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Boundary Value Problems for Doubly Perturbed First Order Ordinary Differential Systems

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Abstract

The aim of this paper is to present new results on existence theory for perturbed BVPs for first order ordinary differential systems. A nonlinear alternative for the sum of a contraction and a compact mapping is used.

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1 INTRODUCTION

This paper is devoted to the question of existence of solutions for a doubly perturbed boundary value problem (BVP) associated with first order ordinary differential systems of the form:

$$x'(t) = A(t)x(t) + F(t, x(t)) + G(t, x(t)), \quad a.e. \quad t \in [0, 1]; \tag{1}$$

$$Mx(0) + Nx(1) = \eta. \tag{2}$$

Here the functions $F, G : [0, 1] \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ are Carathéodory, $A(.)$ is a continuous $(n \times n)$ matrix function, M and N are constant $(n \times n)$ matrices, and $\eta \in \mathbb{R}^n$. Problem (1)-(2) encompasses second order differential equation with periodic condition or Sturm-Liouville nonlinear problem (see the example in Section 3). We shall denote by $\|x\|$ the norm of any element x of the euclidian space $\mathbb{R}^n$ and by $\|A\|$ the norm of any matrix A. The notation $:=$ means throughout to be equal to. In this paper, we shall prove the existence of solutions for Problem (1)-(2) under suitable conditions on the nonlinearities F and G. Our approach will be based, for the existence of solutions, on a fixed point theorem for the sum of a contraction map and a completely continuous map due to Ntouyas and Tsamatos [7] which we recall hereafter; it can be seen as a generalization of Burton and Kirk's Alternative [3]:

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Theorem 1.1 *[7] Let $(X, \|\cdot\|)$ be a Banach space, B_1, B_2 be operators from X into X such that B_1 is a $\gamma-$contraction, and B_2 is completely continuous. Assume also that*

(H) There exists a sphere $B(0,r)$ in X with center 0 and radius r such that for every $y \in B(0,r)$, $r(1-\gamma) \geq \|B_1 0 + B_2 y\|$. Then either

(a) the operator equation $x = (B_1 + B_2)x$ has a solution with $\|x\| \leq r$, or

(b) there exists a point $x_0 \in \partial B(0,r)$ and $\lambda \in (0,1)$ such that $x_0 = \lambda B_1 \left(\frac{x_0}{\lambda}\right) + \lambda B_2 x_0$.

Mappings which are equal to the sum of a contraction and a completely continuous function play an important role in fixed point theory (see [6]). Through Hamerstein operators, one can construct compact mapping and then apply Theorem 1.1 to BVPs associated with second order ODEs (see [2, 4, 6, 8]). In this paper, we extend those results to the case of systems doubly perturbed with contraction and Carathéodory functions satisfying specific growth.

2 Preliminaries

In this section, we introduce notations, and preliminaries used throughout this paper. Recall that $C([0,1], \mathbb{R}^n)$ is the Banach space of all continuous functions from $[0,1]$ into $\mathbb{R}^n$ with the norm

$$\|x\|_0 = \sup \{\|x(t)\| : 0 \leq t \leq 1\}.$$

Let $AC((0,1), \mathbb{R}^n)$ be the space of differentiable functions $x : (0,1) \to \mathbb{R}^n$, which are absolutely continuous.

We denote by $L^1([0,1], \mathbb{R}^n)$ the Banach space of measurable functions x: $[0,1] \longrightarrow \mathbb{R}^n$ which are Lebesgue integrable normed by

$$\|x\|_{L^1} = \int_0^1 \|x(t)\| dt \quad \text{for all} \quad x \in L^1([0,1], \mathbb{R}^n).$$

Recall the following.

Definition 2.1 *A function F: $[0,1] \times \mathbb{R}^n \to \mathbb{R}^n$ is said to be Carathéodory if*

(i) $t \longmapsto F(t,y)$ is measurable for each $y \in \mathbb{R}^n$, and

(ii) $y \longmapsto F(t,y)$ is continuous for almost each $t \in [0,1]$.

Definition 2.2 *Given a Banach space X, we say that a mapping $T : X \to X$ is totally bounded if it maps each bounded subset of X into a relatively compact subset. If, further it is continuous, it is called completely continuous.*

3 EXISTENCE OF SOLUTIONS

In this section, we are concerned with the existence of solutions to Problem (1)-(2). We first state an auxiliary result from linear differential systems theory [1].

Lemma 3.1 *Consider the following linear mixed boundary value problem*

$$x'(t) = A(t)x(t) + h(t), \quad a.e. \quad t \in (0,1), \tag{3}$$

$$Mx(0) + Nx(1) = 0. \tag{4}$$

Let $\Phi(t)$ be a fundamental matrix solution of $x'(t) = A(t)x(t)$, such that $\Phi(0) = I$, the $(n \times n)$ identity matrix. We can easily show that if $det(M + N\Phi(1)) \neq 0$, then the linear inhomogeneous problem (3)-(4) has a unique solution given by

$$x(t) = \int_0^1 k(t,s)h(s)ds$$

where $k(t,s)$ is the Green function defined by

$$k(t,s) = \begin{cases} \Phi(t)J(s), & 0 \leq t \leq s, \\ \Phi(t)\Phi(s)^{-1} + \Phi(t)J(s), & s \leq t \leq 1 \end{cases}$$

and

$$J(t) = -(M + N\Phi(1))^{-1}N\Phi(1)\Phi(t)^{-1}.$$

As for the inhomogeneous boundary conditions, the following Lemma is easily verified:

Lemma 3.2 *Consider the following inhomogeneous linear boundary value problem*

$$x'(t) = A(t)x(t) + h(t), \quad a.e. \ t \in (0,1), \tag{5}$$

$$Mx(0) + Nx(1) = \eta. \tag{6}$$

Let x_h be the solution of the homogeneous boundary value problem (3)-(4). Keeping the same notations as in Lemma 3.1, the solution of Problem (5)-(6) reads

$$x(t) = x_h(t) + \Phi(t)\left(M + N\Phi(1)\right)^{-1}\eta.$$

Next, we transform BVP (1)-(2) into a fixed point problem. Consider the Banach space $X = C([0,1], \mathbb{R}^n)$ endowed with the sup-norm. Let the operator $T : X \longrightarrow X$ be defined by

$$Tx(t) = \int_0^1 k(t,s)[F(s,x(s)) + G(s,x(s))]\,ds.$$

It is clear that fixed points of T are solutions for BVP (1)-(2). Let us introduce the following hypotheses which are assumed hereafter:

- **(H1)** The function $F : [0,1] \times \mathbb{R}^n \to \mathbb{R}^n$ is Carathéodory and satisfies:

$$\exists\, l \in L^1([0,1], \mathbb{R}_+),\ \|F(t,y_1) - F(t,y_2)\| \leq l(t)\|y_1 - y_2\|$$

for almost each $t \in [0,1]$ and all $y_1, y_2 \in \mathbb{R}^n$.

- **(H2)** The function G is continuous and there exist a function $q \in L^1([0,1], \mathbb{R})$ with $q(t) > 0$ for almost each $t \in [0,1]$ and a continuous nondecreasing function $\psi : \mathbb{R}^+ \longrightarrow (0, \infty)$ such that

$$\|G(t,y)\| \leq q(t)\psi(\|y\|) \quad \text{a.e} \quad t \in [0,1] \quad \text{and for all} \quad y \in \mathbb{R}^n.$$

- **(H3)** Set $k^* := \sup_{(t,s) \in [0,1]\times[0,1]} \|k(t,s)\|$ and assume that

$$k^*\|l\|_{L^1} < 1.$$

- **(H4)** Set $F^* := \int_0^1 \|F(s,0)\|\, ds$ and assume there exists $r > 0$ such that

$$r > \frac{k^*\left(F^* + \|q\|_{L^1}\Psi(r)\right)}{1 - k^*\|l\|_{L^1}}. \tag{7}$$

Our main result is:

Theorem 3.1 *Under hypotheses (H1)-(H4), BVP (1)-(2) has at least one solution $x \in AC([0,1], \mathbb{R}^n)$.*

Proof. Define the two operators B_1 and on B_2 on X by

$$B_1x(t) := \int_0^1 k(t,s)F(s,x(s))ds, \quad B_2x(t) := \int_0^1 k(t,s)G(s,x(s))\, ds.$$

We are going to show that the operators B_1 and B_2 satisfy all conditions of Theorem 1.1.

Claim 1. B_1 is a contraction.

Let $x, y \in X$ and $t \in [0,1]$; then

$$\begin{array}{rcl} \|B_1x(t) - B_1y(t)\| & = & \|\int_0^1 k(t,s)\left[F(s,x(s)) - F(s,y(s))\right] ds\| \\ & \leq & \int_0^1 \|k(t,s)\|\|F(s,x(s)) - F(s,y(s))\| \\ & \leq & k^*\|l\|_{L^1}\|x-y\|_0 < \|x-y\|_0. \end{array}$$

Thus

$$\|B_1x - B_1y\|_0 \leq \|x-y\|_0.$$

Claim 2. B_2 **is continuous.**

Let $x_n, x \in X$ such that $x_n \longrightarrow x$ in X, that is

$$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}^*, \quad (n \geq n_0 \Rightarrow \|x_n - x\|_0 < \varepsilon).$$

For each $t \in [0,1]$, we have

$$\begin{array}{rcl} \|B_2x_n(t) - B_2x(t)\| & \leq & \int_0^1 \|k(t,s)\| \cdot \|G(s,x_n(s)) - G(s,x(s))\| \, ds \\ & \leq & k^* \int_0^1 \|G(s,x_n(s)) - G(s,x(s))\| \, ds. \end{array}$$

Since the convergence of a sequence implies its boundedness, there is a number $L > 0$ such that

$$\|x_n(t)\| \leq L, \quad \|x(t)\| \leq L, \quad \forall\, t \in [0,1].$$

Now, the function G is uniformly continuous on the compact set

$$\Big\{(t,x) \in \mathbb{R}^+ \times \mathbb{R}^n,\ t \in [0,1], \|x\| \leq L\Big\}.$$

It follows that

$$\|G(s,x_n(s)) - G(s,x(s))\| \leq \frac{\varepsilon}{k^*}.$$

Therefore, we infer that

$$\|B_2x_n - B_2x\|_0 \leq \varepsilon,\ \forall\, n \geq n_0.$$

The continuity of B_2 is proved.

Claim 3. B_2 **is totally bounded.**

Consider the closed ball $C = \{x \in X; \|x\|_0 \leq M\}$. We prove that the image $B_2(C)$ is relatively compact in X. We have, by (H2)

$$\begin{array}{rcl} \|B_2x(t)\| & = & \| \int_0^1 k(t,s)G(s,x(s))ds\| \\ & \leq & k^* \int_0^1 \|G(s,x(s))\|ds \\ & \leq & k^* \int_0^1 q(s)\psi(\|x(s)\|)ds \\ & \leq & k^*\psi(\|x\|_0)\|q\|_{L^1} \\ & \leq & k^*\psi(M)\|q\|_{L^1}. \end{array}$$

Then $B_2(C)$ is uniformly bounded. In addition, the following estimates hold true:

$$\begin{array}{rcl} \|B_2x(t_2) - B_2x(t_1)\| & = & \| \int_0^1 [k(t_2,s) - k(t_1,s)]G(s,x(s))ds\| \\ & \leq & \int_0^1 \|k(t_2,s) - k(t_1,s)\|q(s)\psi(M)ds \\ & \leq & \psi(M) \int_0^1 q(s)\|k(t_2,s) - k(t_1,s)\| \, ds; \end{array}$$

the right-hand side term tends to 0 as $t_2 \longrightarrow t_2$ for any $x \in C$. Then, $B_2(C)$ is equicontinuous. By the Arzela-Ascoli Theorem, the mapping B_2 is completely continuous on X.

Claim 4. Now, we prove that, under Assumption (7), the second alternative of Theorem 1.1 is not valid.

Consider the sphere $B(0,r)$, r being defined by (H4). For $x \in B(0,r)$, we have

$$\begin{aligned}
\|B_1 0 + B_2 x\|_0 &= \sup_{t\in[0,1]} \| \int_0^1 k(t,s)F(s,0)\,ds + \int_0^1 k(t,s)G(s,x(s))\,ds\| \\
&\leq k^*F^* + k^*\|q\|_{L^1}\Psi(\|x\|_0) \\
&\leq k^*F^* + k^*\|q\|_{L^1}\Psi(r) \\
&< r(1-k^*\|l\|_{L^1}).
\end{aligned}$$

Now, argue by contradiction and assume that there exist $\lambda \in (0,1)$ and $x \in \partial B(0,r)$ with $x = \lambda B_1\left(\frac{x}{\lambda}\right) + \lambda B_2 x$. Then x verifies the estimates

$$\|x(t)\| \leq k^*\|l\|_{L^1}\|x\|_0 + k^*F^* + k^*\|q\|_{L^1}\Psi(\|x\|_0).$$

Hence

$$r = \|x\|_0 \leq \frac{k^*\left(F^* + \|q\|_{L^1}\Psi(r)\right)}{1-k^*\|l\|_{L^1}}$$

contradicting Assumption (7). We then conclude that Assertion (a) in Theorem 1.1 is satisfied, proving the claim of Theorem 3.1.

3.1 Example

Consider the second order boundary value Sturm-Liouville problem

$$-x'' + qx' + rx = f(t,x(t),x'(t)) + g(t,x(t),x'(t)),\ 0 < t < 1 \tag{8}$$

$$a_0 x(0) - a_1 x'(0) = c_0 \tag{9}$$

$$b_0 x(1) + b_1 x'(1) = c_1 \tag{10}$$

where a_0, a_1 and b_0, b_1 are nonnegative real numbers satisfying $a_0 + a_1 > 0$, $b_0 + b_1 > 0$ and $(c_0, c_1) \in \mathbb{R}^2$. The functions f, g: $[0,1] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ are assumed Carathéodory; the function f satisfies Lipschitz condition with respect to the last two arguments while g verifies a growth condition as in Assumption (H2). The functions q, r: $[0,1] \rightarrow \mathbb{R}$ are continuous.

v^t being the transpose of the vector v, we adopt the notations $x' = y$, $X = (x,y)^t$

$$F = (0,-f)^t \quad G = (0,-g)^t$$

as well as

$$A = \begin{pmatrix} 0 & 1 \\ r & q \end{pmatrix}, \quad M = \begin{pmatrix} a_0 & -a_1 \\ 0 & 0 \end{pmatrix}, \quad N = \begin{pmatrix} 0 & 0 \\ b_0 & b_1 \end{pmatrix},$$

and finally $c = (c_0, c_1)^t$.

Problem $(8) - (10)$ is then rewritten under the matrix form

$$\begin{cases} X' = AX + F + G \\ MX(0) + NX(1) = c. \end{cases}$$

Under Assumption (H4) both with $det(M + N\Phi(1)) \neq 0$, Problem $(8) - (10)$ has a solution x.

Remark 3.1 *In case q, r are constant, notice that condition $det(M + N\Phi(1)) \neq 0$ is nothing but $a_0(a_1 e^{r_2} + b_1 r_2 e^{r_2}) \neq b_0(a_1 e^{r_2} + b_1 r_r e^{r_2})$ where r_1 and r_2 are the roots of the characteristic equation $-s^2 + qs + r = 0$.*

4 Existence of Extremal Solutions

In this section we shall prove the existence of maximal and minimal solutions of BVP (1)-(2) under suitable monotonicity conditions on the functions involved in it. We define the usual co-ordinate-wise order relation $\leq$ in $\mathbb{R}^n$ as follows. Let $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$ be any two elements. Then by $x \leq y$, we mean $x_i \leq y_i$ for all $i = 1, ..., n$. We equip the space $X = C([0,1], \mathbb{R}^n)$ with the order relation $\leq$ induced by the natural positive cone $\mathcal{C}$ in X, that is,

$$\mathcal{C} = \{x \in X \mid x(t) \geq 0, \ \forall \, t \in [0,1]\}.$$

It is known that the cone $\mathcal{C}$ is normal in X. Cones and their properties are detailed in [5]. Let $a, b \in X$ be such that $a \leq b$. Then, by an order interval $[a, b]$ we mean a set of points in X given by

$$[a, b] = \{x \in X \mid a \leq x \leq b\}.$$

Definition 4.1 *Let X be an ordered Banach space. A mapping $T : X \to X$ is called isotone increasing if $T(x) \leq T(y)$ for any $x, y \in X$ with $x < y$. Similarly, T is called isotone decreasing if $T(x) \geq T(y)$ whenever $x < y$.*

Definition 4.2 *[5] We say that $x \in X$ is the least fixed point of G in X if $x = Gx$ and $x \leq y$ whenever $y \in X$ and $y = Gy$. The greatest fixed point of G in X is defined similarly by reversing the inequality. If both least and greatest fixed point of G in X exist, we call them extremal fixed point of G in X.*

The following fixed point theorem is due to Heikkila and Lakshmikantham:

Theorem 4.1 *[5] Let $[a,b]$ be an order interval in an order Banach space X and let $Q: [a,b] \to [a,b]$ be a nondecreasing mapping. If each sequence $(Qx_n) \subset Q([a,b])$ converges, whenever (x_n) is a monotone sequence in $[a,b]$, then the sequence of $Q-$iteration of a converges to the least fixed point x_* of Q and the sequence of $Q-$iteration of b converges to the greatest fixed point x^* of Q. Moreover*

$$x_* = \min\{y \in [a,b],\ y \geq Qy\} \text{ and } x^* = \max\{y \in [a,b],\ y \leq Qy\}$$

As a consequence, Dhage and Henderson have proved the following

Theorem 4.2 *[4]. Let K be a cone in the Banach space X and let $[a,b]$ be an order interval in a Banach space and let B_1, B_2: $[a,b] \to X$ be two functions satisfying*

(a) B_1 is a contraction,

(b) B_2 is completely continuous,

(c) B_1 and B_2 are strictly monotone increasing, and

(d) $B_1(x) + B_2(x) \in [a,b],\ \forall\, x \in [a,b]$.

Further if the cone K in X is normal, then the equation $x = B_1(x) + B_2(x)$ has a least fixed point x_* and a greatest fixed point $x^* \in [a,b]$. Moreover $x_* = \lim\limits_{n\to\infty} x_n$ and $x^* = \lim\limits_{n\to\infty} y_n$, where $\{x_n\}$ and $\{y_n\}$ are the sequences in $[a,b]$ defined by

$$x_{n+1} = B_1(x_n) + B_2(x_n),\ x_0 = a \text{ and } y_{n+1} = B_1(y_n) + B_2(y_n),\ y_0 = b.$$

We need the following definitions in the sequel.

Definition 4.3 *A function $v \in AC([0,1], \mathbb{R}^n)$ is called a strict lower solution of BVP (1)-(2) if $v'(t) \leq A(t)v(t) + F(t, v(t)) + G(t, v(t))$ a.e. $t \in [0,1]$, $Mv(0) + Nv(1) \leq \eta$. Similarly a strict upper solution w of BVP (1)-(2) is defined by reversing the order of the above inequalities.*

Definition 4.4 *A solution x_M of BVP (1)-(2) is said to be maximal if for any other solution x of BVP (1)-(2) on $[0,1]$, we have that $x(t) \leq x_M(t)$ for each $t \in [0,1]$. Similarly a minimal solution of BVP (1)-(2) is defined by reversing the order of the inequalities.*

Definition 4.5 *A function $F(t,x)$ is called strictly monotone increasing in x almost everywhere for $t \in J$, if $F(t,x) \leq F(t,y)$ a.e. $t \in J$ for all $x, y \in \mathbb{R}^n$ with $x < y$. Similarly $F(t,x)$ is called strictly monotone decreasing in x almost everywhere for $t \in J$, if $F(t,x) \geq F(t,y)$ a.e. $t \in J$ for all $x, y \in \mathbb{R}^n$ with $x < y$.*

We consider the following assumptions in the sequel.

(H5) The functions $F(t,y)$ and $G(t,y)$ are strictly monotone nondecreasing in y for almost each $t \in [0,1]$.

(H6) The BVP (1)-(2) has a lower solution v and an upper solution w with $v \leq w$.

(H7) The kernel k preserves the order, that is $k(t,s)v(s) \geq 0$ whenever $v \geq 0$.

Remark 4.1 *If we assume that there exist some constant vectors $\underline{y} \leq \overline{y}$ such that for each $t \in [0,1]$*

$$\begin{aligned} A(t)\underline{y} + F(t,\underline{y}) + G(t,\underline{y}) \geq 0, \quad (M+N)\underline{y} \leq \eta, \\ A(t)\overline{y} + F(t,\overline{y}) + G(t,\overline{y}) \leq 0, \quad (M+N)\overline{y} \geq \eta, \end{aligned}$$

then $\underline{y}$, $\overline{y}$ are respectively lower and upper solutions for Problem (1)-(2).

Theorem 4.3 *Assume that Assumptions (H1)-(H6) hold true. Then BVP (1)-(2) has minimal and maximal solutions on $[0,1]$.*

Proof. It can be shown, as in the proof of Theorem 3.1 that B_1 and B_2 are respectively a contraction and compact on $[a,b]$. We shall show that B_1 and B_2 are isotone increasing on $[a,b]$. Let $x, y \in [a,b]$ be such that $x \leq y$, $x \neq y$. Then by Assumptions (H5), (H7), we have for each $t \in [0,1]$

$$\begin{aligned} B_1(x)(t) &= \int_0^1 k(t,s)F(s,x(s))\,ds \\ &\leq \int_0^1 k(t,s)F(s,y(s))\,ds \\ &= B_1(y)(t). \end{aligned}$$

Similarly, $B_2(x) \leq B_2(y)$. Therefore B_1 and B_2 are isotone increasing on $[a,b]$. Finally, let $x \in [a,b]$ be any element. By Assumptions (H6), we deduce that

$$a \leq B_1(a) + B_2(a) \leq B_1(x) + B_2(x) \leq B_1(b) + B_2(b) \leq b,$$

which shows that $B_1(x) + B_2(x) \in [a,b]$ for all $x \in [a,b]$. Thus, the functions B_1 and B_2 satisfy all conditions of Theorem 4.2. It follows that BVP (1)-(2) has maximal and minimal solutions on $[0,1]$. This completes the proof of Theorem 4.3.

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