fixed points of the operator $\widetilde{N}$ are mild solutions of the problem $(3)-(4)$.

For $\phi \in \mathcal{B}$, we consider the function $x(.) : \mathbb{R} \to E$ defined as below by

$$x(t) = \begin{cases} \phi(t), & \text{if } t \le 0; \\ U(t,0)\ \phi(0), & \text{if } t \in J. \end{cases}$$

Then $x_0 = \phi$. For each function $z \in B_{+\infty}$, set

$$y(t) = z(t) + x(t)$$

It is obvious that y satisfies (8) if and only if z satisfies $z_0 = 0$ and

$$\begin{aligned} z(t) &= g(t, z_{\rho(s,z_t+x_t)} + x_{\rho(s,z_t+x_t)}) - U(t,0)g(0,\phi) \\ &\quad + \int_0^t U(t,s)A(s)\ g(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})\ ds \\ &\quad + \int_0^t U(t,s)\ f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})\ ds \\ &\quad + \int_0^t U(t,s)\ h(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})\ ds. \end{aligned}$$

Let

$$B^0_{+\infty} = \{z \in B_{+\infty} : z_0 = 0\}.$$

Define the operator $F, \widetilde{G} : B^0_{+\infty} \to B^0_{+\infty}$ by

$$F(z)(t) = \int_0^t U(t,s)\ f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})\ ds$$

and

$$\begin{aligned}\widetilde{G}(z)(t) &= g(t, z_{\rho(s,z_t+x_t)} + x_{\rho(s,z_t+x_t)}) - U(t,0)g(0,\phi)\\ &+ \int_0^t U(t,s)A(s)\ g(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})\ ds\\ &+ \int_0^t U(t,s)\ h(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})\ ds.\end{aligned}$$

Obviously the operator $\widetilde{N}$ having a fixed point is equivalent to $F + \widetilde{G}$ having one, so it turns to prove that $F + \widetilde{G}$ has a fixed point.

We have shown that the operator F is continuous and compact as in Section 3. It remains to show that the operator $\widetilde{G}$ is a contraction.
Let $z, \overline{z} \in B^0_{+\infty}$. By $(H1)$, $(H4)$, $(H5)$ and $(H7)$, we have for each $t \in [0,n]$ and $n \in \mathbb{N}$

$$\begin{aligned}&|\widetilde{G}(z)(t) - \widetilde{G}(\overline{z})(t)| \leq\\ &\leq |g(t, z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)}) - g(t, \overline{z}_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)})| + \int_0^t \|U(t,s)\|_{B(E)} \times\\ &\quad \times |A(s)[g(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}) - g(s, \overline{z}_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})]|\ ds\\ &\quad + \int_0^t \|U(t,s)\|_{B(E)}\ |h(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}) - h(s, \overline{z}_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})|\ ds\\ &\leq \|A^{-1}(t)\|_{B(E)}\ |A(t)g(t, z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)}) - A(t)g(t, \overline{z}_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)})|\\ &\quad + \int_0^t \widehat{M}\ |A(s)g(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}) - A(s)g(s, \overline{z}_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})|\ ds\\ &\quad + \int_0^t \widehat{M}\ |h(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}) - h(s, \overline{z}_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})|\ ds\\ &\leq \overline{M_0} L_* \|z_{\rho(t,z_t+x_t)} - \overline{z}_{\rho(t,z_t+x_t)}\|_{\mathcal{B}}\\ &\quad + \int_0^t \widehat{M} L_* \|z_{\rho(s,z_s+x_s)} - \overline{z}_{\rho(s,z_s+x_s)}\|_{\mathcal{B}}\ ds\\ &\quad + \int_0^t \widehat{M}\eta(s) \|z_{\rho(s,z_s+x_s)} - \overline{z}_{\rho(s,z_s+x_s)}\|_{\mathcal{B}}\ ds.\end{aligned}$$

Use the inequality (7) to get

$$|\widetilde{G}(z)(t) - \widetilde{G}(\overline{z})(t)| \ \leq\ \overline{M_0} L_* K_n |z(t) - \overline{z}(t)| + \int_0^t \widehat{M} L_* K_n |z(s) - \overline{z}(s)|\ ds$$

$$
\begin{aligned}
&\quad + \int_0^t \widehat{M} K_n \ \eta(s)|z(s)-\overline{z}(s)| \ ds \\
&\leq \ \overline{M_0} L_* K_n |z(t)-\overline{z}(t)| + \int_0^t \widehat{M} K_n [L_* + \eta(s)] |z(s)-\overline{z}(s)| \ ds.
\end{aligned}
$$

Set $\bar{l}_n(t) = \widehat{M} K_n [L_* + \eta(t)]$ for the family of semi-norms $\{\| \cdot \|_{n \in \mathbb{N}}\}$, then

$$
\begin{aligned}
|\widetilde{G}(z)(t) - \widetilde{G}(\overline{z})(t)| &\leq \ \overline{M}_0 L_* K_n \ |z(t)-\overline{z}(t)| + \int_0^t \bar{l}_n(s) \ |z(s)-\overline{z}(s)| ds \\
&\leq \ \left[\overline{M}_0 L_* K_n \ e^{\tau \ L_n^*(t)}\right] \ \left[e^{-\tau \ L_n^*(t)} \ |z(t)-\overline{z}(t)|\right] \\
&\quad + \int_0^t \left[\bar{l}_n(s) \ e^{\tau \ L_n^*(s)}\right] \ \left[e^{-\tau \ L_n^*(s)} \ |z(s)-\overline{z}(s)|\right] ds \\
&\leq \ \overline{M}_0 L_* K_n \ e^{\tau \ L_n^*(t)} \ \|z-\overline{z}\|_n + \int_0^t \left[\frac{e^{\tau \ L_n^*(s)}}{\tau}\right]' \ ds \ \|z-\overline{z}\|_n \\
&\leq \ \left[\overline{M}_0 L_* K_n + \frac{1}{\tau}\right] \ e^{\tau \ L_n^*(t)} \ \|z-\overline{z}\|_n.
\end{aligned}
$$

Therefore

$$
\|\widetilde{G}(z) - \widetilde{G}(\overline{z})\|_n \leq \left[\overline{M_0} L_* K_n + \frac{1}{\tau}\right] \|z-\overline{z}\|_n.
$$

Let us fix $\tau > 0$ and assume that

$$
\overline{M}_0 L_* K_n + \frac{1}{\tau} < 1,
$$

then the operator $\widetilde{G}$ is a contraction for all $n \in \mathbb{N}$.

For applying Theorem (2.8), we must check $(Av2)$ i.e. it remains to show that the set

$$
\widetilde{\Gamma} = \left\{ z \in B^0_{+\infty} \quad : \ z = \lambda F(z) + \lambda \widetilde{G}\left(\frac{z}{\lambda}\right) \quad for \ 0 < \lambda < 1 \right\}
$$

is bounded.

Let $z \in \widetilde{\Gamma}$. By $(H1)-(H2)$, we have for each $t \in [0,n]$

$$
\begin{aligned}
\frac{|z(t)|}{\lambda} \ &\leq \ \|A^{-1}(t)\| |A(t) g(t, z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)})| + \widehat{M} \|A^{-1}(0)\| |A(0) g(0,\phi)| \\
&\quad + \widehat{M} \int_0^t |A(s) g(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})| \ ds \\
&\quad + \widehat{M} \int_0^t p(s) \ \psi\left(\left\| z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)} \right\|\right) \ ds \\
&\quad + \widehat{M} \int_0^t \left| h\left(s, \frac{z_{\rho\left(s, \frac{z_s}{\lambda} + x_s\right)}}{\lambda} + x_{\rho\left(s, \frac{z_s}{\lambda} + x_s\right)}\right) - h(s,0) \right| \ ds \\
&\quad + \widehat{M} \int_0^n |h(s,0)| \ ds.
\end{aligned}
$$

Using assumptions $(H5)-(H6)$ and $(H4)$

$$
\begin{aligned}
\frac{|z(t)|}{\lambda} \leq\ & \overline{M}_0 L(\|z_{\rho(t,z_t+x_t)}+x_{\rho(t,z_t+x_t)}\|_{\mathcal{B}}+1)+\widehat{M}\overline{M}_0 L(\|\phi\|_{\mathcal{B}}+1) \\
& +\widehat{M}L\int_0^t (\|z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}}+1)\ ds \\
& +\widehat{M}\int_0^t\ p(s)\ \psi(\|z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}})\ ds \\
& +\widehat{M}\int_0^t\ \eta(s)\ \left\|\frac{z_{\rho\left(s,\frac{z_s}{\lambda}+x_s\right)}}{\lambda}+x_{\rho\left(s,\frac{z_s}{\lambda}+x_s\right)}\right\|_{\mathcal{B}}\ ds \\
& +\widehat{M}\int_0^t\ |h(s,0)|\ ds \\
\leq\ & \left(\widehat{M}+1\right)\overline{M}_0 L+\widehat{M}Ln+\widehat{M}\overline{M}_0 L\|\phi\|_{\mathcal{B}}+\widehat{M}\int_0^n\ |h(s,0)|\ ds \\
& +\overline{M}_0 L\|z_{\rho(t,z_t+x_t)}+x_{\rho(t,z_t+x_t)}\|_{\mathcal{B}} \\
& +\widehat{M}L\int_0^t\ \|z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}}\ ds \\
& +\widehat{M}\int_0^t\ p(s)\ \psi(\|z_{\rho(s,z_s+x_s)}+x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}})\ ds \\
& +\widehat{M}\int_0^t\ \eta(s)\ \left\|\frac{z_{\rho\left(s,\frac{z_s}{\lambda}+x_s\right)}}{\lambda}+x_{\rho\left(s,\frac{z_s}{\lambda}+x_s\right)}\right\|_{\mathcal{B}}\ ds.
\end{aligned}
$$

Use Proposition (3.4) and inequality (7) to get

$$
\begin{aligned}
\left\|\frac{z_{\rho\left(s,\frac{z_s}{\lambda}+x_s\right)}}{\lambda}+x_{\rho\left(s,\frac{z_s}{\lambda}+x_s\right)}\right\|_{\mathcal{B}} \leq\ & \frac{1}{\lambda}\left\|z_{\rho\left(s,\frac{z_s}{\lambda}+x_s\right)}\right\|_{\mathcal{B}}+\left\|x_{\rho\left(s,\frac{z_s}{\lambda}+x_s\right)}\right\|_{\mathcal{B}} \\
\leq\ & \frac{K_n}{\lambda}|z(s)|+\frac{M_n+\mathcal{L}^\phi}{\lambda}\|z_0\|_{\mathcal{B}} \\
& \quad +K_n|x(s)|+\left(M_n+\mathcal{L}^\phi\right)\|x_0\|_{\mathcal{B}} \\
\leq\ & \frac{K_n}{\lambda}|z(s)|+K_n\|U(s,0)\|_{B(E)}|\phi(0)| \\
& \quad +\left(M_n+\mathcal{L}^\phi\right)\|\phi\|_{\mathcal{B}} \\
\leq\ & \frac{K_n}{\lambda}|z(s)|+K_n\widehat{M}H\|\phi\|_{\mathcal{B}}+\left(M_n+\mathcal{L}^\phi\right)\|\phi\|_{\mathcal{B}} \\
\leq\ & \frac{K_n}{\lambda}|z(s)|+\left(M_n+\mathcal{L}^\phi+K_n\widehat{M}H\right)\|\phi\|_{\mathcal{B}}.
\end{aligned}
$$

Hence

$$
\left\|\frac{z_{\rho\left(s,\frac{z_s}{\lambda}+x_s\right)}}{\lambda}+x_{\rho\left(s,\frac{z_s}{\lambda}+x_s\right)}\right\|_{\mathcal{B}} \leq \frac{K_n}{\lambda}|z(s)|+c_n. \tag{10}
$$

Use the function $u(\cdot)$ and the nondecreasing character of ψ to get

$$\begin{aligned}
\frac{u(t)}{\lambda} \leq\ & \left(\widehat{M}+1\right)\overline{M}_0 L+\widehat{M}Ln+\widehat{M}\overline{M}_0 L\|\phi\|_{\mathcal{B}}+\widehat{M}\int_0^n \ |h(s,0)| \ ds \\
& +\overline{M}_0 L \ \left(K_n u(t)+c_n\right)+\widehat{M}L\int_0^t \ \left(K_n u(s)+c_n\right) \ ds \\
& +\widehat{M}\int_0^t \ p(s) \ \psi\left(K_n u(s)+c_n\right) \ ds+\widehat{M}\int_0^t \ \eta(s) \ \left(\frac{K_n}{\lambda}u(s)+c_n\right) \ ds.
\end{aligned}$$

Then

$$\begin{aligned}
\frac{u(t)}{\lambda} \leq\ & \left(\widehat{M}+1\right)\overline{M}_0 L+\widehat{M}Ln+\overline{M}_0 L\left[M_n+\mathcal{L}^{\phi}+\widehat{M}(1+K_n H)\right]\|\phi\|_{\mathcal{B}} \\
& +\widehat{M}\int_0^n \ |h(s,0)| \ ds+\overline{M}_0 L \ \frac{K_n}{\lambda}u(t)+\widehat{M}L\int_0^t \ \left(\frac{K_n}{\lambda}u(s)+c_n\right) \ ds \\
& +\widehat{M}\int_0^t \ p(s) \ \psi\left(\frac{K_n}{\lambda}u(s)+c_n\right) \ ds+\widehat{M}\int_0^t \ \eta(s) \ \left(\frac{K_n}{\lambda}u(s)+c_n\right) \ ds.
\end{aligned}$$

Set

$$\begin{aligned}
\zeta_n \ :=\ & \left(\widehat{M}+1\right)\overline{M}_0 L+\widehat{M}Ln+\overline{M}_0 L\left[M_n+\mathcal{L}^{\phi}+\widehat{M}(1+K_n H)\right]\|\phi\|_{\mathcal{B}} \\
& +\widehat{M}\int_0^n \ |h(s,0)| \ ds.
\end{aligned}$$

So

$$\begin{aligned}
\frac{K_n}{\lambda}\left(1-\overline{M}_0 L K_n\right)u(t) \ \leq\ & K_n\zeta_n+K_n\widehat{M}\int_0^t \ [L+\eta(s)] \ \left(\frac{K_n}{\lambda}u(s)+c_n\right) \ ds \\
& +K_n\widehat{M}\int_0^t \ p(s) \ \psi\left(\frac{K_n}{\lambda}u(s)+c_n\right) \ ds.
\end{aligned}$$

Set $\xi_n := c_n+\dfrac{K_n\zeta_n}{1-\overline{M}_0 L K_n}$. Then

$$\begin{aligned}
\frac{K_n}{\lambda}u(t)+c_n \ \leq\ & \xi_n+\frac{K_n\widehat{M}}{1-\overline{M}_0 L K_n}\int_0^t \ [L+\eta(s)] \ \left(\frac{K_n}{\lambda}u(s)+c_n\right) \ ds \\
& +\frac{K_n\widehat{M}}{1-\overline{M}_0 L K_n}\int_0^t \ p(s) \ \psi\left(\frac{K_n}{\lambda}u(s)+c_n\right) \ ds.
\end{aligned}$$

We consider the function μ defined by

$$\mu(t)=\sup\left\{\frac{K_n}{\lambda}u(s)+c_n \ \ : \ 0\leq s\leq t\right\}, \ \ 0\leq t\leq +\infty$$

Let $t^\star \in [0,t]$ be such that $\mu(t) = \dfrac{K_n}{\lambda} u(t^\star) + c_n$. By the previous inequality, we have for $t \in [0,n]$

$$\mu(t) \leq \xi_n + \frac{K_n \widehat{M}}{1-\overline{M}_0 L K_n} \int_0^t \, [L+\eta(s)] \; \mu(s) \; ds + \frac{K_n \widehat{M}}{1-\overline{M}_0 L K_n} \int_0^t \, p(s) \; \psi\left(\mu(s)\right) \; ds.$$

Let us take the right-hand side of the above inequality as $v(t)$. Then we have

$$\mu(t) \leq v(t) \quad \forall t \in [0,n].$$

From the definition of v, we get $v(0) = \xi_n$ and

$$v'(t) = \frac{K_n \widehat{M}}{1-\overline{M}_0 L K_n} \; [L+\eta(t)] \; \mu(t) + \frac{K_n \widehat{M}}{1-\overline{M}_0 L K_n} \; p(t) \; \psi\left(\mu(t)\right) \quad \text{a.e. } \; t \in [0,n].$$

Using the nondecreasing character of ψ we have

$$v'(t) \leq \frac{K_n \widehat{M}}{1-\overline{M}_0 L K_n} \; [L+\eta(t)] \; v(t) + \frac{K_n \widehat{M}}{1-\overline{M}_0 L K_n} \; p(t) \; \psi\left(v(t)\right) \quad \text{a.e. } \; t \in [0,n].$$

So using (9) we get for each $t \in [0,n]$

$$\begin{aligned}
\int_{\xi_n}^{v(t)} \frac{ds}{s+\psi(s)} &\leq \frac{K_n \widehat{M}}{1-\overline{M}_0 L K_n} \int_0^t \max(L+\eta(s), p(s)) \; ds \\
&\leq \frac{\widehat{M} K_n}{1-\overline{M}_0 L K_n} \int_0^n \max(L+\eta(s), p(s)) \; ds \\
&< \int_{\xi_n}^{+\infty} \frac{ds}{s+\psi(s)}.
\end{aligned}$$

Thus, for every $t \in [0,n]$, there exists a constant Λ_n such that $v(t) \leq \Lambda_n$ and hence $\mu(t) \leq \Lambda_n$. Since $\|z\|_n \leq \mu(t)$, we have $\|z\|_n \leq \Lambda_n$. This shows that the set $\widetilde{\Gamma}$ is bounded. Then the statement $(Av2)$ in Theorem 2.8 does not hold. The nonlinear alternative of Avramescu implies that $(Av1)$ is satisfied. We deduce that the operator $F + \widetilde{G}$ has a fixed point $z^\star$. Then $y^\star(t) = z^\star(t) + x(t)$, $t \in]-\infty, +\infty[$ is a fixed point of the operator N which is a mild solution of the problem $(3) - (4)$.

5 Examples

To illustrate the previous results, we give in this section two examples.

Example 1. Consider the partial functional differential equation

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,\xi) = \dfrac{\partial^2 u(t,\xi)}{\partial \xi^2} + a_0(t,\xi)u(t,\xi) \\ + \displaystyle\int_{-\infty}^{0} a_1(s-t)u\left[s-\rho_1(t)\rho_2\left(\int_0^\pi a_2(\theta)|u(t,\theta)|^2 d\theta\right),\xi\right] ds \\ + \displaystyle\int_{-\infty}^{0} a_3(s-t)u\left[s-\rho_1(t)\rho_2\left(\int_0^\pi a_2(\theta)|u(t,\theta)|^2 d\theta\right),\xi\right] ds, \\ \qquad\qquad\qquad\qquad\qquad\qquad t \geq 0,\ \xi \in [0,\pi], \\ u(t,0) = u(t,\pi) = 0, \qquad\qquad t \geq 0, \\ u(\theta,\xi) = u_0(\theta,\xi), \qquad\qquad -\infty < \theta \leq 0,\ \xi \in [0,\pi], \end{cases} \tag{11}$$

where $a_0(t,\xi)$ is a continuous function and is uniformly Hölder continuous in t; $a_1, a_3 : \mathbb{R}_- \to \mathbb{R}$ and $a_2 : [0,\pi] \to \mathbb{R}$, $\rho_i : [0,+\infty[\to \mathbb{R}$ are continuous functions for $i = 1,2$.

To study this system, we consider the space $E = L^2([0,\pi],\mathbb{R})$ and the operator $A : D(A) \subset E \to E$ given by $Aw = w''$ with

$$D(A) := \{\ w \in E\ : w'' \in E,\ w(0) = w(\pi) = 0\ \}$$

It is well known that A is the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t\geq 0}$ on E. Furthermore, A has discrete spectrum with eigenvalues $-n^2$, $n \in \mathbb{N}$ and corresponding normalized eigenfunctions given by

$$y_n(\xi) = \sqrt{\frac{2}{\pi}} \sin(n\xi).$$

In addition, $\{y_n : n \in \mathbb{N}\}$ is an orthonormal basis of E and $T(t)x = \displaystyle\sum_{n=1}^{\infty} e^{-n^2 t}(x, y_n)y_n$ for $x \in E$ and $t \geq 0$. It follows from this representation that $T(t)$ is compact for every $t > 0$ and that $\|T(t)\| \leq e^{-t}$ for every $t \geq 0$.

On the domain $D(A)$, we define the operators $A(t) : D(A) \subset E \to E$ by

$$A(t)x(\xi) = Ax(\xi) + a_0(t,\xi)x(\xi).$$

By assuming that $a_0(.)$ is continuous and that $a_0(t,\xi) \leq -\delta_0$ ($\delta_0 > 0$) for every $t \in \mathbb{R}, \xi \in [0,\pi]$, it follows that the system

$$u'(t) = A(t)u(t) \quad t \geq s,$$

$$u(s) = x \in E,$$

has an associated evolution family given by

$$U(t,s)x(\xi) = \left[T(t-s) \exp\left(\int_s^t a_0(\tau,\xi) d\tau \right) x \right](\xi).$$

From this expression, it follows that $U(t,s)$ is a compact linear operator and that

$$\|U(t,s)\| \leq e^{-(1+\delta_0)(t-s)} \text{ for every } (t,s) \in \Delta.$$

Theorem 5.1 *Let $\mathcal{B} = BUC(\mathbb{R}_-; E)$ and $\phi \in \mathcal{B}$. Assume that the condition (H_ϕ) holds. Suppose that the functions $a_1, a_3 : \mathbb{R}_- \to \mathbb{R}$, $a_2 : [0,\pi] \to \mathbb{R}$ and $\rho_i : \mathbb{R}_+ \longrightarrow \mathbb{R}$ for $i = 1,2$ are continuous. Then there exists a mild solution of* (11) *on* $]-\infty, +\infty[$.

Proof. From the assumptions, we have that

$$f(t,\psi)(\xi) = \int_{-r}^0 a_1(s)\psi(s,\xi) ds,$$

$$h(t,\psi)(\xi) = \int_{-r}^0 a_3(s)\psi(s,\xi) ds,$$

$$\rho(s,\psi) = s - \rho_1(s)\rho_2\left(\int_0^\pi a_2(\theta)|\psi(0,\xi)|^2 d\theta \right),$$

are well defined functions, which permit to transform system (11) into the abstract system $(1) - (2)$. Moreover, the functions f and h are bounded and linear. Now, the existence of mild solutions can be deduced from a direct application of Theorem 3.5. From Remark 3.2, we have the following result

Corollary 5.2 *Let $\phi \in \mathcal{B}$ be continuous and bounded. Then there exists a mild solution of* (11) *on* $]-\infty, +\infty[$.

Example 2. Consider the semilinear neutral perturbed evolution equation

$$\begin{cases} \dfrac{\partial}{\partial t}\left[u(t,\xi) - \displaystyle\int_{-\infty}^0 a_3(s-t) u\left(s - \rho_1(t)\rho_2\left(\int_0^\pi a_2(\theta)|u(t,\theta)|^2 d\theta \right), \xi \right) ds \right] \\ = \dfrac{\partial^2 u(t,\xi)}{\partial \xi^2} + a_0(t,\xi)u(t,\xi) \\ + \displaystyle\int_{-\infty}^0 a_1(s-t) u\left(s - \rho_1(t)\rho_2\left(\int_0^\pi a_2(\theta)|u(t,\theta)|^2 d\theta \right), \xi \right) ds \\ + \displaystyle\int_{-\infty}^0 a_4(s-t) u\left(s - \rho_1(t)\rho_2\left(\int_0^\pi a_2(\theta)|u(t,\theta)|^2 d\theta \right), \xi \right) ds, \\ \hfill t \geq 0,\ \xi \in [0,\pi], \\ v(t,0) = v(t,\pi) = 0, \hfill t \geq 0, \\ v(\theta,\xi) = v_0(\theta,\xi), \hfill -\infty < \theta \leq 0,\ \xi \in [0,\pi], \end{cases} \tag{12}$$

where $a_4 : \mathbb{R}_- \to \mathbb{R}$ is a continuous function.

Theorem 5.3 *Let $\mathcal{B} = BUC(\mathbb{R}_-; E)$ and $\phi \in \mathcal{B}$. Assume that the condition (H_ϕ) holds. Suppose that the functions $a_1, a_3, a_4 : \mathbb{R}_- \to \mathbb{R}$, $a_2 : [0, \pi] \to \mathbb{R}$ and $\rho_i : \mathbb{R}_+ \longrightarrow \mathbb{R}$ for $i = 1, 2$ are continuous. Then there exists a mild solution of* (12) *on* $]-\infty, +\infty[$.

Proof. From the assumptions, we have that

$$f(t, \psi)(\xi) = \int_{-\infty}^{0} a_1(s)\psi(s, \xi)ds,$$

$$g(t, \psi)(\xi) = \int_{-\infty}^{0} a_3(s)\psi(s, \xi)ds,$$

$$h(t, \psi)(\xi) = \int_{-r}^{0} a_4(s)\psi(s, \xi)ds,$$

$$\rho(s, \psi) = s - \rho_1(s)\rho_2\left(\int_{0}^{\pi} a_2(\theta)|\psi(0, \xi)|^2 d\theta\right)$$

are well defined functions, which permit to transform system (12) into the abstract system $(3) - (4)$. Moreover, the functions f, g and h are bounded and linear. Now, the existence of mild solutions can be deduced from a direct application of Theorem 4.2.

From Remark 3.2, we have the following result

Corollary 5.4 *Let $\phi \in \mathcal{B}$ be continuous and bounded. Then there exists a mild solution of* (12) *on* $]-\infty, +\infty[$.

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