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Regularity in Orlicz spaces for nondivergence elliptic operators with potentials satisfying a reverse Hölder condition*

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Abstract: The purpose of this paper is to obtain the global regularity in Orlicz spaces for nondivergence elliptic operators with potentials satisfying a reverse Hölder condition.

Keywords: nondivergence elliptic operator; regularity, Orlicz space; potential; reverse Hölder condition.

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1 Introduction

In this paper we consider the following nondivergence elliptic operator

$$Lu \equiv Au + Vu \equiv -\sum_{i,j=1}^{n} a_{ij}(x)u_{x_i x_j} + Vu, \tag{1.1}$$

where $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n (n \geq 3)$, and establish the regularity in Orlicz spaces for (1.1). It will be assumed that the following assumptions on the coefficients of the operator A and the potential V are satisfied

(H_1) $a_{ij} \in L^\infty(\mathbb{R}^n)$ and $a_{ij} = a_{ji}$ for all $i, j = 1, 2, \ldots, n$, and there exists a positive constant Λ such that

$$\Lambda^{-1}|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2$$

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for any $x \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^n$;

(H_2) $a_{ij}(x) \in VMO(\mathbb{R}^n)$, which means that for $i, j = 1, 2, \ldots, n$,

$$\eta_{ij}(r) = \sup_{\rho \leq r} \sup_{x \in \mathbb{R}^n} \left(|B_\rho(x)|^{-1} \int_{B_\rho(x)} \left| a_{ij}(y) - a_{ij}^B \right| dy \right) \to 0,\ r \to 0^+,$$

where $a_{ij}^B = |B_\rho(x)|^{-1} \int_{B_\rho(x)} a_{ij}(y) dy$;

(H_3) $V \in B_q$ for $n/2 \leq q < \infty$, which means that $V \in L^q_{loc}(\mathbb{R}^n), V \geq 0$, and there exists a positive constant c_1 such that the reverse Hölder inequality

$$\left(|B|^{-1} \int_B V(x)^q dx \right)^{1/q} \leq c_1 \left(|B|^{-1} \int_B V(x) dx \right)$$

holds for every ball B in $\mathbb{R}^n$.

Note that when we say $V \in B_\infty$, it means

$$\sup_B V(x) \leq c_1 \left(|B|^{-1} \int_B V(x) dx \right).$$

In fact, if $V \in B_\infty$, then it implies that $V \in B_q$ for $1 < q < \infty$.

Regularity theory for elliptic operators with potentials satisfying a reverse Hölder condition has been studied by many authors (see [4], [9]–[12], [14], [15]). When A is the Laplace operator and $V \in B_q$ $(n/2 \leq q < \infty)$, Shen [10] derived L^p boundedness for $1 < p \leq q$ and showed that the range of p is optimal. If A is the Laplace operator and $V \in B_\infty$, an extension of L^p estimates to the global Orlicz estimates was given by Yao [14] with modifying the iteration-covering method introduced by Acerbi and Mingione [1]. For $a_{ij} \in C^1(\mathbb{R}^n)$ and $V \in B_\infty$, regularity theory in Orlicz spaces for the operators $\sum_{i,j=1}^{n} \partial_{x_i}(a_{ij}\partial_{x_j}) + V$ was proved by Yao [15]. Recently, under the assumptions (H_1)–(H_3), the global $L^p(\mathbb{R}^n)$ estimates for L in (1.1) has been deduced by Bramanti et al [4].

In this paper we will establish global estimates in Orlicz spaces for L which extends results in [4] to the case of the general Orlicz spaces. Our approach is based on an iteration-covering lemma (Lemma 3.1), the technique of "S. Agmon's idea"(see [3], p. 124) and an approximation procedure.

The definitions of Yong functions ϕ, Orlicz spaces $L^\phi(\mathbb{R}^n)$, Orlicz–Sobolev spaces $W^2 L^\phi(\mathbb{R}^n)$, $W^2_V L^\phi(\mathbb{R}^n)$, and their properties will be described in Section 2.

We now state the main result of this paper.

Theorem 1.1 *Let ϕ be a Young function and satisfy the global $\Delta_2 \cap \nabla_2$ condition. Assume that the operator L satisfies the assumptions (H_1), (H_2) and (H_3) for $q \geq \max\{n/2, \alpha_1\}$, $f \in L^\phi(\mathbb{R}^n)$. If $u \in W_V^2 L^\phi(\mathbb{R}^n)$ satisfies*

$$Lu - \mu u = f, \quad x \in \mathbb{R}^n, \tag{1.2}$$

then there exists a constant $C > 0$ such that for any $\mu \gg 1$ large enough, we have

$$\begin{aligned}\mu^{\alpha_2} \int_{\mathbb{R}^n} \phi(|u|)dx + \mu^{\alpha_2/2} \int_{\mathbb{R}^n} \phi(|Du|)dx + \int_{\mathbb{R}^n} \phi(|Vu|)dx + \int_{\mathbb{R}^n} \phi\left(\left|D^2 u\right|\right)dx \\ \leq C \int_{\mathbb{R}^n} \phi(|f|)dx,\end{aligned} \tag{1.3}$$

where the constants α_1 and α_2 appear in Orlicz spaces, see (2.4), C depends only on n, q, Λ, c_1, α_1, α_2 and the VMO moduli of the leading coefficients a_{ij}.

The proof of Theorem 1.1 is based on the following result.

Theorem 1.2 *Under the same assumptions on ϕ, a_{ij}, V, q, f as in Theorem 1.1, let $u \in C_0^\infty(\mathbb{R}^n)$ satisfy $Lu = f$ in $\mathbb{R}^n$. Then there exists a constant $C > 0$ such that*

$$\int_{\mathbb{R}^n} \phi\left(\left|D^2 u\right|\right)dx + \int_{\mathbb{R}^n} \phi(|Vu|)dx \leq C\left\{\int_{\mathbb{R}^n} \phi(|f|)dx + \int_{\mathbb{R}^n} \phi(|u|)dx\right\}, \tag{1.4}$$

where C depends only on n, q, Λ, c_1, a, K and the VMO moduli of a_{ij}.

Note that Theorem 1.2 and Definition 2.9 easily imply the following result by using the monotonicity, convexity of ϕ, (2.2) and Remark 2.7.

Corollary 1.3 *Under the same assumptions on ϕ, a_{ij}, V, q, f as in Theorem 1.1, let $u \in W_V^2 L^\phi(\mathbb{R}^n)$ satisfy $Lu = f$ in $\mathbb{R}^n$. Then there exists a constant $C > 0$ such that*

$$\int_{\mathbb{R}^n} \phi\left(\left|D^2 u\right|\right)dx + \int_{\mathbb{R}^n} \phi(|Vu|)dx \leq C\left\{\int_{\mathbb{R}^n} \phi(|f|)dx + \int_{\mathbb{R}^n} \phi(|u|)dx\right\},$$

where C depends only on n, q, Λ, c_1, a, K and the VMO moduli of a_{ij}.

Remark 1.4 *When we take $\phi(t) = t^p$, $t \geq 0$ for $1 < p < \infty$, then (1.4) is reduced to the classical L^p estimates (see [4, Theorem 1]).*

This paper will be organized as follows. In Section 2 some basic facts about Orlicz spaces and Orlicz–Sobolev spaces are recalled. In Section 3 we prove Theorem 1.2 by describing an iteration-covering lemma (Lemma 3.1) and using the results in [4]. Section 4 is devoted to the proof of Theorem 1.1. We first assume $u \in C_0^\infty(B_{R_0/2})$ satisfying (1.2) and prove that (1.3) is valid by using Theorem 1.2 and "S. Agmon's idea"(see [3], p. 124); then we show that the assumption $u \in C_0^\infty(B_{R_0/2})$ can be removed by an approximation procedure and a covering lemma in [5].

Dependence of constants. Throughout this paper, the letter C denotes a positive constant which may vary from line to line.

2 Preliminaries

We collect here some facts about Orlicz spaces and Orlicz–Sobolev spaces which will be needed in the following. For more properties, we refer the readers to [2] and [8].

We use the following notation:

$$\Phi = \{\phi : [0,+\infty) \to [0,+\infty) \mid \ \phi \text{ is increasing and convex}\}.$$

Definition 2.1 *A function $\phi \in \Phi$ is said to be a Young function if*

$$\phi(0) = 0, \ \lim_{t\to+\infty} \phi\,(t) = +\infty, \ \lim_{t\to 0^+} \frac{\phi(t)}{t} = \lim_{t\to+\infty} \frac{t}{\phi(t)} = 0. \tag{2.1}$$

Definition 2.2 *A Young function ϕ is said to satisfy the global Δ_2 condition denoted by $\phi \in \Delta_2$, if there exists a positive constant K such that for any $t > 0$,*

$$\phi(2t) \le K\phi(t). \tag{2.2}$$

Definition 2.3 *A Young function ϕ is said to satisfy the global ∇_2 condition denoted by $\phi \in \nabla_2$, if there exists a positive constant $a > 1$ such that for any $t > 0$,*

$$\phi(at) \ge 2a\phi(t). \tag{2.3}$$

The following result was obtained in [7].

Lemma 2.4 *If $\phi \in \Delta_2 \cap \nabla_2$, then for any $t > 0$ and $0 < \theta_2 \le 1 \le \theta_1 < \infty$,*

$$\phi(\theta_1 t) \le K{\theta_1}^{\alpha_1}\phi(t) \text{ and } \phi(\theta_2 t) \le 2a{\theta_2}^{\alpha_2}\phi(t), \tag{2.4}$$

where $\alpha_1 = \log_2 K, \alpha_2 = \log_a 2 + 1$ and $\alpha_1 \ge \alpha_2$.

Definition 2.5 *(Orlicz spaces) Given a Young function ϕ, we define the Orlicz class $K^\phi(\mathbb{R}^n)$ which consists of all the measurable functions $g : \mathbb{R}^n \to \mathbb{R}$ satisfying*

$$\int_{\mathbb{R}^n} \phi\,(|g|)dx < \infty$$

and the Orlicz space $L^\phi(\mathbb{R}^n)$ which is the linear hull of $K^\phi(\mathbb{R}^n)$.

In the Orlicz spaces $L^\phi(\mathbb{R}^n)$, we use the following Luxembourg norm

$$\|u\|_{L^\phi(\mathbb{R}^n)} = \inf\left\{k > 0 : \int_{\mathbb{R}^n} \phi\left(|u|/k\right) dx \leqslant 1\right\}. \tag{2.5}$$

The space $L^\phi(\mathbb{R}^n)$ equipped with the Luxembourg norm $\|\cdot\|_{L^\phi(\mathbb{R}^n)}$ is a Banach space. In general, $K^\phi \subset L^\phi$. Moreover, if ϕ satisfies the global Δ_2 condition, then $K^\phi = L^\phi$ and C_0^∞ is dense in L^ϕ (see [2], pp. 266–274).

Definition 2.6 *(Convergence in mean) A sequence $\{u_k\}$ of functions in $L^\phi(\mathbb{R}^n)$ is said to converge in mean to $u \in L^\phi(\mathbb{R}^n)$ if*

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} \phi(|u_k(x) - u(x)|)dx = 0.$$

Remark 2.7 *(see [2], p. 270)*

(i) The norm convergence in $L^\phi(\mathbb{R}^n)$ implies the mean convergence.

(ii) If $\phi \in \Delta_2$, then the mean convergence implies the norm convergence.

Definition 2.8 *(Orlicz–Sobolev spaces) The Orlicz–Sobolev space $W^2L^\phi(\mathbb{R}^n)$ is the set of all functions u which satisfy $|D^\alpha u(x)| \in L^\phi(\mathbb{R}^n)$ for $0 \le |\alpha| \le 2$. The norm is defined by*

$$\|u\|_{W^2L^\phi(\mathbb{R}^n)} = \|u\|_{L^\phi(\mathbb{R}^n)} + \|Du\|_{L^\phi(\mathbb{R}^n)} + \left\|D^2u\right\|_{L^\phi(\mathbb{R}^n)},$$

where $Du(x) = \{u_{x_i}\}_{i=1}^n$, $D^2u(x) = \{u_{x_ix_j}\}_{i,j=1}^n$, $\|Du\|_{L^\phi(\mathbb{R}^n)} = \sum_{i=1}^n \|u_{x_i}\|_{L^\phi(\mathbb{R}^n)}$, $\|D^2u\|_{L^\phi(\mathbb{R}^n)} = \sum_{i,j=1}^n \left\|u_{x_ix_j}\right\|_{L^\phi(\mathbb{R}^n)}$.

The following definition is analogous to the definition of the space $W_V^{2,p}(\mathbb{R}^n)$ introduced by Bramanti, Brandolini, Harboure and Viviani in [4].

Definition 2.9 *The space $W_V^2L^\phi(\mathbb{R}^n)$ is the closure of $C_0^\infty(\mathbb{R}^n)$ in the norm*

$$\|u\|_{W_V^2L^\phi(\mathbb{R}^n)} = \|u\|_{W^2L^\phi(\mathbb{R}^n)} + \|Vu\|_{L^\phi(\mathbb{R}^n)}.$$

Remark 2.10 *(see e.g. [13]) If $g \in L^\phi(\mathbb{R}^n)$, then $\int_{\mathbb{R}^n} \phi\left(|g|\right)dx$ can be easily rewritten in an integral form*

$$\int_{\mathbb{R}^n} \phi\left(|g|\right)dx = \int_0^\infty |\{x \in \mathbb{R}^n : |g| > t\}| d\left[\phi\left(t\right)\right]. \tag{2.6}$$

As usual, we denote by $B_R(x)$ the open ball in $\mathbb{R}^n$ of radius R centered at x and $B_R = B_R(0)$.

3 Proof of Theorem 1.2

Before the proof of Theorem 1.2, some notions and two useful lemmas are given. Let us introduce the notation

$$p = \frac{1+\alpha_2}{2} > 1.$$

For $u \in C_0^\infty(\mathbb{R}^n)$ satisfying $Lu = f$, set

$$\lambda_0^p = \int_{\mathbb{R}^n} |Vu|^p dx + \varepsilon^{-p} \left(\int_{\mathbb{R}^n} |f|^p dx + \int_{\mathbb{R}^n} |u|^p dx \right),$$

where $\varepsilon \in (0,1)$ is a small enough constant to be determined later. Let

$$u_\lambda = \frac{u}{\lambda_0 \lambda} \text{ and } f_\lambda = \frac{f}{\lambda_0 \lambda}, \text{ for any } \lambda > 0.$$

Then u_λ satisfies $Lu_\lambda = f_\lambda$. For any ball B in $\mathbb{R}^n$, we use the notations

$$J_\lambda[B] = \frac{1}{|B|} \int_B |Vu_\lambda|^p dx + \frac{1}{\varepsilon^p |B|} \left(\int_B |f_\lambda|^p dx + \int_B |u_\lambda|^p dx \right)$$

and

$$E_\lambda(1) = \{x \in \mathbb{R}^n : |Vu_\lambda| > 1\}.$$

The following lemma is just an analogous version of the result given in [15, Lemma 2.2]. Here the selection of λ_0 and the condition of V are different from [15].

Lemma 3.1 *(Iteration-covering lemma) For any $\lambda > 0$, there exists a family of disjoint balls $\{B_{\rho_{x_i}}(x_i)\}$ with $x_i \in E_\lambda(1)$ and $\rho_{x_i} = \rho(x_i, \lambda) > 0$ such that*

$$J_\lambda[B_{\rho_{x_i}}(x_i)] = 1, \ J_\lambda[B_\rho(x_i)] < 1 \ \text{ for any } \ \rho > \rho_{x_i}, \tag{3.1}$$

and

$$E_\lambda(1) \subset \bigcup_{i \geq 1} B_{5\rho_{x_i}}(x_i) \bigcup F, \tag{3.2}$$

where F is a zero measure set. Moreover,

$$\left|B_{\rho_{x_i}}(x_i)\right| \leq \frac{3^{p-1}}{3^{p-1}-1} \left\{ \int_{\left\{x \in B_{\rho_{x_i}}(x_i) : |Vu_\lambda| > \frac{1}{3}\right\}} |Vu_\lambda|^p dx \right.$$
$$\left. + \varepsilon^{-p} \int_{\left\{x \in B_{\rho_{x_i}}(x_i) : |f_\lambda| > \frac{\varepsilon}{3}\right\}} |f_\lambda|^p dx + \varepsilon^{-p} \int_{\left\{x \in B_{\rho_{x_i}}(x_i) : |u_\lambda| > \frac{\varepsilon}{3}\right\}} |u_\lambda|^p dx \right\}. \tag{3.3}$$

We omit the proof of Lemma 3.1 because it is actually similar to that of [15, Lemma 2.2].

In analogy with [4, Theorem 13], the following lemma holds by using [4, Theorem 2, Theorem 3], and standard techniques involving cutoff functions and the interpolation inequality (see e.g. [6]).

Lemma 3.2 *Under the assumptions* (H_1)*–*(H_3)*, for any* $\gamma \in (1, q]$*, there exists a positive constant* C *such that for any* x_i*,* ρ_{x_i} *as in Lemma 3.1 and* $u \in C_0^\infty(\mathbb{R}^n)$*,*

$$\int_{B_{5\rho_{x_i}}(x_i)} |Vu|^\gamma dx \le C \left\{ \int_{B_{10\rho_{x_i}}(x_i)} |Lu|^\gamma dx + \int_{B_{10\rho_{x_i}}(x_i)} |u|^\gamma dx \right\},$$

where C *depends only on* n*,* γ*,* q*,* c_1*,* Λ *and the VMO moduli of* a_{ij}*.*

Proof of Theorem 1.2. In order to prove (1.4), the first step is to check the following estimate

$$\int_{\mathbb{R}^n} \phi(|Vu|)dx \le C \left(\int_{\mathbb{R}^n} \phi(|f|)dx + \int_{\mathbb{R}^n} \phi(|u|)dx \right). \tag{3.4}$$

Since $u \in C_0^\infty(\mathbb{R}^n)$, then there exists some constant $R_0 > 0$ such that u is compactly supported in B_{R_0}. It follows from $q \ge \max\{n/2, \alpha_1\}$ and (2.4) that

$$\begin{aligned}
\int_{\mathbb{R}^n} \phi(|Vu|)dx &= \int_{\{x\in\mathbb{R}^n:|Vu|\ge 1\}} \phi(|Vu|)\, dx + \int_{\{x\in\mathbb{R}^n:|Vu|\le 1\}} \phi(|Vu|)\, dx \\
&\le K\phi(1) \int_{\mathbb{R}^n} |Vu|^{\alpha_1} dx + 2a\phi(1) \int_{\mathbb{R}^n} |Vu|^{\alpha_2} dx \\
&\le C \left(\sup_{B_{R_0}} |u|^{\alpha_1} + \sup_{B_{R_0}} |u|^{\alpha_2} \right) \left(\int_{B_{R_0}} |V|^{\alpha_1} dx + \int_{B_{R_0}} |V|^{\alpha_2} dx \right) \\
&< \infty,
\end{aligned}$$

that is $|Vu| \in L^\phi(\mathbb{R}^n)$. Hence by (2.6), it yields

$$\int_{\mathbb{R}^n} \phi(|Vu|)dx = \int_0^\infty |\{x \in \mathbb{R}^n : |Vu| > \lambda_0\lambda\}| d[\phi(\lambda_0\lambda)].$$

Due to (3.2),

$$|\{x \in \mathbb{R}^n : |Vu| > \lambda_0\lambda\}| \le \sum_{i=1}^\infty \left|\left\{x \in B_{5\rho_{x_i}}(x_i) : |Vu_\lambda| > 1\right\}\right|.$$

Thus the key is to estimate $\left|\left\{x \in B_{5\rho_{x_i}}(x_i) : |Vu_\lambda| > 1\right\}\right|$. Applying Lemma 3.2, (3.1) and (3.3) we deduce

$$
\begin{aligned}
&\left|\left\{x \in B_{5\rho_{x_i}}(x_i) : |Vu_\lambda| > 1\right\}\right| \\
&\le \int_{B_{5\rho_{x_i}}(x_i)} |Vu_\lambda|^p dx \\
&\le C\left\{\int_{B_{10\rho_{x_i}}(x_i)} |f_\lambda|^p dx + \int_{B_{10\rho_{x_i}}(x_i)} |u_\lambda|^p dx\right\} \\
&\le \varepsilon^p C(p,n)\left|B_{\rho_{x_i}}(x_i)\right| \\
&\le C(p,n)\left\{\varepsilon^p \int_{\left\{x\in B_{\rho_{x_i}}(x_i):|Vu_\lambda|>\frac{1}{3}\right\}} |Vu_\lambda|^p dx + \int_{\left\{x\in B_{\rho_{x_i}}(x_i):|f_\lambda|>\frac{\varepsilon}{3}\right\}} |f_\lambda|^p dx\right. \\
&\qquad\left. + \int_{\left\{x\in B_{\rho_{x_i}}(x_i):|u_\lambda|>\frac{\varepsilon}{3}\right\}} |u_\lambda|^p dx\right\}.
\end{aligned}
$$

Set $\tilde{\lambda} = \lambda_0\lambda$ and observe that

$$
\begin{aligned}
\int_{\mathbb{R}^n} \phi(|Vu|)dx &= \int_0^\infty \left|\left\{x \in \mathbb{R}^n : |Vu| > \tilde{\lambda}\right\}\right| d[\phi(\tilde{\lambda})] \\
&\le C(p,n)\varepsilon^p \int_0^\infty \tilde{\lambda}^{-p}\left\{\int_{\left\{x\in\mathbb{R}^n:|Vu|>\tilde{\lambda}/3\right\}} |Vu|^p dx\right\} d[\phi(\tilde{\lambda})] \\
&\quad + C(p,n)\int_0^\infty \tilde{\lambda}^{-p}\left\{\int_{\left\{x\in\mathbb{R}^n:|f|>\varepsilon\tilde{\lambda}/3\right\}} |f|^p dx\right\} d[\phi(\tilde{\lambda})] \\
&\quad + C(p,n)\int_0^\infty \tilde{\lambda}^{-p}\left\{\int_{\left\{x\in\mathbb{R}^n:|u|>\varepsilon\tilde{\lambda}/3\right\}} |u|^p dx\right\} d[\phi(\tilde{\lambda})] \\
&=: C(p,n)(\varepsilon^p I_1 + I_2 + I_3).
\end{aligned}
$$

By Fubini's theorem, integration by parts and (2.4), it implies that

$$
\begin{aligned}
I_1 &= \int_{\mathbb{R}^n} |Vu|^p \left\{\int_0^{3|Vu|} \frac{d\phi(\tilde{\lambda})}{\tilde{\lambda}^p}\right\} dx \\
&= \frac{1}{3^p}\int_{\mathbb{R}^n} \phi(3|Vu|)dx + p\int_{\mathbb{R}^n} |Vu|^p \left\{\int_0^{3|Vu|} \frac{\phi(\tilde{\lambda})}{\tilde{\lambda}^{p+1}} d\tilde{\lambda}\right\} dx \\
&\le \frac{1}{3^p}\int_{\mathbb{R}^n} \phi(3|Vu|)dx + \frac{2ap}{3^p(\alpha_2 - p)}\int_{\mathbb{R}^n} \phi(3|Vu|)dx \\
&\le C(n,p,a,K)\int_{\mathbb{R}^n} \phi(|Vu|)dx.
\end{aligned}
$$

Similarly,

$$
I_2 \le C(n,p,a,K)\varepsilon^{p-\alpha_1}\int_{\mathbb{R}^n} \phi(|f|)dx
$$

and

$$I_3 \le C(n,p,a,K)\varepsilon^{p-\alpha_1} \int_{\mathbb{R}^n} \phi(|u|)dx.$$

Therefore,

$$\int_{\mathbb{R}^n} \phi(|Vu|)dx \le C\left\{\varepsilon^p \int_{\mathbb{R}^n} \phi(|Vu|)dx + \varepsilon^{p-\alpha_1} \int_{\mathbb{R}^n} \phi(|f|)dx + \varepsilon^{p-\alpha_1} \int_{\mathbb{R}^n} \phi(|u|)dx\right\}.$$

Choosing a suitable ε such that $C(n,p,a,K)\varepsilon^p < \frac{1}{2}$, (3.4) is obtained.

Next, taking into account [16, Theorem 2.8], the convexity of ϕ, (2.2) and (3.4), we have

$$\begin{aligned}
\int_{\mathbb{R}^n} \phi\left(\left|D^2u\right|\right)dx &\le C\int_{\mathbb{R}^n} \phi\left(|f-Vu|\right)dx \\
&\le \frac{C}{2}\int_{\mathbb{R}^n} \phi\left(|2f|\right)dx + \frac{C}{2}\int_{\mathbb{R}^n} \phi\left(|2Vu|\right)dx \\
&\le \frac{KC}{2}\int_{\mathbb{R}^n} \phi\left(|f|\right)dx + \frac{KC}{2}\int_{\mathbb{R}^n} \phi\left(|Vu|\right)dx \\
&\le C\left\{\int_{\mathbb{R}^n} \phi\left(|f|\right)dx + \int_{\mathbb{R}^n} \phi\left(|u|\right)dx\right\}. \qquad (3.5)
\end{aligned}$$

Thus, (3.5) implies (1.4). The proof is finished. □

4 Proof of Theorem 1.1

By the technique of "S. Agmon's idea"(see [3], p. 124) and Theorem 1.2, we first prove the following lemma.

Lemma 4.1 *Under the same assumptions on ϕ, a_{ij}, V, q, f as in Theorem 1.1, let $u \in C_0^\infty(B_{R_0/2})$ satisfy the following equation*

$$Lu - \mu u = f,\ x \in \mathbb{R}^n.$$

Then for any $\mu \gg 1$ large enough,

$$\begin{aligned}
&\mu^{\alpha_2}\int_{\mathbb{R}^n} \phi\left(|u|\right)dx + \mu^{\alpha_2/2}\int_{\mathbb{R}^n} \phi\left(|Du|\right)dx + \int_{\mathbb{R}^n} \phi\left(|Vu|\right)dx + \int_{\mathbb{R}^n} \phi\left(\left|D^2u\right|\right)dx \\
&\le C\int_{\mathbb{R}^n} \phi\left(|Lu-\mu u|\right)dx = C\int_{\mathbb{R}^n} \phi\left(|f|\right)dx, \qquad (4.1)
\end{aligned}$$

where the constant C is independent of μ, and R_0, α_2 are the constants in the proofs of Theorem 1.2 and (2.4), respectively.

Proof Let $\xi \in C_0^\infty(-R_0/2, R_0/2)$ be a cutoff function (not identically zero) and set

$$\tilde{u}(z) = \tilde{u}(x,t) = \xi(t)\cos(\sqrt{\mu}t)u(x) \tag{4.2}$$

and

$$\tilde{L}\tilde{u}(z) = L\tilde{u} + \tilde{u}_{tt}, \tag{4.3}$$

where $\mu \geq 1$ will be chosen later, then $\tilde{u}(z) \in C_0^\infty(B_{R_0/2} \times (-R_0/2, R_0/2))$. It is easy to verify that the coefficients matrix

$$\begin{pmatrix} (a_{ij})_{n\times n} & 0 \\ 0 & 1 \end{pmatrix}$$

of the operator $\tilde{L}$ still satisfies the assumptions (H_1) and (H_2). Furthermore, in view of (4.2) and (4.3) we find that

$$\tilde{L}\tilde{u}(z) = \tilde{f}(z), \tag{4.4}$$

where

$$\tilde{f}(z) = \xi(t)\cos(\sqrt{\mu}t)(Lu - \mu u) + (\xi''(t)\cos(\sqrt{\mu}t) - 2\sqrt{\mu}\xi'(t)\sin(\sqrt{\mu}t))u. \tag{4.5}$$

For the sake of convenience, we use the following notation

$$D_{zz}^2\tilde{u}(z) = \{D_{xx}^2\tilde{u}(z), \tilde{u}_{xt}(z), \tilde{u}_{tt}(z)\},$$

where

$$D_{xx}^2\tilde{u}(z) = \{\tilde{u}_{x_ix_j}\}_{i,j=1}^n \quad \text{and} \quad \tilde{u}_{xt} = \{\tilde{u}_{x_it}\}_{i=1}^n.$$

Applying Theorem 1.2 to (4.4),

$$\begin{aligned} &\int_{\mathbb{R}^{n+1}} \phi\left(\left|D_{zz}^2\tilde{u}\right|\right)dxdt + \int_{\mathbb{R}^{n+1}} \phi\left(|V\tilde{u}|\right)dxdt \\ &\leq C\left\{\int_{\mathbb{R}^{n+1}} \phi\left(\left|\tilde{f}\right|\right)dxdt + \int_{\mathbb{R}^{n+1}} \phi\left(|\tilde{u}|\right)dxdt\right\}. \end{aligned} \tag{4.6}$$

If $\left|\xi(t)\cos(\sqrt{\mu}t)\right| > 0$, by (2.4) we have

$$\begin{aligned} \phi\left(\left|D^2u(x)\right|\right) &= \phi\left(\left|(\xi(t)\cos(\sqrt{\mu}t))^{-1}\xi(t)\cos(\sqrt{\mu}t)D^2u(x)\right|\right) \\ &\leq K|\xi(t)\cos(\sqrt{\mu}t)|^{-\alpha_1}\phi\left(\left|\xi(t)\cos(\sqrt{\mu}t)D^2u(x)\right|\right). \end{aligned}$$

This and (4.2) yield

$$\begin{aligned}
&\int_{\mathbb{R}^n} \phi\left(\left|D^2u(x)\right|\right)dx \\
&= \left(\int_{\mathbb{R}} K^{-1}|\xi(t)\cos(\sqrt{\mu}t)|^{\alpha_1}dt\right)^{-1} \\
&\quad \times \int_{\mathbb{R}^{n+1}} K^{-1}|\xi(t)\cos(\sqrt{\mu}t)|^{\alpha_1}\phi\left(\left|D^2u(x)\right|\right)dxdt \\
&\le C\int_{\mathbb{R}^{n+1}} K^{-1}|\xi(t)\cos(\sqrt{\mu}t)|^{\alpha_1}\phi\left(\left|D^2u(x)\right|\right)dxdt \\
&= C\int_{\left\{(x,t)\in\mathbb{R}^{n+1}\left|\left|\xi(t)\cos(\sqrt{\mu}t)\right|>0\right.\right\}} K^{-1}|\xi(t)\cos(\sqrt{\mu}t)|^{\alpha_1}\phi\left(\left|D^2u(x)\right|\right)dxdt \\
&\le C\int_{\mathbb{R}^{n+1}} \phi\left(\left|D^2_{xx}\tilde{u}(z)\right|\right)dxdt \\
&\le C\int_{\mathbb{R}^{n+1}} \phi\left(\left|D^2_{zz}\tilde{u}(z)\right|\right)dxdt. \qquad (4.7)
\end{aligned}$$

Similarly to (4.7) we get

$$\int_{\mathbb{R}^n} \phi\left(|Vu|\right)dx \le C\int_{\mathbb{R}^{n+1}} \phi\left(|V\tilde{u}(z)|\right)dxdt. \qquad (4.8)$$

Using (2.4),

$$\phi\left(|Du(x)|\right) \le K|\xi(t)\sin(\sqrt{\mu}t)|^{-\alpha_1}\phi\left(|\xi(t)\sin(\sqrt{\mu}t)Du|\right).$$

Thus,

$$\begin{aligned}
\int_{\mathbb{R}^n} \phi\left(|Du(x)|\right)dx &\le C\int_{\mathbb{R}^{n+1}} \phi\left(|\xi(t)\sin(\sqrt{\mu}t)Du|\right)dxdt \\
&\le C\sum_{i=1}^{n}\int_{\mathbb{R}^{n+1}} \phi\left(\mu^{-1/2}\left|\xi'(t)\cos(\sqrt{\mu}t)u_{x_i} - \tilde{u}_{x_it}\right|\right)dxdt \\
&\le C\mu^{-\alpha_2/2}\left(\int_{\mathbb{R}^n} \phi\left(|Du|\right)dx + \int_{\mathbb{R}^{n+1}} \phi\left(|\tilde{u}_{xt}|\right)dxdt\right).
\end{aligned}$$

By choosing $\mu \gg 1$ large enough, we obtain the following

$$\begin{aligned}
\mu^{\alpha_2/2}\int_{\mathbb{R}^n} \phi\left(|Du(x)|\right)dx &\le C\int_{\mathbb{R}^{n+1}} \phi\left(|\tilde{u}_{xt}(z)|\right)dxdt \\
&\le C\int_{\mathbb{R}^{n+1}} \phi\left(\left|D^2_{zz}\tilde{u}(z)\right|\right)dxdt. \qquad (4.9)
\end{aligned}$$

Since

$$-\mu\xi(t)\cos(\sqrt{\mu}t)u(x) = \tilde{u}_{tt}(z) - \left(\xi''(t)\cos(\sqrt{\mu}t) - 2\sqrt{\mu}\xi'(t)\sin(\sqrt{\mu}t)\right)u(x),$$

we get

$$\begin{aligned}\mu^{\alpha_2}\int_{\mathbb{R}^n}\phi(|u(x)|)dx &\le C\int_{\mathbb{R}^{n+1}}\phi(|\tilde{u}_{tt}(z)|)dxdt\\ &\le C\int_{\mathbb{R}^{n+1}}\phi\left(\left|D^2_{zz}\tilde{u}(z)\right|\right)dxdt. \end{aligned}\tag{4.10}$$

Combining (4.5)–(4.10) and noting that

$$-\sqrt{\mu}\xi'(t)\sin(\sqrt{\mu}t)u(x)=\left((\xi'(t)\cos(\sqrt{\mu}t))_t-\xi''(t)\cos(\sqrt{\mu}t)\right)u(x),$$

we immediately find that

$$\begin{aligned}&\mu^{\alpha_2}\int_{\mathbb{R}^n}\phi(|u|)dx+\mu^{\alpha_2/2}\int_{\mathbb{R}^n}\phi(|Du|)dx+\int_{\mathbb{R}^n}\phi(|Vu|)dx+\int_{\mathbb{R}^n}\phi\left(\left|D^2u\right|\right)dx\\ &\le C\left\{\int_{\mathbb{R}^{n+1}}\phi\left(\left|D^2_{zz}\tilde{u}\right|\right)dxdt+\int_{\mathbb{R}^{n+1}}\phi(|V\tilde{u}|)dxdt\right\}\\ &\le C\left\{\int_{\mathbb{R}^{n+1}}\phi\left(\left|\tilde{f}\right|\right)dxdt+\int_{\mathbb{R}^{n+1}}\phi(|\tilde{u}|)dxdt\right\}\\ &\le C\left(\int_{\mathbb{R}^n}\phi(|Lu-\mu u|)dx+\int_{\mathbb{R}^n}\phi(|u|)dx\right).\end{aligned}$$

The desired estimate (4.1) follows by taking $\mu\gg 1$ large enough. The lemma is proved. □

Furthermore, we shall show that the assumption $C_0^\infty(B_{R_0/2})$ can be removed. A covering lemma in a locally invariant quasimetric space was proved by Bramanti et al. in [5]. Since the Euclidean space $\mathbb{R}^n$ is a special locally invariant quasimetric space, the covering lemma also holds in $\mathbb{R}^n$. For the convenience to readers, we describe it as follows.

Lemma 4.2 *For given R_0 and any $\kappa>1$, there exist $R_1\in(0,R_0/2)$, a positive integer M and a sequence of points $\{x_i\}_{i=1}^\infty\subset\mathbb{R}^n$ such that*

$$\mathbb{R}^n=\bigcup_{i=1}^\infty B_{R_1}(x_i);$$

$$\sum_{i=1}^\infty\chi_{B_{\kappa R_1}(x_i)}(y)\le M\quad\textit{for any}\quad y\in\mathbb{R}^n,$$

where $\chi_{B_{\kappa R_1}(x_i)}(y)$ is the characteristic function of $B_{\kappa R_1}(x_i)$, that is, the function equal to 1 in $B_{\kappa R_1}(x_i)$ and 0 in $\mathbb{R}^n\backslash B_{\kappa R_1}(x_i)$.

Proof of Theorem 1.1. Let $\rho(x)$ be a cutoff function on $B_{R_0/2}$ relative to B_{R_1}, namely, $\rho(x) \in C_0^\infty(B_{R_0/2})$, $0 \le \rho(x) \le 1$ and $\rho(x) \equiv 1$ on B_{R_1}, where R_1 is as in Lemma 4.2. For any fixed $x_0 \in \mathbb{R}^n$, we set

$$u^0(x) = u(x)\rho(x - x_0) =: u(x)\rho^0(x) \tag{4.11}$$

and observe that

$$Lu^0(x) - \mu u^0(x) = f\rho^0 - 2a_{ij}u_{x_i}\rho^0_{x_j} - a_{ij}u\rho^0_{x_ix_j} =: f^0.$$

By Definition 2.9, there exists a sequence $\{u_k\}$ of functions in $C_0^\infty(\mathbb{R}^n)$ such that

$$\|u_k - u\|_{W^2L^\phi(\mathbb{R}^n)} + \|Vu_k - Vu\|_{L^\phi(\mathbb{R}^n)} \to 0, \text{ as } k \to \infty. \tag{4.12}$$

It follows from Remark 2.7 that

$$\int_{\mathbb{R}^n} \phi(|u_k - u|)dx + \int_{\mathbb{R}^n} \phi(|D(u_k - u)|)dx + \int_{\mathbb{R}^n} \phi(\left|D^2(u_k - u)\right|)dx$$
$$+ \int_{\mathbb{R}^n} \phi(V\,|u_k - u|)dx \to 0, \text{ as } k \to \infty. \tag{4.13}$$

Let $u_k^0 = u_k\rho^0$. Then using the properties of ρ, the monotonicity, convexity of ϕ, (4.13), (2.4) and Remark 2.7, we obtain

$$\left\|u_k^0 - u^0\right\|_{W^2L^\phi(\mathbb{R}^n)} + \left\|Vu_k^0 - Vu^0\right\|_{L^\phi(\mathbb{R}^n)} \to 0, \quad \text{as} \quad k \to \infty. \tag{4.14}$$

Set

$$f_k = Lu_k - \mu u_k \;\text{ and }\; f_k^0 = Lu_k^0 - \mu u_k^0.$$

It follows by (H_1) and (4.12) that

$$\left\|f_k^0 - f^0\right\|_{L^\phi(\mathbb{R}^n)}$$
$$\le \left\|Lu_k^0 - Lu^0\right\|_{L^\phi(\mathbb{R}^n)} + \mu\left\|u_k^0 - u^0\right\|_{L^\phi(\mathbb{R}^n)} \to 0, \, as\, k \to \infty. \tag{4.15}$$

Hence, by (4.14), (4.15), Lemma 4.1 and Remark 2.7 we have

$$\mu^{\alpha_2}\int_{\mathbb{R}^n} \phi\left(\left|u^0\right|\right)dx + \mu^{\alpha_2/2}\int_{\mathbb{R}^n} \phi\left(\left|Du^0\right|\right)dx + \int_{\mathbb{R}^n} \phi\left(\left|Vu^0\right|\right)dx$$
$$+ \int_{\mathbb{R}^n} \phi\left(\left|D^2u^0\right|\right)dx$$
$$\le C\int_{\mathbb{R}^n} \phi\left(\left|f^0\right|\right)dx$$
$$\le C\left\{\int_{B_{R_0/2}(x_0)} \phi\left(|f|\right)dx + \int_{B_{R_0/2}(x_0)} \phi\left(|u|\right)dx + \int_{B_{R_0/2}(x_0)} \phi\left(|Du|\right)dx\right\}. \tag{4.16}$$

Note that (4.11) and (2.4) yield

$$\int_{\mathbb{R}^n} \phi\left(\left|\rho^0 Du\right|\right)dx \le C\left\{\int_{\mathbb{R}^n} \phi\left(\left|Du^0\right|\right)dx + \int_{\mathbb{R}^n} \phi\left(\left|uD\rho^0\right|\right)dx\right\} \tag{4.17}$$

and

$$\int_{\mathbb{R}^n} \phi\left(\left|\rho^0 D^2u\right|\right)dx \le C\left\{\int_{\mathbb{R}^n} \phi\left(\left|D^2u^0\right|\right)dx + \int_{\mathbb{R}^n} \phi\left(\left|uD^2\rho^0\right|\right)dx + \int_{\mathbb{R}^n} \phi\left(\left|Du\cdot D\rho^0\right|\right)dx\right\}. \tag{4.18}$$

Then combining (4.16), (4.17) and (4.18) implies that

$$\begin{aligned}
&\mu^{\alpha_2}\int_{B_{R_0/2}(x_0)} \phi\left(\left|\rho^0 u\right|\right)dx + \mu^{\alpha_2/2}\int_{B_{R_0/2}(x_0)} \phi\left(\left|\rho^0 Du\right|\right)dx \\
&\quad + \int_{B_{R_0/2}(x_0)} \phi\left(\left|\rho^0 Vu\right|\right)dx + \int_{B_{R_0/2}(x_0)} \phi\left(\left|\rho^0 D^2u\right|\right)dx \\
&\le C\left\{\int_{B_{R_0/2}(x_0)} \phi\left(|f|\right)dx + \mu^{\alpha_2/2}\int_{B_{R_0/2}(x_0)} \phi\left(|u|\right)dx + \int_{B_{R_0/2}(x_0)} \phi\left(|Du|\right)dx\right\}.
\end{aligned}$$

Therefore, by the above inequality and Lemma 4.2 we deduce that

$$\begin{aligned}
&\mu^{\alpha_2}\int_{\mathbb{R}^n} \phi\left(|u|\right)dx + \mu^{\alpha_2/2}\int_{\mathbb{R}^n} \phi\left(|Du|\right)dx + \int_{\mathbb{R}^n} \phi\left(|Vu|\right)dx + \int_{\mathbb{R}^n} \phi\left(\left|D^2u\right|\right)dx \\
&\le \sum_{i=1}^{\infty}\left\{\mu^{\alpha_2}\int_{B_{R_1}(x_i)} \phi\left(\left|\rho^0 u\right|\right)dx + \mu^{\alpha_2/2}\int_{B_{R_1}(x_i)} \phi\left(\left|\rho^0 Du\right|\right)dx\right. \\
&\qquad \left. + \int_{B_{R_1}(x_i)} \phi\left(\left|\rho^0 Vu\right|\right)dx + \int_{B_{R_1}(x_i)} \phi\left(\left|\rho^0 D^2u\right|\right)dx\right\} \\
&\le C\sum_{i=1}^{\infty}\left\{\int_{B_{R_0/2}(x_i)} \phi(|f|)dx + \mu^{\alpha_2/2}\int_{B_{R_0/2}(x_i)} \phi(|u|)dx\right. \\
&\qquad \left. + \int_{B_{R_0/2}(x_i)} \phi(|Du|)dx\right\} \\
&\le C\left\{\int_{\mathbb{R}^n} \phi\left(|f|\right)dx + \mu^{\alpha_2/2}\int_{\mathbb{R}^n} \phi\left(|u|\right)dx + \int_{\mathbb{R}^n} \phi\left(|Du|\right)dx\right\}.
\end{aligned}$$

(1.3) is obtained by taking $\mu \gg 1$ large enough. The theorem is proved. □

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