Electronic Journal of Qualitative Theory of Differential Equations
2006, No. **7**, 1-14; **http://www.math.u-szeged.hu/ejqtde/**

Positive Solutions of Second Order Boundary Value Problems With Changing Signs Carathéodory Nonlinearities

Abdelkader Boucherif *
King Fahd University of Petroleum and Minerals
Department of Mathematical Sciences, Box 5046
Dhahran, 31261, Saudi Arabia
e-mail: aboucher@kfupm.edu.sa

Johnny Henderson
Department of Mathematics
Baylor University, Waco, TX 76798-7328, USA
e-mail: Johnny_Henderson@baylor.edu

Abstract

In this paper we investigate the existence of positive solutions of two-point boundary value problems for nonlinear second order differential equations of the form $(py')'(t)+q(t)y(t) = f(t, y(t), y'(t))$, where f is a Carathéodory function, which may change sign, with respect to its second argument, infinitely many times.

2000 Mathematics Subject Classification: 34B15, 34B18
Keywords: positive solutions, Carathéodory function, topological transversality theorem, upper and lower solutions.

1 Introduction

We are interested in the existence of positive solutions of the two-point boundary value problem,

$$\begin{aligned} &(py')'(t) + q(t)y(t) = f(t, y(t), y'(t)), \quad 0 < t < 1, \\ &y(0) = y(1) = 0. \end{aligned} \tag{1.1}$$

*corresponding author

Problems of this type arise naturally in the description of physical phenomena, where only positive solutions, that is, solutions y satisfying $y(t) > 0$ for all $t \in (0,1)$, are meaningful. It is well known that Krasnoselskii's fixed point theorem in a cone has been instrumental in proving existence of positive solutions of problem (1.1). Most of the previous works deal with the case $p(t) = 1$, $q(t) = 0$, for all $t \in [0,1]$, and assume that f is nonnegative, that f does not depend on y', and that $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous and satisfies, either

$$\liminf_{u\to 0+} \min_{0\le t\le 1} \frac{f(t,u)}{u} = +\infty, \text{ and } \limsup_{u\to +\infty} \max_{0\le t\le 1} \frac{f(t,u)}{u} = 0 \text{ (sublinear case)},$$

or

$$\limsup_{u\to o+} \max_{0\le t\le 1} \frac{f(t,u)}{u} = 0, \text{ and } \liminf_{u\to +\infty} \min_{0\le t\le 1} \frac{f(t,u)}{u} = +\infty \text{ (superlinear case)}.$$

See for instance [1], [7], [12], [13], [16] and the references therein. The above conditions have been relaxed in [17] and [18], where the authors remove the condition f is nonnegative, and they consider the behavior of f relative to π^2. Notice that π^2 is the first eigenvalue of the operator $u \to -u''$, subject to the boundary conditions in (1.1). The arguments in [17] and [18] are based on fixed point index theory in a cone. When the nonlinear term depends also on the first derivative of y, we refer the interested reader to [2], [3], [19]. The authors in [2], [8] and [20] deal with a singular problem. Several papers are concerned with the problem of the existence of multiple solutions. See for instance [3], [13], [14], [15] and [21]. However, our assumptions are simple and more general. In fact, we obtain a multiplicity result as a byproduct of our main result, with no extra assumptions. We exploit the fact that the nonlinearity changes sign with respect to its second argument. We do not rely on cone preserving mappings. Also, the sign of the Green's function of the corresponding linear homogeneous problem plays no role in our study. However, we assume the existence of positive lower and upper solutions. We provide an example to motivate our assumptions. Our results complement and generalize those obtained in [21].

2 Topological Transversality Theory

In this section, we recall the most important notions and results related to the topological transversality theory due to Granas. See Granas-Dugundji [10] for the details of the theory.

Let X be a Banach space, $\mathcal{C}$ a convex subset of X and U an open set in $\mathcal{C}$.

(i) $g : X \to X$ is compact if $\overline{g(X)}$ is compact

(ii) $H : [0,1] \times X \to X$ is a compact homotopy if H is a homotopy and for all $\lambda \in [0,1]$, $H(\lambda, \cdot) : X \to X$ is compact.

(iii) $g : \overline{U} \to \mathcal{C}$ is called admissible if g is compact and has no fixed points on $\Gamma = \partial U$.

Let $\mathcal{M}_\Gamma(\overline{U}, \mathcal{C})$ denote the class of all admissible maps from $\overline{U}$ to $\mathcal{C}$.

(iv) A compact homotopy H is admissible if, for each $\lambda \in [0,1]$, $H(\lambda, \cdot)$ is admissible.

(v) Two mappings g and h in $\mathcal{M}_\Gamma(\overline{U}, \mathcal{C})$ are homotopic if there is an admissible homotopy $H : [0,1] \times \overline{U} \to \mathcal{C}$ such that $H(0, \cdot) = g$ and $H(1, \cdot) = h$.

(vi) $g \in \mathcal{M}_\Gamma(\overline{U}, \mathcal{C})$ is called inessential if there is a fixed point free compact map $h : \overline{U} \to \mathcal{C}$ such that $g|_\Gamma = h|_\Gamma$. Otherwise, g is called essential

Lemma 2.1 *Let d be an arbitrary point in U and $g \in \mathcal{M}_\Gamma(\overline{U}, \mathcal{C})$ be the constant map $g(x) \equiv d$. Then g is essential.*

Lemma 2.2 *$g \in \mathcal{M}_\Gamma(\overline{U}, \mathcal{C})$ is inessential if and only if g is homotopic to a fixed point free compact map.*

Theorem 2.1 *Let g, $h \in \mathcal{M}_\Gamma(\overline{U}, \mathcal{C})$ be homotopic maps. Then g is essential if and only if h is essential.*

3 Preliminaries

3.1 Function Spaces.

Let I denote the real interval $[0,1]$, and let $\mathbb{R}_+$ denote the set of all nonnegative real numbers. For $k = 0, 1, \ldots,$ $C^k(I)$ denotes the space of all functions $u : I \to \mathbb{R}$, whose derivatives up to order k are continuous. For $u \in C^k(I)$ we define its norm by $\|u\| = \sum_{i=0}^{k} \{\max \left|u^{(i)}(t)\right| : t \in I\}$. Equipped with this norm $C^k(I)$ is a Banach space. When $k = 0$, we shall use the notation $\|u\|_0$ for the norm of $u \in C(I)$. Also $C_0^1(I)$ shall denote the space $\{y \in C^1(I) : y(0) = y(1) = 0\}$. It can be easily shown that $(C_0^1(I), \ \|\cdot\|)$ is a Banach space.

A real valued function f defined on $I \times \mathbb{R}^2$ is said to be an L^1-Caratheodory function if it satisfies the following conditions

(i) $f(t,\cdot)$ is continuous for almost all $t \in I$

(ii) $f(\cdot, z)$ is measurable for all $z \in \mathbb{R}^2$

(iii) for each $\rho > 0$, there exists $h_\rho \in L^1(I;\mathbb{R}_+)$ such that $\|z\|_{\mathbb{R}^2} \le \rho$ implies $|f(t,z)| \le h_\rho(t)$ for almost all $t \in I$.

3.2 A Linear Problem

Consider the following linear boundary value problem

$$\begin{cases} (py')'(t) + q(t)y(t) = h(t), & t \in (0,1), \\ y(0) = 0, \quad y(1) = 0, \end{cases}$$

where the coefficient functions p and q satisfy

> **(H0)** $p \in C^1(I)$, $q \in C(I)$, $p(t) \ge p_0 > 0$ for all $t \in I$, $q(t) \le p_0\pi^2$ with strict inequality on a subset of I with positive measure.

Lemma 3.1 *Assume that* (H0) *is satisfied. Then for any nontrivial* $y \in C_0^1(I)$*, we have*

$$\int_0^1 [p(t)y'(t)^2 - q(t)y(t)^2]dt > 0.$$

Proof. It follows from (H0) that

$$\int_0^1 [p(t)y'(t)^2 - q(t)y(t)]dt > p_0 \int_0^1 [y'(t)^2 - \pi^2 y(t)^2]dt.$$

Consider the functional $\chi : C_0^1(I) \to \mathbb{R}$ defined by

$$\chi(y) = \int_0^1 [y'(t)^2 - \pi^2 y(t)^2]dt.$$

Results from the classical calculus of variations (see [6]) shows that $\chi(y) \ge 0$ for all $y \in C_0^1(I)$. Hence, the conclusion of the Lemma holds.

Lemma 3.2 *If* (H0) *is satisfied, then the linear homogeneous problem* $(py')'(t) + q(t)y(t) = 0$, $y(0) = y(1) = 0$, *has only the trivial solution.*

Proof. Assume on the contrary that the problem has a nontrivial solution y_0. Then, we have

$$0 = \int_0^1 [(py_0')'(t) + q(t)y_0(t)]y_0(t)dt.$$

But, Lemma 3.2 implies that

$$\int_0^1 [(py_0')'(t) + q(t)y_0(t)]y_0(t)dt = -[\int_0^1 [(p(t)y_0'(t)^2 - q(t)y_0(t)^2]dt < 0.$$

This contradiction shows that $y_0 = 0$, and the proof is complete.

As a consequence of Lemma 3.2, the corresponding Green's function, $G(t,s)$, exists and the linear nonhomogeneous problem $(py')'(t)+q(t)y(t) = h(t), \quad y(0) = y(1) = 0$, has a unique solution, given by $y(t) = \int_0^1 G(t,s)h(s)ds$.

Define a linear operator $L : W^{2,1}(I) \cap C_0^1(I) \to L^1(I)$ by

$$Ly(t) := (py')'(t) + q(t)y(t), \qquad t \in I.$$

It follows from the above that L is one-to-one and onto, and $L^{-1} : L^1(I) \to W^{2,1}(I) \cap C_0^1(I)$ is defined by

$$L^{-1}h(t) := \int_0^1 G(t,s)h(s)ds.$$

By the compactness of the embedding $W^{2,1}(I) \hookrightarrow C^1(I)$, the operator L^{-1} maps $L^1(I)$ into $C_0^1(I)$ and is compact.

4 Main Results

Consider the nonlinear problem (1.1)

$$\begin{cases} (py')'(t) + q(t)y(t) = f(t, y(t), y'(t)), & t \in (0,1), \\ y(0) = 0, \quad y(1) = 0. \end{cases}$$

The nonlinearity $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is an L^1-Carathéodory function and satisfies

(H1) There exist positive functions $\alpha \le \beta$ in $C_0^1(I)$ such that for all $t \in I$,

(i) $L\alpha\,(t) \ge f\,(t, \alpha(t), \alpha'(t))$, $L\beta\,(t) \le f\,(t, \beta(t), \beta'(t))$

(ii) $f\,(t, \alpha(t), \alpha'(t)) > 0 > f\,(t, \beta(t), \beta'(t))$

(H2) there exist $C > 0$, $Q \in L^1(I; \mathbb{R}_+)$ and $\Psi : [0, +\infty) \to [1, +\infty)$ continuous and nondecreasing with $\dfrac{1}{\Psi}$ integrable over bounded intervals and $\int_0^{+\infty} \dfrac{du}{\Psi(u)} = +\infty$, such that $|f(t,y,z)| \leq \Psi(|z|)(Q(t)+C|z|)$ for all $t \in [0,1]$, $\alpha \leq y \leq \beta$, $z \in \mathbb{R}$.

Remarks.

(i) $\alpha \leq y \leq \beta$ means $\alpha(t) \leq y(t) \leq \beta(t)$ for all $t \in I$.

(ii) Condition (H2) is known as a Nagumo-Wintner condition, and is more general than the usual Nagumo or Nagumo-Bernstein conditions.

Theorem 4.1 *Assume* (H1) *and* (H2) *are satisfied. Then* (1.1) *has at least one positive solution* $y \in [\alpha, \beta]$.

Proof. The proof will be given in several steps. Consider $\delta(y) = \delta(\alpha, y, \beta) = \max(\alpha, \min(y, \beta))$.

Since our arguments are based on the topological transversality theory, we consider the following one-parameter family of problems,

$$\begin{cases} (py')'(t) + q(t)y(t) = \lambda f_1(t, y(t), y'(t)), & t \in (0,1), \\ y(0) = 0, \quad y(1) = 0, \end{cases} \tag{1.2_λ}$$

where $0 \leq \lambda \leq 1$ and

$$f_1(t, y(t), y'(t)) = f(t, \delta(y(t)), \delta(y)'(t)). \tag{1.3}$$

Notice that (1.2_0) has only the trivial solution. Hence, we shall consider only the case $0 < \lambda \leq 1$.

Step.1. All possible solutions of (1.2_λ) satisfy $y(t) \leq \beta(t)$ for all $t \in I$.

Let $y \neq 0$ be a possible solution of (1.2_λ), and suppose on the contrary that there is a $\tau \in (0,1)$ such that $y(\tau) > \beta(\tau)$. Then, there exists a maximal interval $I_1 = (a,b)$ such that $\tau \in I_1$ and $y(t) > \beta(t)$ for all $t \in I_1$.

It follows that $\delta(y(t)) = \delta(\alpha(t), y(t), \beta(t)) = \beta(t)$ and $\delta(y)'(t) = \beta'(t)$ for all $t \in (a,b)$.

Let $z(t) := y(t) - \beta(t)$. Then, we have $z(t) > 0$ for all $t \in I_1$.

We have two possibilities.

(i) $\overline{I_1} \subset I$. Then $z(a) = 0$, $z(b) = 0$ and $z(t) > 0$ for all $t \in (a,b)$. Hence, by (H1), for all $t \in (a,b)$

$$\begin{aligned} Lz(t) &= Ly(t) - L\beta(t) = \\ \lambda f_1(t, y(t), y'(t)) - L\beta(t) &= \lambda f(t, \delta(y(t)), \delta(y)'(t)) - L\beta(t) \\ &\geq \lambda f(t, \delta(y(t)), \delta(y)'(t)) - f(t, \beta(t), \beta'(t)) \\ &= (\lambda - 1) f(t, \beta(t), \beta'(t)) \geq 0, \end{aligned}$$

so that

$$\int_{I_1} Lz(t)z(t)dt = \int_a^b Lz(t)z(t)dt > 0.$$

On the other hand, Lemma 3.2 implies that

$$\int_a^b Lz(t)z(t)dt < 0.$$

This is a contradiction.

(ii) $\overline{I_1} = I$. In this case $z(0) = 0,\ \ z(1) = 0$ and $z(t) > 0$ for all $t \in (0,1)$. It follows that

$$\int_0^1 Lz(t)z(t)dt > 0,$$

But again

$$\int_0^1 Lz(t)z(t)dt < 0,$$

and again, we arrive at a contradiction.

Therefore, we conclude

$$y(t) \le \beta(t) \ \text{ for all } t \in I.$$

Similarly, we can prove that $\alpha(t) \le y(t)$ for all $t \in I$.

Hence, we have shown that any solution y of (1.2_λ) satisfies

$$\alpha\,(t) \le y(t) \le \beta\,(t) \quad \text{for all } t \in I. \tag{1.4}$$

But, for all y satisfying (1.4) f_1 and f coincide. So, for $\lambda = 1$, we have that all solutions of (1.2_1) are solutions of (1.1). Moreover solutions of (1.2_λ) satisfy the a priori bound, independently of λ,

$$\|y\|_0 \le K_0 := \|\beta\|_0\,.$$

Step.2. A priori bound on the derivative y' for solutions y of (1.2_λ) satisfying the inequality (1.4).

Define $K_1 > 0$ by the formula $\int_0^{K_1} \dfrac{du}{\Psi(u)} > \dfrac{1}{p_0}[\|Q_0\|_{L^1} + 2(C + \|p'\|_0)\ K_0]$, (this is possible because of the property of Ψ), where $Q_0(t) = Q(t) + \|q\|_0\,K_0$.

We want to show that $|y'(t)| \le K_1$ for all $t \in I$. Suppose, on the contrary that there exists τ_1 such that $|y'(\tau_1)| > K_1$. Then, there exists an interval $[\mu, \xi] \subset [0,1]$

such that the following situations occur:

$$
\begin{array}{lll}
\text{(i)} & y'(\mu)=0, \quad y'(\xi)=K_1, \quad 0<y'(t)<K_1 & \mu<t<\xi, \\
\text{(ii)} & y'(\mu)=K_1, \quad y'(\xi)=0, \quad 0<y'(t)<K_1 & \mu<t<\xi, \\
\text{(iii)} & y'(\mu)=0, \quad y'(\xi)=-K_1, \ -K_1<y'(t)<0 & \mu<t<\xi, \\
\text{(iv)} & y'(\mu)=-K_1, \quad y'(\xi)=0, \ -K_1<y'(t)<0 & \mu<t<\xi.
\end{array}
$$

We study the first case. The others can be handled in a similar way. For (i), the differential equation in (1.2_λ) and condition (H2) imply

$$
|Ly(t)| \leq \Psi\left(y'(t)\right)\left(Q(t)+Cy'(t)\right) \qquad \mu \leq t \leq \xi. \tag{1.5}
$$

Since

$$
\begin{aligned}
|Ly(t)| &= |p(t)y''(t)+p'(t)y'(t)+q(t)y(t)| \\
&\geq |p(t)y''(t)| - |p'(t)y'(t)| - |q(t)y(t)|,
\end{aligned}
$$

we have

$$
\begin{aligned}
|Ly(t)| &\geq |p(t)y''(t)| - \|p'\|_0 |y'(t)| - \|q\|_0 K_0 \\
&\geq p_0 y''(t) - \|p'\|_0 |y'(t)| - \|q\|_0 K_0.
\end{aligned}
$$

It follows from (1.5) that

$$
\begin{aligned}
p_0 y''(t) &\leq \Psi\left(y'(t)\right)\left(Q(t)+Cy'(t)\right) \ + \|p'\|_0 |y'(t)| + \|q\|_0 K_0 \\
&\leq Q(t)\Psi\left(y'(t)\right) + Cy'(t)\Psi\left(y'(t)\right) + \|p'\|_0 |y'(t)| + \|q\|_0 K_0.
\end{aligned}
$$

Notice that $\Psi(z) \geq 1$ for all $z \geq 0$, so that

$$
\begin{aligned}
p_0 y''(t) &\leq \Psi\left(y'(t)\right)\left[\left(Q(t)+\|q\|_0 K_0\right) + y'(t)\left(C+\|p'\|_0\right)\right] \\
&\leq \Psi\left(y'(t)\right)\left(Q_0(t) + \left(C+\|p'\|_0\right) y'(t)\right).
\end{aligned}
$$

This implies

$$
\frac{y''(t)}{\Psi\left(y'(t)\right)} \leq \frac{1}{p_0}\left(Q_0(t) + \left(C+\|p'\|_0\right) y'(t)\right) \qquad \text{for} \quad \mu \leq t \leq \xi.
$$

Integration from μ to ξ, and a change of variables (see [8, Lemma A.10]) lead to

$$\int_0^{K_1} \frac{du}{\Psi(u)} \leq \frac{1}{p_0}[\|Q_0\|_{L^1} + 2\left(C + \|p'\|_0\right) K_0.$$

This clearly contradicts the definition of K_1.

Taking into consideration all the four cases above, we see that

$$|y'(t)| \leq K_1 \quad \text{ for all } t \in I.$$

Let

$$K_2 := \max\{K_1, \|\alpha'\|_0, \|\beta'\|_0\}.$$

Then, any solution y of (1.2_λ) satisfying the inequality (1.4), is such that its first derivative y' will satisfy the a priori bound

$$|y'(t)| \leq K_2 \quad \text{ for all } t \in I.$$

As a consequence of Step 1 and Step 2 above, we deduce that any solution y of (1.2_λ) satisfies

$$\|y\| \leq K := K_0 + K_2. \tag{1.6}$$

Since f is an L^1-Caratheodory function, it follows from (1.6) that there exists $h_K \in L^1(I : \mathbb{R}_+)$ such that $|f(t, y(t), y'(t))| \leq h_K(t)$, for almost every $t \in I$. Now, the differential equation in (1.1) implies there exists $\phi \in L^1(I : \mathbb{R}_+)$, depending on only $p, ||p'||_0, ||q||_0, h_K$ such that $y''(t) \leq \phi(t)$ for almost every $t \in I$. In particular, $y'' \in L^1(I : \mathbb{R}_+)$.

Step.3. Existence of solutions of (1.2_λ).

If $G(t, s)$ is the Green's function corresponding to the linear homogeneous problem $(py')'(t) + q(t)y(t) = 0$, $y(0) = 0 = y\,(1)$, then problem (1.2_λ) is equivalent to

$$y(t) = \lambda \int_0^1 G(t, s) f_1(s, y(s), y'(s))ds \tag{1.7}$$

Let

$$F_1 : C^1(I) \to L^1(I)$$

be defined by, for all $t \in I$,

$$F_1(y)(t) = f_1(t, y(t), y'(t)).$$

For all $\lambda \in [0, 1]$, $t \in I$, define

$$H : [0,1] \times C_0^1(I) \to C_0^1(I)$$

by

$$H(\lambda, y) = \lambda L^{-1} F_1(y),$$

$$\text{i.e.} \quad H(\lambda, y)(t) = \lambda \int_0^1 G(t,s) f_1(s, y(s), y'(s)) ds. \tag{1.8}$$

Since L^{-1} is compact and F_1 is continuous it follows that $H(\lambda, \cdot)$ is a compact operator. It is easily seen that $H(\cdot, \cdot)$ is uniformly continuous in λ.

Let

$$U := \{y \in C_0^1(I) : \quad \|y\| < 1 + K\}.$$

It is clear from Steps 1 and 2 above and the choice of U that there is no $y \in \partial U$ such that $H(\lambda, y) = y$ for $\lambda \in [0,1]$. This shows that $H(\lambda, \cdot) : \overline{U} \to C_0^1(I)$ is an admissible homotopy; i.e. a compact homotopy without fixed points on ∂U, the boundary of U.

Therefore, $H(\lambda, \cdot) : \overline{U} \to C_0^1(I)$ is an admissible homotopy between the constant map $H(0, \cdot) = 0$ and the compact map $H(1, \cdot)$. Since $0 \in U$, we have that $H(0, \cdot)$ is essential. By the topological transversality theorem of Granas, $H(1, \cdot)$ is essential. This implies that it has a fixed point in U, and this fixed point is a solution of (1.2_1). Since solutions of (1.2_1) are solutions of (1.1) we conclude that (1.1) has at least one solution, which is necessarily positive because of (1.4).

This completes the proof of the main result.

Remark. It is possible to obtain a uniqueness result if we assume, in addition to (H1) and (H2), the following condition:

(H3) There exists a constant M, such that $q(t) + M \leq q_0 \pi^2$, with strict inequality on a subset of I with positive measure, and $f(t, y_1, z) - f(t, y_2, z) \geq -M(y_1 - y_2)$ for all $t \in I$, $z \in \mathbb{R}$ and $\alpha \leq y_2 \leq y_1 \leq \beta$.

Theorem 4.2 *Assume that the conditions* (H1), (H2) *and* (H3) *hold. Then* (1.1) *has a unique positive solution* $y \in [\alpha, \beta]$.

Proof. Theorem 4.1 guarantees the existence of at least one solution $y \in [\alpha, \beta]$. For contradiction, suppose there are two solutions $u, v \in [\alpha, \beta]$.

Assume first that $u(t) \leq v(t)$. Then

$$w(t) := v(t) - u(t) \geq 0 \qquad \text{for all } t \in I.$$

Since

$$Lv(t) - Lu(t) = f(t, v(t), v'(t)) - f(t, u(t), u'(t)) \qquad \text{for all } t \in I,$$

assumption (H3) implies that

$$Lw(t) \geq -Mw(t) \ \text{ for all } t \in I,$$

or

$$(L + M)w(t) \geq 0 \text{ for all } t \in I. \tag{$*.1$}$$

Suppose, next that $v(t) \leq u(t)$ for all $t \in I$. Then $-w(t) \geq 0$ for all $t \in I$. This yields

$$(L + M)(-w(t)) \geq 0 \text{ for all } t \in I. \tag{$*.2$}$$

Comparing ($*$.1) and ($*$.2) we see that $(L+M)w(t) = 0$ for all $t \in I$.The assumption on M and Lemma 3.3 imply that $w(t) = 0$ for all $t \in I$. Therefore

$$u(t) = v(t) \ \text{ for all } t \in I.$$

This proves uniqueness.

5 Multiplicity of Solutions

In this section we use the previous result to get multiplicity of solutions of problem (1.1).

Theorem 5.1 *Assume* $f : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ *is an* L^1*-Caratheodory function and satisfies:*

(H4) *there are sequences* $\{\alpha_j\}, \{\beta_j\}$ *of positive functions in* $C_0^1(I)$ *such that for all* $j = 1, 2, \ldots,$

(i) $0 < \alpha_j < \beta_j \leq \alpha_{j+1}$,

(ii) $L\alpha_j(t) \geq f(t, \alpha_j(t), \alpha_j'(t))$, $L\beta_j(t) \leq f(t, \beta_j(t), \beta_j'(t))$,

(iii) $f(t, \alpha_j(t), \alpha_j'(t)) > 0 > f(t, \beta_j(t), \beta_j'(t))$, $t \in [0, 1]$,

(iv) *the condition* (H2) *holds on* $[0, 1] \times [\alpha_j, \beta_j] \times \mathbb{R}$.

Then (1.1) *has infinitely many positive solutions* y_j *such that* $\alpha_j \leq y_j \leq \beta_j$.

6 Example

Assume $p(t) = 1$ and $q(t) = 0$ for all $t \in [0, 1]$ and consider the following problem,

$$\begin{cases} y''(t) = \phi(t)\,(1 + y'(t)^2)g(y(t)), & 0 < t < 1, \\ y(0) = y(1) = 0, \end{cases} \tag{1.9}$$

where $\phi \in L^1(I)$ and $g : \mathbb{R} \to \mathbb{R}$ is continuous and has an infinite number of positive simple zeros. This is the case if we assume the existence of an increasing sequence $\{a_j\}_{j\in\mathbb{N}}$ of positive numbers such that

$$g(a_j)\; g(a_{j+1}) < 0 \qquad \text{for } j = 0, 1, \ldots.$$

We know of no applicable previous published works. However, f satisfies condition (H4) of Theorem 5.1, hence Problem (1.9) has infinitely many positive solutions.

Remark. A typical example for g is $g(y) = \sin y$, whose positive zeros form an infinite sequence $\{n\pi;\, n = 1.2, ...\}$.

It is clear that the differential equation

$$y''(t) = \phi(t)\,(1 + y'(t)^2)\sin(y(t))$$

has infinitely many positive solutions, $y_n(t) = n\pi,\;\; n \geq 1$.

The function f, defined by

$$f\,(t, y, z) = 2t(1 + z^2)\sin y(t), \qquad 0 \leq t \leq 1,$$

changes sign infinitely many times. In fact we have

$$f(t, \alpha_j, z) > 0 \quad \text{for} \quad \alpha_j = \left(\frac{1}{2} + 2j\right)\pi, \qquad , j = 0, 1, 2, \ldots,$$

and

$$f(t, \beta_j, z) < 0 \quad \text{for} \quad \beta_j = \left(\frac{3}{2} + 2j\right)\pi, \qquad , j = 0, 1, 2, \ldots.$$

Acknowledgement. The authors would like to thank an anonymous referee for comments that led to significant improvement of the original manuscript. Also, A. Boucherif is grateful to King Fahd University of Petroleum and Minerals for its constant support.

References

[1] R. P. Agarwal, D. O'Regan and P. J. Y. Wong, *Positive Solutions of Differential, Difference and Integral Equations*, Kluwer, Boston, MA, 1999.

[2] R. P. Agarwal, D. O'Regan and S. Stanek, Positive solutions of singular problems with sign changing Caratheodory nonlinearities depending on x', *J. Math. Anal. Appl.* **279** (2003), 597-616.

[3] Z. Bai and W. Ge, Existence of three positive solutions for some second order boundary value problems, *Comput. Math. Appl.* **48** (2004), 699-707.

[4] A. Boucherif, Positive solutions of second order boundary value problems, Preprint.

[5] X. Cheng and C. Zhong, Existence of positive solutions for a second order ordinary differential system, *J. Math. Anal. Appl.* **312** (2005), 14-23.

[6] R. Courant and D. Hilbert, *Methods of Mathematical Physics*, Interscience, New York, 1962.

[7] L. H. Erbe and H. Wang, On the existence of positive solutions of ordinary differential equations, *Proc. Amer. Math. Soc.* **120** (1994), 743-748.

[8] L. H. Erbe and R. M. Mathsen, Positive solutions for singular nonlinear boundary value problems, *Nonlin. Anal.* **46** (2001), 979-986.

[9] M. Frigon, Theoremes d'existence de solutions d'inclusions differentielles, in *Topological Methods in Differential Equations and Inclusions* (A. Granas and M. Frigon, Eds.), Kluwer, Dordrecht, 1995, 51-87.

[10] A. Granas and J. Dugundji, *Fixed Point Theory*, Springer-Verlag, Berlin, 2003.

[11] A. Granas and Z. E. Guennoun, Quelques resultats dans la théorie de Bernstein-Carathéodory de l'équation $y'' = f(t, y, y')$, *C. R. Acad. Sci. Paris*, **t. 306**, Serie I, (1988), 703-706.

[12] D. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, New York, 1988.

[13] J. Henderson and H. B. Thompson, Multiple symmetric positive solutions for a second order boundary value problem, *Proc. Amer. Math. Soc.* **128** (2000), 2373-2379.

[14] E. R. Kaufmann and N. Kosmatov, A multiplicity result for a boundary value problem with infinitely many singularities, *J. Math. Anal. Appl.* **269** (2002), 444-453.

[15] N. Kosmatov, A note on the singular Sturm-Liouville problem with infinitely many solutions, *Electronic J. Diff. Equations* **2002** (2002), No.80, 1-10.

[16] M. A. Krasnoselskii, *Positive Solutions of Operator Equations*, Noordhoff, Holland, 1964.

[17] F. Li, Z. Liang and Qi Zhang, Existence of solutions to a class of nonlinear second order two-point boundary value problems, *J. Math. Anal. Appl.* **312** (2005), 357-373.

[18] Y. Li, Positive solutions of second order boundary value problems with sign changing nonlinear term, *J. Math. Anal. Appl.* **282** (2003), 232-240.

[19] Y. Li, On the existence and nonexistence of positive solutions for nonlinear Sturm-Liouville boundary value problems, *J. Math. Anal. Appl.* **304** (2005), 74-86.

[20] J. Sun and G. Zhang, Nontrivial solutions of singular superlinear Sturm-Liouville problems, *J. Math. Anal. Appl.* **313** (2006), 518-536.

[21] X. Yang, Multiple positive solutions of second order differential equations, *Nonlin. Anal.* **62** (2005), 107-116.

(Received December 5, 2005)