NON-AUTONOMOUS BIFURCATION IN IMPULSIVE SYSTEMS

M. U. AKHMET* AND A. KASHKYNBAYEV

ABSTRACT. This is the first paper which considers non-autonomous bifurcations in impulsive differential equations. Impulsive generalizations of the non-autonomous pitchfork and transcritical bifurcation are discussed. We consider scalar differential equation with fixed moments of impulses. It is illustrated by means of certain systems how the idea of pullback attracting sets remains a fruitful concept in the impulsive systems. Basics of the theory are provided.

Asymptotic behavior of fixed points and analysis of bifurcation is of great importance in the qualitative theory of differential equations. In autonomous ordinary differential equations this theory is well developed. As in the autonomous systems, non-autonomous bifurcation is understood as a qualitative change in the structure and stability of the invariant sets of the system. However, to implement this concept in non-autonomous systems, locally defined notions of attractive and repulsive solutions are needed. There are currently qualitative studies which are devoted to non-autonomous bifurcation theory by treating attractors called pullback attractors [11, 12, 23, 25, 26, 29, 31, 35]. The theory of pullback attraction is not concerned with the asymptotic behavior of the solution as $t \to \infty$ for fixed t_0, but as $t_0 \to -\infty$ for fixed t [8, 11, 13, 15–18, 25, 28, 30, 32, 33]. This approach requires the discussion of bifurcation in non-autonomous differential equations by defining various types of stability and instability.

Investigation of states of dynamical systems which are not constant in time leads to non-autonomous problems in the form of the equation of perturbed motion. If this model depends on parameters, it is the

main object of non-autonomous bifurcation theory to describe qualitative changes when these parameters are varied. Extending non-autonomous bifurcation theory to impulsive systems is a contemporary problem.

Many evolutionary processes in the real world are characterized by sudden changes at certain times. These changes are called impulsive phenomena [1, 9, 19, 27, 34], which are widespread in modeling in mechanics, electronics, biology, neural networks, medicine, and social sciences [1,4,7]. An impulsive differential equation is one of the basic instruments to understand better the role of discontinuity for the real world problems. Therefore, there are qualitative studies on asymptotic behavior of impulsive systems [1,3,5,9,27,34]. There are also many studies which deal with bifurcation theory either in autonomous differential equations [1, 2, 6] or periodic equations with fixed moments of impulses [10, 20, 21]. However, differential equations with fixed moments of impulses are naturally non-autonomous differential equations. Consequently, one cannot construct the theory similar to autonomous systems of ordinary differential equations. Thus, in order to achieve results on fixed moments, it is crucial to extend the idea of pullback attraction to impulsive systems for non-autonomous differential equations. Although the theory of impulsive differential equations is very developed nowadays, there are no results concerning analogues of equations studied in [8,11,13,17,18,25,26,28,32,33]. This appears to be due to the absence of papers concerning concrete systems analyzing the existence of non-autonomous bifurcations. It is hoped that the present paper fills this gap.

Langa et al. in [29] and Caraballo and Langa in [11] present the canonical non-autonomous ODE example of a pitchfork bifurcation,

$$\dot{x} = ax - b(t)x^3. \tag{1}$$

Next, Langa et al. in [31] investigate the non-autonomous form of the canonical transcritical example,

$$\dot{x} = \lambda a(t)x - b(t)x^2. \tag{2}$$

Throughout Section 2 we make use of definitions of pullback attracting sets and pullback stability for impulsive differential equations which are the same as for ODE. The main novelty of this paper is to give impulsive extensions of the systems (1) and (2) with appropriate definitions of pullback attracting sets. This is the very first step towards the bifurcation of

non-autonomous differential equations with impulses. We present three systems which illustrate the given definitions. The first system (Section 3) is the impulsive extension of a non-autonomous pitchfork bifurcation,

$$
\begin{aligned}
\dot{x} &= a(t)x - b(t)x^3, \\
\Delta x|_{t=\theta_i} &= -x + \frac{x}{\sqrt{c_i + d_i x^2}}.
\end{aligned}
\tag{3}
$$

In Theorem 1 we have obtained impulsive extension for the results of Caraballo and Langa in [11] and Langa et al. in [29]. Next, in Section 4, we investigate the non-autonomous transcritical bifurcation in the impulsive system

$$
\begin{aligned}
\dot{x} &= a(t)x - b(t)x^2, \\
\Delta x|_{t=\theta_i} &= -x + \frac{x}{c_i + d_i x}.
\end{aligned}
\tag{4}
$$

In particular, in Theorem 2 and Theorem 3, we give impulsive extension for results of Langa et al. in [31] for equation (2). Finally, in Section 5 we consider bifurcation in the non-order-preserving system

$$
\begin{aligned}
\dot{x} &= a(t)x - b(t)x^3, \\
\Delta x|_{t=\theta_i} &= -x - \frac{x}{\sqrt{c_i + d_i x^2}}.
\end{aligned}
\tag{5}
$$

In the conclusion part, we summarize the results and consider how the theory might be further developed in a systematic way.

1. Preliminaries

In this section we introduce concepts of attractive and repulsive solutions, which are used to analyze asymptotic behavior of impulsive non-autonomous systems. This paper is concerned with systems of the type

$$
\begin{aligned}
\dot{x} &= f(t,x), \\
\Delta x|_{t=\theta_i} &= J_i(x),
\end{aligned}
\tag{6}
$$

where $\Delta x|_{t=\theta_i} := x(\theta_i+) - x(\theta_i)$, $x(\theta_i+) = \lim_{t \to \theta_i^+} x(t)$. The system (6) is defined on the set $\Omega = \mathbb{R} \times \mathbb{Z} \times G$ where $G \subseteq \mathbb{R}^n$. θ is a nonempty sequence with the set of indexes $\mathbb{Z}$, set of integers, such that $|\theta_i| \to \infty$ as $|i| \to \infty$. Let $\phi(t, t_0, x_0)$ be solution of (6). In this paper, we treat only scalar impulsive differential equations such that $\phi(t, t_0, x_0)$ is continuable on $\mathbb{R}$. Solutions are unique both forwards and backwards in time and

$J_i(x)$ is order-preserving so that the whole system (6) is order-preserving, i.e.,

$$x_0 > y_0 \Rightarrow x(t, t_0, x_0) > y(t, t_0, y_0) \quad \text{for all} \quad t, t_0 \in \mathbb{R}$$

allowing $x(t)$ or $y(t)$ to be $\pm\infty$ if necessary.

We say that the function $\phi : \mathbb{R} \to \mathbb{R}^n$ is from the set $PC(\mathbb{R}, \theta)$, where $\theta = \{\theta_i\}$ is an infinite set such that $|\theta_i| \to \infty$ as $|i| \to \infty$, if:

- ϕ is left continuous on $\mathbb{R}$;
- it is continuous everywhere except possibly points of θ where it has discontinuities of the first kind.

Developing the theory for non-autonomous impulsive differential equations by following the same route as for autonomous systems poses a problem. Indeed, for generic non-autonomous system we would not expect to find any fixed points: if x_0 is the fixed point, then this would require that $f(x_0, t) = 0$ and $J_i(x_0) = 0$ for all $i \in \mathbb{Z}$ and $t \in \mathbb{R}$. Instead, we replace fixed points to the notion of a complete trajectory. The piecewise continuous map $x : \mathbb{R} \to G$ is said to be a complete trajectory if $X(t, t_0)x(t_0) = x(t)$ for all $t, t_0 \in I$ where $X(t, t_0)$ is the solution operator for (6). We investigate appearances and disappearances of complete trajectories that are stable and unstable in the pullback sense. Note that complete trajectories are particular examples of invariant sets. A time varying family of sets $\Sigma(t)$ is invariant if $\phi(t, t_0, \Sigma(t_0)) = \Sigma(t)$ for all $t, t_0 \in \mathbb{R}$. That is, if $x(t_0) \in \Sigma(t_0)$, then $\phi(t, t_0, x(t_0)) \in \Sigma(t)$. In order to study non-autonomous bifurcation with impulses we should define corresponding concepts of stability. In this paper, we use Hausdorff semi-distance between sets A and B as $\text{dist}(A, B) = \sup_{a \in A} \inf_{b \in B} d(a, b)$

1.1. **Attraction.** Asymptotic properties of continuous dynamics and dynamics with discontinuity are the same. Therefore, we shall use notion of pullback attracting sets without any change from $[8, 11, 13, 15\text{–}18, 24, 25, 28, 30, 32, 33, 35]$. In autonomous system, an invariant set Σ is attracting if there exists a neighborhood N of Σ such that

$$\text{dist}(\phi(t, 0, x_0), \Sigma) \to 0 \quad \text{as} \quad t \to \infty \quad \text{for all} \quad x_0 \in N \tag{7}$$

where initial time is not important, we may take it arbitrary. For this case it is true that $X(t, t_0) = X(t - t_0, 0)$. The concept of attraction for

autonomous systems is equivalent to the existence of a neighborhood N of Σ for each fixed $t \in \mathbb{R}$,

$$\text{dist}(\phi(t, t_0, x_0), \Sigma) \to 0 \ \ \text{as} \ \ t_0 \to -\infty \ \ \text{for all} \ \ x_0 \in N. \tag{8}$$

This is the idea of pullback attraction [24, 33], which does not involve running time backwards. Instead, we consider taking measurements in an experiment *now* (at time t) which began at some time in the past (at time $t_0 < t$). That is, we are interested in asymptotic behavior as $t_0 \to -\infty$ for fixed t.

Pullback attraction is a natural tool to study non-autonomous systems because it provides us to consider asymptotic behavior without having to consider sets $\Sigma(t)$ that are moving, since final time is fixed. This approach has many applications in stochastic differential equations [17, 18], ODEs [24, 25] and PDEs [14, 16, 32].

Definition 1. [24] An invariant set $\Sigma(\cdot)$ is called (locally) pullback attracting if for every $t \in \mathbb{R}$ there exists a $\delta(t) > 0$ such that if

$$\lim_{t_0 \to -\infty} (\text{dist}(x_0, \Sigma(t_0)) < \delta(t), \ \text{then} \ \lim_{t_0 \to -\infty} \text{dist}(\phi(t, t_0, x_0), \Sigma(t)) = 0.$$

$$\tag{9}$$

It is crucial that δ is not allowed to depend on t_0, otherwise every invariant set would be pullback attracting due to continuous dependence on initial conditions. If $\lim_{t_0 \to -\infty} \text{dist}(\phi(t, t_0, x_0), \Sigma(t)) = 0$ for every $t \in \mathbb{R}$ and every $x_0 \in \mathbb{R}^n$ then $\Sigma(\cdot)$ is said to be globally pullback attracting.

1.2. **Stability.** The above discussion helps to define asymptotic stability, which has two parts. One of them is attraction and another one is stability. In this part, we define stability in non-autonomous case in the pullback sense.

Definition 2. [29] An invariant set $\Sigma(\cdot)$ is pullback stable if for every $t \in \mathbb{R}$ and $\epsilon > 0$ there exists a $\delta(t) > 0$ such that for any $t_0 < t, x_0 \in N(\Sigma(t_0), \delta(t))$ implies that $\phi(t, t_0, x_0) \in N(\Sigma(t), \epsilon)$.

An invariant set $\Sigma(\cdot)$ is said to be locally (globally) pullback asymptotically stable if it is pullback stable and locally (globally) pullback attracting. As in the scalar non-autonomous differential equations, pullback

attraction implies pullback stability for complete trajectories of scalar impulsive systems.

Lemma 1. [31] *Let $y(t)$ be a complete trajectory in a non-autonomous scalar impulsive differential equation that is locally pullback attracting; then, this trajectory is also pullback stable.*

The proof of this lemma, given by Langa et al. in [31], is the same for impulsive systems. This lemma allows us to consider only pullback attraction properties of complete trajectories rather than their pullback stability properties.

1.3. **Instability.** Local pullback instability is defined as the converse of pullback stability. An invariant set $\Sigma(\cdot)$ is called locally pullback unstable if it is not pullback stable, i.e., if there exists a $t \in \mathbb{R}$ and $\epsilon > 0$ such that for each $\delta > 0$, there exists a $t_0 < t$ and $x_0 \in N(\Sigma(t_0), \delta)$ such that $\mathrm{dist}(\phi(t, t_0, x_0), \Sigma(t)) > \epsilon$. However, we make use of the idea "unstable set" defined by Crauel for the random dynamical systems which is more natural concept from a dynamics point of view.

Definition 3. [16] If $\Sigma(\cdot)$ is an invariant set then the unstable set of Σ, $U_{\Sigma(\cdot)}$ is defined as

$$U_{\Sigma(\cdot)} = \{u : \lim_{t \to -\infty} \mathrm{dist}(\phi(t, t_0, u), \Sigma(t)) = 0\}.$$

We say that $\Sigma(\cdot)$ is asymptotically unstable if for some t we have

$$U_{\Sigma(t)} \neq \Sigma(t).$$

Since we always have $\Sigma(t) \subset U_{\Sigma(t)}$ when $\Sigma(\cdot)$ is invariant, the last definition says that $\Sigma(t)$ is a strict subset of $U_{\Sigma(t)}$. In this case we will say that $U_{\Sigma(t)}$ is non-trivial. The power of this definition comes from the following result.

Proposition 1. [29] *If $\Sigma(\cdot)$ is asymptotically unstable then it is also locally pullback unstable and cannot be locally pullback attracting.*

This result proven by Langa et al. in [29] is valid for impulsive systems. Most ideas of instability are related to the behavior of solutions $\phi(t)$ as $t \to -\infty$. Note that the idea of the asymptotic instability defined above is a time-reversed definition of 'forward attraction'. Alternatively,

it is possible to define instability as a time-reversed version of pullback attraction.

Definition 4. [31] An invariant set $\Sigma(\cdot)$ is (locally) pullback repelling if it is (locally) pullback attracting for time-reversed system, i.e., if for every $t \in \mathbb{R}$ and every $x_0 \in \mathbb{R}^n$,

$$\lim_{t_0 \to \infty} \text{dist}(\phi(t, t_0, x_0), \Sigma(t)) = 0.$$

2. The pitchfork bifurcation

In this section, we study generalization of the system (1) with fixed moments of impulses. Consider the system

$$\dot{x} = a(t)x - b(t)x^3, \tag{10a}$$

$$\Delta x|_{t=\theta_i} = -x + \frac{x}{\sqrt{c_i + d_i x^2}}, \tag{10b}$$

where $a, b \in PC(\mathbb{R}, \theta)$. Assume that there exist constants A, B, C and D such that

$$|a(t)| < A < \infty \quad \text{and} \quad 0 < c_i \leq C < \infty, \tag{11}$$

and

$$0 < b_0 \leq b(t) < B < \infty \quad \text{and} \quad 0 < d_i \leq D < \infty, \tag{12}$$

for $i \in \mathbb{Z}$ and $t \in \mathbb{R}$. We suppose that there exist positive numbers $\underline{\theta}$ and $\overline{\theta}$ such that

$$\underline{\theta} \leq \theta_{i+1} - \theta_i \leq \overline{\theta}. \tag{13}$$

Moreover, there exists the limit

$$\lim_{t-s \to \infty} \frac{2 \int_s^t a(u)du - \sum_{s \leq \theta_i < t} \ln c_i}{t - s} = \gamma. \tag{14}$$

By means of substitution $y = \frac{1}{x^2}$, the system (10) is converted to the linear impulsive system

$$\begin{aligned} \dot{y} &= -2a(t)y + 2b(t), \\ \Delta y|_{t=\theta_i} &= (c_i - 1)y + d_i. \end{aligned} \tag{15}$$

In what follows, we discuss the system (15) to analyze the system (10). Since $c_i \neq 0$, the transition matrix of the associated homogeneous part of (15), according to [1], is the following:

$$Y(t,s) = e^{-2\int_s^t a(u)du} \prod_{s \leq \theta_i < t} c_i = e^{-\frac{2\int_s^t a(u)du - \sum_{s \leq \theta_i < t} \ln c_i}{t-s}(t-s)}, \quad t \geq s. \quad (16)$$

Lemma 2. *If $\alpha > \gamma > \beta > 0$, then there exist positive numbers M and m such that*

$$me^{-\alpha(t-s)} \leq Y(t,s) \leq Me^{-\beta(t-s)}, \quad t \geq s. \quad (17)$$

Proof. By relation (14), there exists T such that if $t - s \geq T$, then $\beta < \frac{2\int_s^t a(u)du - \sum_{s \leq \theta_i < t} \ln c_i}{t-s} < \alpha$. Consequently, by means of (11) and (13), it is true that

$$M = \sup_{0 \leq t-s \leq T} e^{-2\int_s^t a(u)du} \prod_{s \leq \theta_i < t} c_i$$

and

$$m = \inf_{0 \leq t-s \leq T} e^{-2\int_s^t a(u)du} \prod_{s \leq \theta_i < t} c_i.$$

Hence,

$$me^{-\alpha(t-s)} \leq Y(t,s) = e^{-2\int_s^T a(u)du + \sum_{s \leq \theta_i < T} \ln c_i} e^{-2\int_T^t a(u)du + \sum_{T \leq \theta_i < t} \ln c_i}$$
$$\leq Me^{-\beta(t-s)}, \quad t \geq s.$$

The lemma is proved. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Theorem 1. *Assume that (11), (12) and (14) hold for the system (10). Then, for $\gamma < 0$ the origin is globally asymptotically pullback stable, and for $\gamma > 0$ the origin is asymptotically unstable and there appear positive and negative, $\beta(t,\gamma)$ and $-\beta(t,\gamma)$ respectively, locally asymptotically pullback complete trajectories such that*

$$\beta^2(t,\gamma) = \frac{1}{2\int_{-\infty}^t Y(t,s)b(s)ds + \sum_{\theta_i < t} Y(t,\theta_i+)d_i}.$$

Proof. Equation (10b) can be rewritten as $x(\theta_i+) = \frac{x(\theta_i)}{\sqrt{c_i + d_i x^2(\theta_i)}}$. To show that equation (10) is order-preserving, it is sufficient that the jump

equation satisfies $x(\theta_i+) > y(\theta_i+)$ for $x(\theta_i) > y(\theta_i)$. In other words, we must show that $\frac{x(\theta_i)}{\sqrt{c_i+d_ix^2(\theta_i)}} > \frac{y(\theta_i)}{\sqrt{c_i+d_iy^2(\theta_i)}}$. Defining $f(x) = \frac{x}{\sqrt{c_i+d_ix^2}}$, one can check that $f'(x) > 0$. Since uniqueness is assumed and the equation is order-preserving, for $x_0 \neq 0$ we have $x(t) \neq 0$. Therefore, by substitution $y = \frac{1}{x^2}$, we see that the solution of the system (10), according to [1, 34], satisfies the integral equation

$$y(t, t_0, y_0) = Y(t, t_0)y_0 + 2\int_{t_0}^t Y(t, s)b(s)ds + \sum_{t_0 \leq \theta_i < t} Y(t, \theta_i+)d_i. \quad (18)$$

By means of (14), one can see that the asymptotic behavior of $y(t, t_0, y_0)$ depends on the sign of γ.

Consider the case $\gamma < 0$. From (18) it follows that $y(t, t_0, y_0) \to \infty$ as $t_0 \to -\infty$. So, $x(t, t_0, x_0) \to 0$ as $t_0 \to -\infty$, implying that all solutions are attracted both forwards and pullback to the point $\{0\}$, since this is also limit of (18) as $t \to \infty$.

If $\gamma > 0$, then from (18) it follows that $y(t, t_0, y_0) \to 0$ as $t \to \infty$ implying that all solutions are unbounded as $t \to \infty$. However, as $t_0 \to -\infty$ we have

$$\lim_{t_0 \to -\infty} y(t, t_0, y_0) = 2\int_{-\infty}^t Y(t, s)b(s)ds + \sum_{\theta_i < t} Y(t, \theta_i+)d_i. \quad (19)$$

The last equation implies that

$$\lim_{t_0 \to -\infty} x^2(t, t_0, x_0) = \beta^2(t, \gamma) = \frac{1}{2\int_{-\infty}^t Y(t, s)b(s)ds + \sum_{\theta_i < t} Y(t, \theta_i+)d_i}$$

where $s, \theta_i \in (-\infty, t]$. By means of (13) and Lemma 2, one can show that

$$
\begin{aligned}
0 \quad &< \frac{2mb_0}{\alpha} < 2\int_{-\infty}^t Y(t, s)b(s)ds + \sum_{\theta_i < t} Y(t, \theta_i+)d_i \\
&< \frac{2BM}{\beta} + DM\sum_{\theta_i < t} e^{-\beta(t-\theta_i)} \leq \frac{2BM}{\beta} + DM\sum_{i=0}^\infty e^{-i\beta\underline{\theta}} \\
&= \frac{2BM}{\beta} + DM\frac{1}{1 - e^{\beta\underline{\theta}}} < \infty.
\end{aligned}
$$

Thus, $\beta^2(t, \gamma)$ is bounded both from above and from below. To check that $\beta(t, \gamma)$ is a complete trajectory, it would be enough to check that

$\eta(t) = \frac{1}{\beta^2(t,\gamma)}$ satisfies (15).

$$
\begin{aligned}
\dot{\eta} &= -4a(t) \int_{-\infty}^{t} Y(t,s)b(s)ds + 2Y(t,t)b(t) - 2a(t) \sum_{\theta_i < t} Y(t,\theta_i+)d_i \\
&= -2a(t) \left\{ 2 \int_{-\infty}^{t} Y(t,s)b(s)ds + \sum_{\theta_i < t} Y(t,\theta_i+)d_i \right\} + 2b(t) \\
&= -2a(t)\eta + 2b(t).
\end{aligned}
$$

To show that $\eta(t)$ satisfies the equation jumps, we note for fixed j it is true that $Y(\theta_j+,s) - Y(\theta_j,s) = (c_j-1)Y(\theta_j,s)$; so that $Y(\theta_j+,s) = c_j Y(\theta_j,s)$. Then,

$$
\begin{aligned}
\Delta \eta(t)|_{t=\theta_j} =& \,\eta(\theta_j+) - \eta(\theta_j) \\
=& \,2 \int_{-\infty}^{\theta_j+} Y(\theta_j+,s)b(s)ds + \sum_{\theta_i < \theta_j+} Y(\theta_j+,\theta_j+)d_j \\
& - 2 \int_{-\infty}^{\theta_j} Y(\theta_j,s)b(s)ds - \sum_{\theta_i < \theta_j} Y(\theta_j,\theta_j+)d_j \\
=& \,2c_j \int_{-\infty}^{\theta_j} Y(\theta_j,s)b(s)ds - 2 \int_{-\infty}^{\theta_j} Y(\theta_j,s)b(s)ds + d_j \\
& + \sum_{\theta_i < \theta_j} c_j Y(\theta_j,\theta_j+)d_j - \sum_{\theta_i < \theta_j} Y(\theta_j,\theta_j+)d_j \\
=& \,(c_j - 1) \left\{ 2 \int_{-\infty}^{\theta_j} Y(\theta_j,s)b(s)ds + \sum_{\theta_i < \theta_j} Y(\theta_j,\theta_j+)d_j \right\} + d_j \\
=& \,(c_j - 1)\eta(\theta_j) + d_j.
\end{aligned}
$$

Construction of $\beta(t,\gamma)$ ensures that it is pullback attracting. Thus, Lemma 1 implies that $\beta(t,\gamma)$ is pullback stable. Moreover, since the system (10) is order-preserving, for $\gamma > 0$ all trajectories with $x_0 > 0$ are pullback attracted to $\beta(t,\gamma)$ and all trajectories with $x_0 < 0$ are pullback attracted to $-\beta(t,\gamma)$ as it is illustrated in Figure 1. By means of (18), it

follows that

$$x^2(t, t_0, x_0) = \frac{1}{y(t, t_0, y_0)}$$

$$= \frac{1}{Y(t, t_0)x_0^{-2} + 2\int_{t_0}^t Y(t, s)b(s)ds + \sum_{t_0 \le \theta_i < t} Y(t, \theta_i+)d_i}$$

$$= \frac{1}{Y(t, t_0)(x_0^{-2} - \beta^{-2}(t_0)) + 2\int_{-\infty}^t Y(t, s)b(s)ds + \sum_{\theta_i < t} Y(t, \theta_i+)d_i}.$$

If $|x_0| < \beta(t_0)$ so that $x^{-2} - \beta^{-2}(t_0) > 0$, then $x(t)$ converges to 0 as $t \to -\infty$ implying that origin is asymptotically unstable. $\qquad \square$

Remark 1. We do not consider formal impulsive analogue of equation (1),

$$\begin{aligned} \dot{x} &= a(t)x - b(t)x^3, \\ \Delta x|_{t=\theta_i} &= c_i x + d_i x^3, \end{aligned} \tag{20}$$

since it is not possible to find explicit solution of the system (20).

Example 1. Let $a(t) \equiv a$, $c_i \equiv c$, and $\theta_i = ih$ for the system (10) with $h > 0$. That is,

$$\begin{aligned} \dot{x} &= ax - b(t)x^3, \\ \Delta x|_{t=ih} &= -x + \frac{x}{\sqrt{c+d_i x^2}}. \end{aligned} \tag{21}$$

Then $\gamma = 2a - \frac{1}{h}\ln c$. By means of $y = \frac{1}{x^2}$, the system (21) is converted to the linear impulsive system

$$\begin{aligned} \dot{y} &= -2ay + 2b(t), \\ \Delta y|_{t=ih} &= (c-1)y + d_i. \end{aligned} \tag{22}$$

Asymptotic behavior of (22) depends on the sign of $2a - \frac{1}{h}\ln c = \gamma$, and results of Theorem 1 are true for the system (21). If, in particular, $c = 1$ and $d_i = 0$, then there is no equation of jumps in the system (21). Moreover, $\gamma = 2a$ so that the asymptotic behavior of (22) depends on the sign of a. Thus, results of Theorem 1 are generalizations of the results obtained in the studies of Langa et al. in [29] and Caraballo and Langa in [11].
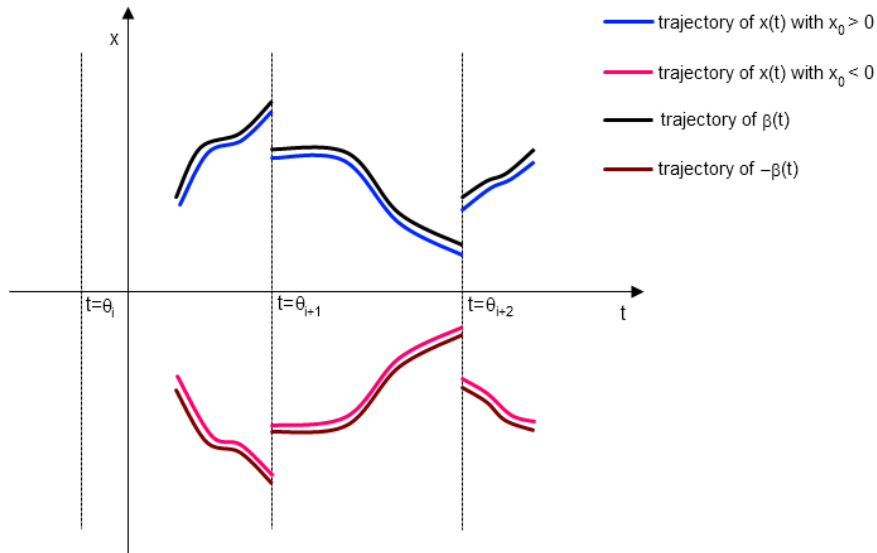
Figure 1. Asymptotic behavior of the system (10).

3. The transcritical bifurcation

Consider the impulsive system

$$\dot{x} = a(t)x - b(t)x^2, \tag{23a}$$

$$\Delta x|_{t=\theta_i} = -x + \frac{x}{c_i + d_i x}, \tag{23b}$$

where $c_i > 0, d_i \in \mathbb{R}, i \in \mathbb{Z}, a, b \in PC(\mathbb{R}, \theta)$. Differently from the previous section, the function a can be unbounded. However, as in the previous section, we suppose that there exist positive numbers $\underline{\theta}$ and $\overline{\theta}$ such that

$\underline{\theta} \leq \theta_{i+1} - \theta_i \leq \overline{\theta}$, and there exists the limit

$$\lim_{t-s\to\infty} \frac{\int_s^t a(u)du - \sum_{s\leq\theta_i<t} \ln c_i}{t-s} = \gamma. \qquad (24)$$

The functions b and d_i are asymptotically positive as $t \to -\infty$, i.e., there exist constants $\underline{b}$ and $\underline{d}$ such that

$$b(t) \geq \underline{b} > 0 \ \text{ for all } \ t \leq T^-, \ \text{ and } \ d_i \geq \underline{d} > 0 \ \text{ for all } \ \theta_i \leq T^-. \qquad (25)$$

By means of substitution $x = \frac{1}{y}$, the system (23) is converted to the linear impulsive differential equation

$$\begin{aligned}
\dot{y} &= -a(t)y + b(t), \\
\Delta y|_{t=\theta_i} &= (c_i - 1)y + d_i.
\end{aligned} \qquad (26)$$

The transition matrix of the associated homogeneous part of the system (26), according to [1], is

$$Y(t,s) = e^{-\int_s^t a(u)du} \prod_{s\leq\theta_i<t} c_i = e^{-\frac{\int_s^t a(u)du - \sum_{s\leq\theta_i<t} \ln c_i}{t-s})(t-s)}, \ \ t \geq s. \qquad (27)$$

Assume that there exists a $\gamma_0 > 0$ such that

$$0 < m_\gamma \leq x_\gamma(t) = \frac{1}{\int_{-\infty}^t Y(t,s)b(s)ds + \sum_{\theta_i<t} Y(t,\theta_i+)d_i} \leq M_\gamma \qquad (28)$$

for all $t \in \mathbb{R}, \ \ i \in \mathbb{Z}, \ \ 0 < \gamma < \gamma_0$, and

$$\liminf_{t_0\to-\infty} \frac{Y(t,t_0)}{\int_{t_0}^t Y(t,s)b(s)ds + \sum_{t_0\leq\theta_i<t} Y(t,\theta_i+)d_i} \geq m_\gamma > 0 \qquad (29)$$

for all $-\gamma_0 < \gamma < 0$.

Theorem 2. *Assume that the above conditions hold for equation (23). Then, for $-\gamma_0 < \gamma < 0$ the origin is locally pullback attracting in $\mathbb{R}$; and for $0 < \gamma < \gamma_0$ the origin is asymptotically unstable and the trajectory $x_\gamma(t)$ is locally pullback attracting.*

Proof. Equation (23b) can be rewritten as $x(\theta_i+) = \frac{x(\theta_i)}{c_i+d_i x(\theta_i)}$. To show that (23) is order-preserving, it is enough to show that the jump equation satisfies $\frac{x(\theta_i)}{c_i+d_i x(\theta_i)} > \frac{y(\theta_i)}{c_i+d_i y(\theta_i)}$ if $x(\theta_i) > y(\theta_i)$. Considering $f(x) = \frac{x}{c_i+d_i x}$, one can check that $f'(x) > 0$. Next, by introducing the transformation

$x = \frac{1}{y}$ for equation (23), we see that the solution of the impulsive system (26), according to [1, 34], satisfies the integral equation

$$y(t, t_0, y_0) = Y(t, t_0)y_0 + \int_{t_0}^{t} Y(t, s)b(s)ds + \sum_{t_0 \leq \theta_i < t} Y(t, \theta_i+)d_i. \quad (30)$$

Transforming backwards we have

$$x(t, t_0, x_0) = \frac{1}{Y(t, t_0)x_0^{-1} + \int_{t_0}^{t} Y(t, s)b(s)ds + \sum_{t_0 \leq \theta_i < t} Y(t, \theta_i+)d_i}. \quad (31)$$

By means of (24), one can see that the asymptotic behavior of (31) depends on the sign of γ.

Consider the case when $\gamma > 0$.

If $x_0 > 0$, then as $t_0 \to -\infty$, (31) implies that

$$\lim_{t_0 \to -\infty} x(t, t_0, x_0) = x_\gamma(t) = \frac{1}{\int_{-\infty}^{t} Y(t, s)b(s)ds + \sum_{\theta_i < t} Y(t, \theta_i+)d_i} \quad (32)$$

as long as the solution exists on the interval $[t_0, t]$. To ensure the existence, it is sufficient to have

$$Y(\tau, t_0)x_0^{-1} + \int_{t_0}^{\tau} Y(\tau, s)b(s)ds + \sum_{t_0 \leq \theta_i < \tau} Y(\tau, \theta_i+)d_i > 0 \quad (33)$$

for $\tau \in [t_0, t]$. Let us show that (33) holds if we require $x_0 < (1+\alpha_t)x_\gamma(t_0)$ for some $\alpha_t > 0$.

$$Y(\tau, t_0)x_0^{-1} + \int_{t_0}^{\tau} Y(\tau, s)b(s)ds + \sum_{t_0 \leq \theta_i < \tau} Y(\tau, \theta_i+)d_i$$

$$> \frac{1}{1+\alpha_t}\left\{\int_{-\infty}^{t_0} Y(\tau, s)b(s)ds + \sum_{\theta_i < t_0} Y(\tau, \theta_i+)d_i\right\}$$

$$+ \int_{t_0}^{\tau} Y(\tau, s)b(s)ds + \sum_{t_0 \leq \theta_i < \tau} Y(\tau, \theta_i+)d_i$$

$$= \int_{-\infty}^{\tau} Y(\tau, s)b(s)ds + \sum_{\theta_i < \tau} Y(\tau, \theta_i+)d_i$$

$$- \frac{\alpha_t}{1+\alpha_t}\left\{\int_{-\infty}^{t_0} Y(\tau, s)b(s)ds + \sum_{\theta_i < t_0} Y(\tau, \theta_i+)d_i\right\} > 0$$

for all $t_0 \leq \tau \leq t$. Taking into account the assumption (25), it suffices to show that the last expression holds for any τ from the interval $[T^-, t]$. This can be done by choosing $\alpha_t > 0$ appropriately. Hence, choosing $\delta(t) = \alpha_t m_\gamma$ and implementing Definition 1, it follows that $x_\gamma(t)$ is locally pullback attracting.

Since $x(t) \equiv 0$ and $x_\gamma(t)$ are solutions and the system is order-preserving, any solution with $0 < x_0 < x_\gamma(t_0)$ exists for all $t \leq t_0$. Moreover, assumption (28) implies that

$$0 < \int_{-\infty}^{t} Y(t,s)b(s)ds + \sum_{\theta_i < t} Y(t, \theta_i+)d_i < \infty.$$

Thus, from equation (31) and relation (24), it follows that $x(t, t_0, x_0) \to 0$ as $t \to -\infty$, which implies that the origin is asymptotically unstable.

If $x_0 < 0$, then for t_0 sufficiently large and negative $x(\tau, t_0, x_0)$ blow up for some $\tau \geq t_0$. To see this, note that $Y(t, t_0)x_0^{-1}$ is negative and tends to zero as $t_0 \to -\infty$, while $\int_{t_0}^{t} Y(t,s)b(s)ds + \sum_{t_0 \leq \theta_i < t} Y(t, \theta_i+)d_i$ is positive and bounded below. As a result, $x(\tau, t_0, x_0) \to -\infty$ in a finite time as the denominator of (31) tends to zero for some $\tau \geq t_0$.

Consider the case $\gamma < 0$.

From equation (31) and relation (24), it follows that $x(t, t_0, x_0) \to 0$ as $t_0 \to -\infty$ for any $x_0 \neq 0$ as long as $x(\tau, t_0, x_0)$ exists for all $\tau \in [t_0, t]$.

For $x_0 > 0$, it is sufficient to show that

$$Y(\tau, t_0)x_0^{-1} + \int_{t_0}^{\tau} Y(\tau, s)b(s)ds + \sum_{t_0 \leq \theta_i < \tau} Y(\tau, \theta_i+)d_i > 0 \qquad (34)$$

for $\tau \in [t_0, t]$. By means of (25), inequality (34) is satisfied if

$$Y(\tau, t_0)x_0^{-1} + \int_{T^-}^{\tau} Y(\tau, s)b(s)ds + \sum_{T^- \leq \theta_i < \tau} Y(\tau, \theta_i+)d_i > 0 \qquad (35)$$

for $\tau \in [T^-, t]$. Because of assumption (24), for t_0 small enough $Y(\tau, t_0)$ is bounded below on $(-\infty, T^-]$. Thus, (34) is satisfied provided that

$$x_0 < \frac{\inf_{t_0 \leq T^-} Y(\tau, t_0)}{\sup_{\tau \in [T^-, t]} |\int_{T^-}^{\tau} Y(\tau, s)b(s)ds + \sum_{T^- \leq \theta_i < \tau} Y(\tau, \theta_i+)d_i|}. \qquad (36)$$

For $x_0 < 0$ the argument requires condition (29), which implies that there exists a μ_t such that

$$\frac{Y(\tau, t_0)}{\int_{t_0}^{\tau} Y(\tau, s)b(s)ds + \sum_{t_0 \le \theta_i < \tau} Y(\tau, \theta_i+)d_i} \ge \frac{m_\gamma}{2} \qquad (37)$$

for all $t_0 \le \mu_t$. Now, it is sufficient to show that

$$Y(\tau, t_0)x_0^{-1} + \int_{t_0}^{\tau} Y(\tau, s)b(s)ds + \sum_{t_0 \le \theta_i < \tau} Y(\tau, \theta_i+)d_i < 0 \qquad (38)$$

for $\tau \in [t_0, t]$. Denote $I(t_0, \tau) = \int_{t_0}^{\tau} Y(\tau, s)b(s)ds + \sum_{t_0 \le \theta_i < \tau} Y(\tau, \theta_i+)d_i$.
If $I(t_0, \tau) < 0$, then (38) is satisfied. If $I(t_0, \tau) > 0$, then we require

$$|x_0| < \frac{Y(\tau, t_0)}{\int_{t_0}^{\tau} Y(\tau, s)b(s)ds + \sum_{t_0 \le \theta_i < \tau} Y(\tau, \theta_i+)d_i},$$

which has the right-hand side of this expression is bounded below by $\frac{m_\gamma}{2}$ using (37). Therefore, for each t there exists a μ_t such that if $t_0 \le \mu_t$ and $|x_0|$ is sufficiently small, the solution exists on $[t_0, t]$ and, hence, the origin is locally pullback attracting. The theorem is proved. $\qquad \square$

Next, we want to formulate an impulsive extension of the system (23), which is related to the forward attraction. We assume that the functions b and d_i are asymptotically positive as $t \to \infty$, and the 'balance condition' (28) is valid. That is,

$$b(t) \ge \bar{b} > 0 \text{ for all } t \ge T^+, \text{ and } d_i \ge \bar{d} > 0 \text{ for all } \theta_i \ge T^+. \qquad (39)$$

$$0 < m_\gamma \le x_\gamma(t) = \frac{1}{\int_{-\infty}^{t} Y(t, s)b(s)ds + \sum_{\theta_i < t} Y(t, \theta_i+)d_i} \le M_\gamma \qquad (40)$$

for all $t \in \mathbb{R}$, $0 < \gamma < \gamma_0$.

Theorem 3. *Assume the above conditions hold for equation (23). Then, for $-\gamma_0 < \gamma < 0$ the origin is locally forward attracting, and for $0 < \gamma < \gamma_0$ the trajectory $x_\gamma(t)$ is locally forward attracting. In addition, if*

$$0 < m_\gamma \le x_\gamma(t) = \frac{1}{\int_{t}^{\infty} Y(t, s)b(s)ds + \sum_{\theta_i < t} Y(t, \theta_i+)d_i} \le M_\gamma \qquad (41)$$

for all $t \in \mathbb{R}, \gamma < 0$, then for $-\gamma_0 < \gamma < 0$ the trajectory $x_\gamma(t)$ is both asymptotically unstable and locally pullback repelling.

Proof. If $\gamma < 0$, the origin is locally forward attracting when x_0 is sufficiently small, since condition (39) implies that

$$\inf_{t \geq t_0} \left\{ \int_{t_0}^t Y(t,s)b(s)ds + \sum_{t_0 \leq \theta_i < t} Y(t,\theta_i+)d_i \right\} > -\infty.$$

If $\gamma > 0$, the trajectory $x_\gamma(t)$ is locally forward attracting. To see this, we notice that

$$\left(\frac{1}{x(t)} - \frac{1}{x_\gamma(t)} \right) = Y(t,t_0) \left(\frac{1}{x_0} - \frac{1}{x_\gamma(t_0)} \right).$$

Therefore,

$$|x(t) - x_\gamma(t)| = \frac{x_\gamma(t)x(t)}{x_\gamma(t_0)x_0} e^{\left(\frac{-\int_{t_0}^t a(u)du + \sum_{t_0 \leq \theta_i < t} \ln c_i}{t - t_0} \right)(t-t_0)} |x_0 - x_\gamma(t_0)|. \quad (42)$$

Using the balance condition (40) with $x_0 > 0$ implies that

$$x(t) = \frac{1}{Y(t,t_0)x_0^{-1} + \int_{t_0}^t Y(t,s)b(s)ds + \sum_{t_0 \leq \theta_i < t} Y(t,\theta_i+)d_i}$$

$$\leq M_\gamma \frac{\int_{-\infty}^t Y(t,s)b(s)ds + \sum_{\theta_i < t} Y(t,\theta_i+)d_i}{Y(t,t_0)x_0^{-1} + \int_{t_0}^t Y(t,s)b(s)ds + \sum_{t_0 \leq \theta_i < t} Y(t,\theta_i+)d_i}$$

$$= M_\gamma \frac{Y(t,t_0)x_\gamma^{-1}(t_0) + \int_{t_0}^t Y(t,s)b(s)ds + \sum_{t_0 \leq \theta_i < t} Y(t,\theta_i+)d_i}{Y(t,t_0)x_0^{-1} + \int_{t_0}^t Y(t,s)b(s)ds + \sum_{t_0 \leq \theta_i < t} Y(t,\theta_i+)d_i}.$$

Condition (39) guarantees that the integral and the sum in the numerator and denominator are positive for t sufficiently large. So, from the last expression it follows that

$$\limsup_{t \to \infty} x(t) \leq M_\gamma \max \left\{ 1, \frac{x_0}{x_\gamma(t_0)} \right\}.$$

Therefore, any solution with $x_0 > 0$ is bounded as $t \to \infty$. Hence, from (42) it follows that $x_\gamma(t)$ is forward attracting as long as solutions exist.

Next, we show that solutions dot not blow up for $x_0 < (1 + \alpha_{t_0})x_\gamma(t_0)$.

$$Y(t, t_0)x_0^{-1} + \int_{t_0}^t Y(t, s)b(s)ds + \sum_{t_0 \le \theta_i < t} Y(t, \theta_i+)d_i$$

$$> \frac{1}{1 + \alpha_{t_0}} \left\{ \int_{-\infty}^{t_0} Y(t, s)b(s)ds + \sum_{\theta_i < t_0} Y(t, \theta_i+)d_i \right\}$$

$$+ \int_{t_0}^t Y(t, s)b(s)ds + \sum_{t_0 \le \theta_i < t} Y(t, \theta_i+)d_i$$

$$= \int_{-\infty}^t Y(t, s)b(s)ds + \sum_{\theta_i < t} Y(t, \theta_i)d_i$$

$$- \frac{\alpha_{t_0}}{1 + \alpha_{t_0}} \left\{ \int_{-\infty}^{t_0} Y(t, s)b(s)ds + \sum_{\theta_i < t_0} Y(t, \theta_i+)d_i \right\}.$$

The last expression is positive for sufficiently small α_{t_0} because of the assumption (39). Therefore, $x_\gamma(t)$ is locally forward attracting.

Under the final assumption (41), the results follow by making the transformations

$$\gamma \mapsto -\gamma, \quad x \mapsto -x, \quad \theta \mapsto -\theta \quad \text{and} \quad t \mapsto -t.$$

$\square$

Remark 2. In this paper, we do not consider the formal impulsive analogue of (2),

$$\begin{aligned}\dot{x} &= a(t)x - b(t)x^2, \\ \Delta x|_{t=\theta_i} &= c_i x + d_i x^2,\end{aligned} \tag{43}$$

since it is not possible to find explicit solution of the system (43).

Example 2. Let $a(t) \equiv a$, $c_i \equiv c$, and $\theta_i = ih$ for the system (23) with $h > 0$. That is,

$$\begin{aligned}\dot{x} &= ax - b(t)x^2, \\ \Delta x|_{t=ih} &= -x + \frac{x}{c+d_i x}.\end{aligned} \tag{44}$$

Then $\gamma = a - \frac{1}{h} \ln c$. By means of $y = \frac{1}{x}$, the system (21) is converted to the linear impulsive system

$$\begin{aligned}\dot{y} &= -ay + b(t), \\ \Delta y|_{t=ih} &= (c-1)y + d_i.\end{aligned} \qquad (45)$$

Asymptotic behavior of (45) depends on the sign of γ, and results of Theorem 2 and Theorem 3 are true for the system (44). If $c = 1$ and $d_i = 0$, then $\gamma = a$ and there is no equation of jumps in the system (44).

4. Bifurcation in the non-order-preserving system

In the continuous differential equations requiring uniqueness implies that a system is order-preserving. However, in impulsive systems order-
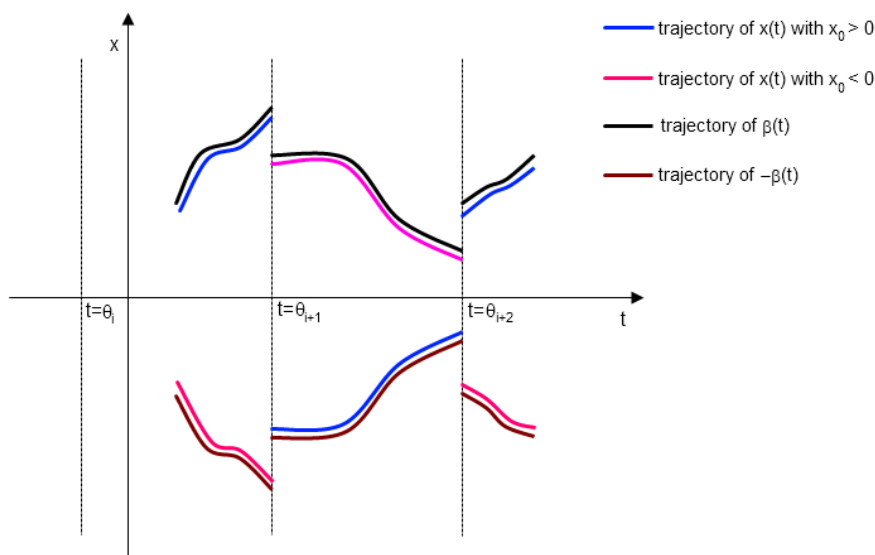

FIGURE 2. Asymptotic behavior of the system (5).

preservation is violated even for the scalar case if we do not impose any

condition on the jump equation. In this section, we want to consider a non-order-preserving system and analyze bifurcation phenomena. Let us consider the system (5) which differs only by the jump equation from the system (10). We assume the same conditions for (5) as for (10). The impulsive equation of (5) can be rewritten as $x(\theta_i+) = -\frac{x(\theta_i)}{\sqrt{c_i+d_i x^2(\theta_i)}}$. Defining $f(x) = -\frac{x}{\sqrt{c_i+d_i x^2}}$, one can check that $f'(x) < 0$. Although uniqueness of solutions is assumed, the system (5) is non-order-preserving due to the jump equations. However, by means of transformation $y = \frac{1}{x^2}$, the system (5) is also transformed into the system (15). Therefore, the results of Theorem 1 are also true for the system (5). Exceptionally, since the system (5) is non-order-preserving, for $\gamma > 0$ all trajectories of the system (5) are in the neighborhood of $|\beta(t,\gamma)|$ and alternatively change their position from neighborhood the of $\beta(t,\gamma)$ to the neighborhood of $-\beta(t,\gamma)$ as it is shown in Figure 2.

5. CONCLUSION

The pitchfork and the transcritical bifurcations are considered for non-autonomous impulsive differential equations. Explicitly solvable models with the specific equations of jump have been considered. This allowed us to categorize one-dimensional bifurcations in impulsive systems which are order-preserving. Moreover, the non-order-preserving system is studied.

This theory could be developed in many ways. One can consider formal impulsive analogues for the pitchfork bifurcation for the system (20), and corresponding formal impulsive analogue for the system (43), for the transcritical bifurcation without finding explicit solution similarly to that done in [35]. Non-autonomous saddle-node bifurcation remains unconsidered even for one-dimensional impulsive systems. Finally, general theory of higher-dimensional bifurcation results with impulses has to be developed.

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MIDDLE EAST TECHNICAL UNIVERSITY, DEPARTMENT OF MATHEMATICS, 06531
ANKARA, TURKEY

CORRESPONDING AUTHOR* TEL.:+90 312 210 5355, FAX:+90 312 210 2972
E-mail address: marat@metu.edu.tr

MIDDLE EAST TECHNICAL UNIVERSITY, DEPARTMENT OF MATHEMATICS, 06531
ANKARA, TURKEY
E-mail address: ardaky@gmail.com