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MONOTONE INCREASING MULTI-VALUED CODENSING RANDOM OPERATORS AND RANDOM DIFFERENTIAL INCLUSIONS

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Abstract

In this paper, some random fixed point theorems for monotone increasing, condensing and closed multi-valued random operators are proved. They are then applied to first order ordinary nonconvex random differential inclusions for proving the existence of solutions as well as the existence of extremal solutions under certain monotonicity conditions.

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1 Introduction

Let $(\Omega, \mathcal{A})$ be a measurable space and let X be a separable Banach space with norm $\|\cdot\|$. Let β_X denote the Borel σ-algebra of open subsets of X. A function $x : \Omega \to X$ is called measurable if

$$x^{-1}(B) = \{\omega \in \Omega \mid x(\omega) \in B\} \in \mathcal{A} \tag{1.1}$$

for all $B \in \beta_X$. The set of all measurable functions form the set Ω into X is denoted by $\mathcal{M}(\Omega, X)$. Let $\mathcal{P}(X)$ denote the class of all subsets of X, called the power set of X. Denote

$$\mathcal{P}_p(X) = \{A \subset X \mid A \text{ is non-empty and has the property } p\}. \tag{1.2}$$

Here, p may be p =closed (in short cl) or p =convex (in short cv) or p =bounded (in short bd) or p =compact (in short cp). Thus $\mathcal{P}_{cl}(X)$, $\mathcal{P}_{cv}(X)$, $\mathcal{P}_{bd}(X)$ and $\mathcal{P}_{cp}(X)$ denote, respectively, the classes of all closed, convex, bounded and compact subsets of X. Similarly, $\mathcal{P}_{cl,bd}(X)$ and $\mathcal{P}_{cv,cp}(X)$ denote, respectively, the classes of closed-bounded and compact-convex subsets of X.

A correspondence $T : X \to \mathcal{P}_p(X)$ is called a multi-valued mapping or multi-valued operator on X into X. A point $u \in X$ is called a fixed point of T if $u \in Tu$, and the set of all fixed points of T in X is denoted by $\mathcal{F}_T$.

Let $T : X \to \mathcal{P}_p(X)$ be a multi-valued mapping. T is called bounded if $\overline{\bigcup T(S)}$ is bounded subset of X for all bounded subsets S of X. T is called compact if $\overline{T(X)}$ is a compact subset of X. Again, T is called totally bounded if $T(S)$ is totally bounded subset of X for all bounded sets S in X. It is clear that every compact mapping is totally bounded, but the converse may not be true. However, these two notions are equivalent on bounded subsets of X. T is called an upper semi-continuous at $x \in X$ if for each open set V in X containing $f(x)$, there exists a neighborhood $N(x)$ in X such that $\bigcup T(N(x)) \subset V$. T is called upper semi-continuous on X if it is upper semi-continuous at each point of X. An upper semi-continuous multi-valued mapping T on X is also called a closed multi-valued mapping on X. Finally, T is called completely continuous on X if it is upper semi-continuous and totally bounded on X. It is known that if T is a closed multi-valued mapping with compact values on X, then if we have sequences $\{x_n\}$ and $\{y_n\}$ in X such that $x_n \to x_*$, $y_n \to y_*$ and $y_n \in Tx_n$, $n \in \mathbb{N}$, then $y_* \in Tx_*$. The converse of this statement holds if T is a compact multi-valued mapping on X. The details of all these definitions appear in Hu and Papageorgiou [11].

A multi-valued mapping $T : \Omega \to \mathcal{P}_p(X)$ is called measurable (respectively weakly measurable) if

$$T^{-1}(B) = \{\omega \in \Omega \mid T(\omega) \cap B \neq \emptyset\} \in \mathcal{A} \tag{1.3}$$

for all closed (respectively open) subsets B in X. A multi-valued mapping $T : \Omega \times X \to \mathcal{P}_p(X)$ is called a multi-valued random operator if $T(\cdot, x)$ is measurable for each $x \in X$, and we write $T(\omega, x) = T(\omega)x$. A measurable function $\xi : \Omega \to X$ is called a random fixed point of the multi-valued random operator $T(\omega)$ if $\xi(\omega) \in T(\omega)\xi(\omega)$ for all $\omega \in \Omega$. The set of all random fixed points of the multi-valued random operator $T(\omega)$ is denoted by $\mathcal{F}_{T(\omega)}$. A multi-valued random operator $T : \Omega \to \mathcal{P}_p(X)$ is called bounded, totally bounded, compact, closed, completely continuous if the multi-valued mapping $T(\omega, \cdot)$ is bounded, totally bounded, compact, closed, completely continuous for each $\omega \in \Omega$, respectively.

A non-empty closed subset K of the Banach space X is called a cone in X if (i) $K + K \subseteq K$, (ii) $\lambda K \subseteq K$, $\lambda \in \mathbb{R}$, $\lambda \geq 0$ and (iii) $\{-K\} \cap K = \{0\}$, where 0 is the zero element of X. A cone K is called normal in X if the norm $\|\cdot\|$ is semi-monotone on K. It is known that if the cone is normal, then every order bounded set in X is bounded in norm. Again, a cone K in X is called regular if every monotone order bounded sequence in X converges in norm. The details of cones and their properties may be found in Heikkilä and Lakshmikatham [10].

We define an order relation $\leq$ in X with the help of the cone K in X as follows. Let $x, y \in X$. Then we define

$$x \leq y \iff y - x \in K. \tag{1.4}$$

The Banach space X together with the order relation $\leq$ becomes an ordered Banach space. Let $a, b \in X$ be such that $a \leq b$. Then by an order interval $[a, b]$ we mean a set in X defined by

$$[a, b] = \{x \in X \mid a \leq x \leq b\}. \tag{1.5}$$

Let $a, b : \Omega \to X$ be two measurable functions. By $a \leq b$ on Ω, we mean $a(\omega) \leq b(\omega)$ for all $\omega \in \Omega$. Then the sector $[a, b]$ defined by

$$\begin{aligned} [a, b] &= \{x \in X \mid a(\omega) \leq x \leq b(\omega) \ \text{ for all } \ \omega \in \Omega\} \\ &= \bigcap_{\omega \in \Omega} [a(\omega), b(\omega)] \end{aligned} \tag{1.6}$$

is called the random order interval in X.

The Kuratowskii measure $\alpha(S)$ and the Hausdorff measure $\beta(S)$ of noncompactness of a bounded set S in a Banach space X are the nonnegative real numbers defined by

$$\alpha(S) = \inf \Big\{ r > 0 : S \subset \bigcup_{i=1}^{n} S_i, \text{ and } \operatorname{diam}(S_i) \leq r, \forall\, i \Big\} \tag{1.7}$$

and

$$\beta(S) = \inf \Big\{ r > 0 : S \subset \bigcup_{i=1}^{n} \mathcal{B}_i(x_i, r), \text{ for some } x_i \in X \Big\}, \tag{1.8}$$

where $\mathcal{B}_i(x_i, r) = \{x \in X \mid d(x, x_i) < r\}$.

The details of Hausdorff measure of noncompactness and its properties appear in Deimling [2], Hu and Papageorgiou [11], and the references therein.

Definition 1.1 *A multi-valued mapping $Q : X \to \mathcal{P}_{bd}(X)$ is called β-**condensing** if for any $S \in \mathcal{P}_{bd}(X)$, we have that $\beta(Q(S)) < \beta(S)$ for $\beta(S) > 0$.*

It is known that compact and contraction multi-valued maps are condensing, but the converse may not be true. The following results are well-known in the literature..

Lemma 1.1 (Akhmerov et al. [1]) *Let α and β be respectively the Kuratowskii and Hausdorff measure of noncompactness in a Banach space X, then for any bounded set S in X, we have $\alpha(S) \leq 2\,\beta(S)$.*

Lemma 1.2 (Akhmerov et al. [1]) *If $A : X \to X$ is a single-valued Lipschitz mapping with the Lipschitz constant k, then we have $\alpha(A(S)) \leq k\,\alpha(S)$ for any bounded subset S of X.*

The study of random fixed point theorems for the monotone increasing, completely continuous multi-valued random mappings on the random order intervals is initiated in Dhage [6] using the properties of cones in ordered Banach spaces. In the present work, we establish some random fixed point theorems for monotone increasing, condensing and closed multi-valued random operators on random intervals under suitable conditions. In the case of multi-valued mappings, there are different types of monotonicity conditions, namely, right or left monotone increasing and strict monotonicity etc. In the following section, we formulate the random fixed point theorems for multi-valued random operators for each of these monotonicity criteria.

2 Multi-valued Random Fixed Point Theory

Let the Banach space X be equipped with the order relation $\leq$ and define the order relation in $\mathcal{P}_p(X)$ as follows:

Let $A, B \in \mathcal{P}_p(X)$. Then by $A \overset{i}{\leq} B$ we mean "for every $a \in A$ there exists a $b \in B$ such that $a \leq b$." Again $A \overset{d}{\leq} B$ means "for each $b \in B$ there exists $a \in A$ such that $a \leq b$". Further, we have $A \overset{id}{\leq} B \iff A \overset{i}{\leq} B$ and $A \overset{d}{\leq} B$. Finally, $A \leq B$ implies that $a \leq b$ for all $a \in A$ and $b \in B$. Note that if $A \leq A$, then it follows that A is a singleton set. The details appear in Dhage [4] and references therein.

Let $T : \Omega \times X \to \mathcal{P}_p(X)$ be a multi-valued random operator and let

$$S_{T(\omega)}(x) = \{u \in \mathcal{M}(\Omega, X) \mid u(\omega) \in T(\omega)x \text{ for all } \omega \in \Omega\}. \tag{2.1}$$

The set $S_{T(\omega)}(x)$ is called the set of measurable selectors of the multi-valued random operator $T(\omega)$ at x. The key result in formulating random fixed point theorems concerning the existence of measurable selector for a multi-valued mapping is the following:

Theorem 2.1 (Kuratowskii and Ryll-Nardzewski [14]) *If the multi-valued operator $T : \Omega \times X \to \mathcal{P}_p(X)$ is measurable with closed values, then T has a measurable selector.*

Remark 2.1 *Note that if $T : \Omega \times X \to \mathcal{P}_{cl}(X)$ is a multi-valued random operator, then the set $S_{T(\omega)}(x)$ is non-empty for each $x \in X$.*

Now we are ready to formulate random fixed point theorems for different types of monotone increasing multi-valued random operators on separable Banach spaces.

2.1 Right monotone increasing multi-valued random operators

Definition 2.1 *A multi-valued random operator $T : \Omega \times X \to \mathcal{P}_{cl}(X)$ is called right monotone increasing if for each $\omega \in \Omega$ we have that $S_{T(\omega)}(x) \overset{i}{\leq} S_{T(\omega)}(y)$ for all $x, y \in X$ for which $x \leq y$.*

Our first random fixed point theorem for the right monotone increasing condensing multi-valued random operators is as follows:

Theorem 2.2 *Let $(\Omega, \mathcal{A})$ be a measurable space and let $[a, b]$ be a random interval in a separable Banach space X. If $T : \Omega \times [a, b] \to \mathcal{P}_{cl}([a, b])$ is a condensing, upper semi-continuous right monotone increasing multi-valued random operator and the cone K in X is normal, then $T(\omega)$ has a random fixed point in $[a, b]$.*

Proof : Define a monotone increasing sequence $\{x_n(\omega)\}$ of measurable functions in $[a, b]$ defined by

$$x_0(\omega) = a(\omega),\ x_{n+1}(\omega) \in T(\omega)x_n,\ n = 0, 1, 2, \dots, \tag{2.2}$$

which does exist in view of the right monotonicity of the multi-valued random operator $T(\omega)$. If $x_r(\omega) = x_{r+1}(\omega)$ for some $r \in \mathbb{N}$, then $u(\omega) = x_r(\omega)$ is the required random fixed point of the multi-valued random operator $T(\omega)$. Assume that $x_n(\omega) \neq x_{n+1}(\omega)$ for each $n \in \mathbb{N}$. Then we have

$$x_0(\omega) = a(\omega) < x_1(\omega) < x_2(\omega) < \dots < x_n(\omega) < \dots \leq b(\omega) \tag{2.3}$$

for all $\omega \in \Omega$. Since the cone K is normal in X, the random order interval $[a, b]$ is norm-bounded subset of X. Denote

$$A = \{x_0, \dots, x_n \dots\}.$$

Then,

$$A = \{x_0\} \cup \{x_1, \dots, x_n \dots\} \subseteq \{x_0\} \cup Q(\omega)(A).$$

If $\beta(A) \neq 0$, then we have

$$\beta(A) \leq \max\{\beta(\{x_0\}), \beta(Q(\omega)(A))\} < \beta(A)$$

which is a contradiction, and so $\beta(A) = 0$. Hence $\overline{A}$ is compact. As a result, $\lim_{n\to\infty} x_n(\omega) = x_*(\omega)$ exists for all $\omega \in \Omega$. By the upper semi-continuity of $T(\omega)$, one has $x_*(\omega) \in T(\omega)x_*(\omega)$. The measurability of $x_*(\omega)$ follows from the fact that the strong limit of the sequence of measurable functions is measurable. Hence, the multi-valued random operator $T(\omega)$ has a random fixed point in $[a, b]$. This completes the proof. □

Corollary 2.1 (Dhage [8]) *Let $(\Omega, \mathcal{A})$ be a measurable space and let $[a, b]$ be a random interval in a separable Banach space X. If $T : \Omega \times [a, b] \to \mathcal{P}_{cl}([a, b])$ is a compact, upper semi-continuous right monotone increasing multi-valued random operator and the cone K in X is normal, then $T(\omega)$ has a random fixed point in $[a, b]$.*

To prove the next result, we need the following lemma in the sequel.

Lemma 2.1 (Dhage [9]) *Let $(\Omega, \mathcal{A}, \mu)$ be a complete σ-finite measure space and let X be a separable Banach space. If $F, G : \Omega \to \mathcal{P}_{cl}(X)$ are two multi-valued random operators, then the sum $F + G$ defined by $(F + G)(\omega) = F(\omega) + G(\omega)$ is again a multi-valued random operator on Ω.*

Definition 2.2 *A multi-valued mapping $T : X \to \mathcal{P}_p(X)$ is said to be a multi-valued Lipschitz if there exists a real number $\lambda > 0$ such that*

$$d_H(Tx, Ty) \leq \lambda \|x - y\| \tag{2.4}$$

for all $x, y \in X$ and the constant λ is called the Lipschitz constant of T on X. Furthermore, if $\lambda < 1$, then T is called a contraction on X with the contraction constant λ. Similarly, a multi-valued random operator $T : \Omega \times X \to \mathcal{P}_p(X)$ is called multi-valued Lipschitz if $T(\omega)$ is a multi-valued Lipschitz on X for each $\omega \in \Omega$. Moreover, if T is a multi-valued random contraction on X if $T(\omega)$ is a multi-valued contraction on X for each $\omega \in \Omega$.

Corollary 2.2 *Let $(\Omega, \mathcal{A}, \mu)$ be a complete σ-finite measure space and let $[a, b]$ be a random interval in a separable Banach space X. Let $A, B : \Omega \times [a, b] \to \mathcal{P}_{cl}(X)$ be two right monotone increasing multi-valued random operators satisfying for each $\omega \in \Omega$,*

(a) $A(\omega)$ is a multi-valued contraction,

(b) $B(\omega)$ is completely continuous, and

(c) $A(\omega)x + B(\omega)x \in [a, b]$ for all $x \in [a, b]$.

Furthermore, if the cone K in X is normal, then the random operator inclusion $x \in A(\omega)x + B(\omega)x$ has a random solution in $[a, b]$.

Proof : Define a multi-valued mapping $T : \Omega \times X \to \mathcal{P}_{cl}(X)$ by

$$T(\omega)x = A(\omega)x + B(\omega)x. \tag{2.5}$$

By Lemma 2.1, the sum of two measurable multi-valued operators $A(\omega)$ and $B(\omega)$ is again measurable (see Dhage [9]). Hence, $T(\omega)$ defines a multi-valued random operator $T : \Omega \times [a, b] \to \mathcal{P}_{cl}([a, b])$ in view of hypothesis (c). We will show that $T(\omega)$ is upper semi-continuous and a condensing multi-valued random operator on $[a, b]$. Let $S \subset [a, b]$. Since the cone K in X is normal, the random order interval $[a, b]$ is bounded in norm. Hence, S and $T(\omega)(S)$ is bounded for each $\omega \in \Omega$. Therefore, we have

$$\beta(T(\omega)(S)) \leq \beta(A(\omega)(S)) + \beta(B(\omega)(S)) \leq \lambda\beta(A(\omega)(S))$$

for all $\omega \in \Omega$. Again, since $A(\omega)$ is a multi-valued contraction, it is continuous in the Hausdorff metric d_H on X. Hence, $A(\omega)$ is upper semi-continuous on $[a, b]$ for each $\omega \in \Omega$. Furthermore, the sum of two upper semi-continuous multi-valued operators is upper semi-continuous, and so $T(\omega)$ is upper semi-continuous on $[a, b]$ for all $\omega \in \Omega$. Now we apply Theorem 2.2 to yield that the random operator inclusion $x \in A(\omega)x + B(\omega)x$ has a random solution in $[a, b]$. This completes the proof. $\square$

Corollary 2.3 *Let $(\Omega, \mathcal{A}, \mu)$ be a complete σ-finite measure space and let $[a, b]$ be a random interval in a separable Banach space X. Assume that $A : \Omega \times [a, b] \to X$ is nondecreasing and $B : \Omega \times [a, b] \to \mathcal{P}_{cl}(X)$ is a right monotone increasing multi-valued random operator satisfying for each $\omega \in \Omega$,*

(a) *$A(\omega)$ is a single-valued contraction with the contraction constant $\lambda < 1/2$,*

(b) *$B(\omega)$ is completely continuous, and*

(c) *$A(\omega)x + B(\omega)x \in [a, b]$ for all $x \in [a, b]$.*

Furthermore, if the cone K in X is normal, then the random operator inclusion $x \in A(\omega)x + B(\omega)x$ has a random solution in $[a, b]$.

Remark 2.2 Hypothesis (c) of Corollary 2.3 holds if the random operators $A(\omega)$ and $B(\omega)$ are right monotone increasing and the elements a and b satisfy $a \leq A(\omega)a+B(\omega)a$ and $A(\omega)b + B(\omega)b \leq b$ for all $\omega \in \Omega$.

2.2 Strict monotone increasing multi-valued random operators

We need the following definition in the sequel.

Definition 2.3 *A multi-valued random operator $T : \Omega \times X \to \mathcal{P}_p(X)$ is called strict monotone increasing if for each $\omega \in \Omega$, $T(\omega)x \leq T(\omega)y$ for all $x, y \in X$ for which $x < y$. Similarly, the multi-valued random operator $T(\omega)$ is called monotone decreasing if for each $\Omega \in \omega$, $T(\omega)x \geq T(\omega)y$ for all $x, y \in X$ for which $x < y$. Finally, $T(\omega)$ is called monotone if it is a either monotone increasing or monotone decreasing multi-valued random operator on X.*

Remark 2.3 *We note that every strict monotone increasing multi-valued random operator is right monotone increasing, but the converse may not be true.*

Below we prove some random fixed point theorems for strict monotone increasing multi-valued random operators on separable ordered Banach spaces.

Theorem 2.3 *Let $(\Omega, \mathcal{A})$ be a measurable space and let $[a, b]$ be a random order interval in a separable Banach space X. If $T : \Omega \times [a, b] \to \mathcal{P}_{cl}([a, b])$ is a strict monotone increasing, upper semi-continuous and condensing multi-valued random operator and the cone K in X is normal, then $T(\omega)$ has the least random fixed point $x_*(\omega)$ and the greatest random fixed point $y^*(\omega)$ in $[a, b]$ and the sequences $\{x_n(\omega)\}$ and $\{y_n(\omega)\}$ defined by*

$$x_0(\omega) = a(\omega),\ x_{n+1}(\omega) \in T(\omega)x_n,\ n = 0, 1, 2, \ldots, \tag{2.6}$$

and

$$y_0(\omega) = b(\omega),\ y_{n+1}(\omega) \in T(\omega)y_n,\ n = 0, 1, 2, \ldots, \tag{2.7}$$

converge to $x_*(\omega)$ and $y^*(\omega)$ respectively.

Proof : Define a monotone increasing sequence $\{x_n(\omega)\}$ of measurable functions in $[a,b]$ by

$$x_0(\omega) = a(\omega),\ x_{n+1}(\omega) \in T(\omega)x_n,\ n = 0, 1, 2, \ldots,$$

which does exist by virtue of the strict monotone increasing multi-valued random operator $T(\omega)$. If $x_r(\omega) = x_{r+1}(\omega)$ for some $r \in \mathbb{N}$, then $u(\omega) = x_r(\omega)$ is the required random fixed point of the multi-valued random operator $T(\omega)$. Assume that $x_n(\omega) \neq x_{n+1}(\omega)$ for each $n \in \mathbb{N}$. Then, we have

$$x_0(\omega) = a(\omega) < x_1(\omega) < x_2(\omega) < \ldots < x_n(\omega) < \ldots \leq b(\omega)$$

for all $\omega \in \Omega$. Proceeding with the arguments as in the proof of Theorem 2.1, it is proved that the sequence $\{x_n(\omega)\}$ converges increasingly to the random fixed point $x_*(\omega)$ of $T(\omega)$. Similarly, the sequence $\{y_n(\omega)\}$ of monotone decreasing measurable functions converges decreasingly to the random fixed point $y^*(\omega)$ of $T(\omega)$. Next we show that $x_*(\omega)$ and $y^*(\omega)$ are respectively the least and the greatest random fixed point of the multi-valued random operator $T(\omega)$ on X. Let $x(\omega)$ be any random fixed point of $T(\omega)$ in $[a,b]$. By the strict monotonicity of $T(\omega)$, we have

$$a(\omega) = x_0(\omega) \leq x_1(\omega) \leq \ldots \leq x_n(\omega) \leq x(\omega) \leq y_n(\omega) \leq \ldots \leq y_1(\omega) \leq y_0(\omega) = b(\omega)$$

for all $\omega \in \Omega$. Hence, $x_*(\omega) \leq x(\omega) \leq y^*(\omega)$ for all $\omega \in \Omega$. Thus $x_*(\omega)$ and $y^*(\omega)$ are respectively the least and the greatest random fixed point of the multi-valued random operator $T(\omega)$ in $[a,b]$. This completes the proof. □

Corollary 2.4 (Dhage [8]) *Let $(\Omega, \mathcal{A})$ be a measurable space and let $[a,b]$ be a random order interval in a separable Banach space X. If $T : \Omega \times [a,b] \to \mathcal{P}_{cl}([a,b])$ is a strict monotone increasing completely continuous multi-valued random operator and the cone K in X is normal, then $T(\omega)$ has the least random fixed point $x_*(\omega)$ and the greatest random fixed point $y^*(\omega)$ in $[a,b]$. Moreover, the sequences $\{x_n(\omega)\}$ and $\{y_n(\omega)\}$ defined by (2.6) and (2.7) converge to $x_*(\omega)$ and $y^*(\omega)$ respectively.*

Corollary 2.5 *Let $(\Omega, \mathcal{A}, \mu)$ be a complete σ-finite measure space and let $[a,b]$ be a random order interval in a separable Banach space X. Let $A, B : \Omega \times [a,b] \to \mathcal{P}_{cl}(X)$ be two strict monotone increasing multi-valued random operators satisfying for each $\omega \in \Omega$,*

(a) $A(\omega)$ is multi-valued contraction,

(b) $B(\omega)$ is completely continuous, and

(c) $A(\omega)x + B(\omega)x \in [a,b]$ for all $x \in [a,b]$.

If the cone K in X is normal, then the random operator inclusion $x \in A(\omega)x + B(\omega)x$ has the least random solution x_* and the greatest random solution y^* in $[a,b]$. Moreover, the sequences $\{x_n(\omega)\}$ and $\{y_n(\omega)\}$ defined by

$$x_0(\omega) = a(\omega),\ x_{n+1}(\omega) \in A(\omega)x_n + B(\omega)x_n,\ n = 0, 1, 2, \ldots; \tag{2.8}$$

and

$$y_0(\omega) = b(\omega),\ y_{n+1}(\omega) \in A(\omega)y_n + B(\omega)y_n,\ n = 0, 1, 2, \ldots, \tag{2.9}$$

converge to $x_*(\omega)$ *and* $y^*(\omega)$ *respectively.*

Proof : The proof is similar to that of Corollary 2.2 and the conclusion now follows by an application of Theorem 2.3. $\square$

Corollary 2.6 *Let* $(\Omega, \mathcal{A}, \mu)$ *be a complete* σ*-finite measure space and let* $[a, b]$ *be a random interval in a separable Banach space* X*. Let* $A : \Omega\times[a, b] \to X$ *be nondecreasing and* $B : \Omega \times [a, b] \to \mathcal{P}_{cl}(X)$ *be an right monotone increasing multi-valued random operator satisfying for each* $\omega \in \Omega$*,*

(a) $A(\omega)$ *is a single-valued contraction with the contraction constant* $\lambda < 1/2$*,*

(b) $B(\omega)$ *is completely continuous, and*

(c) $A(\omega)x + B(\omega)x \in [a, b]$ *for all* $x \in [a, b]$*.*

Furthermore, if the cone K *in* X *is normal, then the random operator inclusion* $x \in A(\omega)x + B(\omega)x$ *has a least random solution* $x_*(\omega)$ *and a greatest random solution* $y^*(\omega)$ *in* $[a, b]$*. Moreover, the sequences* $\{x_n(\omega)\}$ *and* $\{y_n(\omega)\}$ *defined by (2.8) and (2.9) converge to* $x_*(\omega)$ *and* $y^*(\omega)$ *respectively.*

Remark 2.4 Hypothesis (c) of Corollary 2.5 holds if the random operators $A(\omega)$ and $B(\omega)$ are strict monotone increasing and the elements a and b satisfy $a \le A(\omega)a+B(\omega)a$ and $A(\omega)b + B(\omega)b \le b$ for all $\omega \in \Omega$.

Remark 2.5 We remark that in most of random fixed point theorems of topological nature for single as well as multi-valued random operators, the separability hypothesis of the underlined Banach space is indispensable, but the case with random fixed point theorems of algebraic nature for such operators is quite different. Here, we do not require the separability hypothesis for the validity of the algebraic random fixed point theorems of this paper. Note that our Theorems 2.2 and 2.3 have wide range of applications to random differential and integral inclusions for proving the existence as well as the existence of extremal solutions under suitable conditions. In the following section we establish the existence theorems for certain perturbed random differential inclusions under mixed Lipschitz, Carathéodory and monotonic conditions of the multi-functions involved in them.

3 Random Differential Inclusions

In this section, we discuss initial value problems of ordinary first order random differential inclusions for existence as well as existence of the extremal random solutions between the given strict lower and strict upper random solutions.

Let $(\Omega, \mathcal{A}, \mu)$ be a complete σ-finite measure space and let $\mathbb{R}$ be the real and and let $J = [0, T]$ be a closed and bounded interval in $\mathbb{R}$. Consider the initial value problem of first order ordinary random differential inclusion (in short RDI),

$$\left.\begin{aligned} x'(t,\omega) &\in F(t, x(t,\omega), \omega) + G(t, x(t,\omega), \omega) \text{ a.e. } t \in J \\ x(0,\omega) &= q(\omega) \end{aligned}\right\} \tag{3.1}$$

for all $\omega \in \Omega$, where $q : \Omega \to \mathbb{R}$ is measurable and $F, G : J \times \mathbb{R} \times \Omega \to \mathcal{P}_p(\mathbb{R})$.

By a *random solution* for the RDI (3.1) we mean a measurable function $x : \Omega \to AC(J, \mathbb{R})$ satisfying for each $\omega \in \Omega$, $x'(t,\omega) = v_1(t,\omega) + v_2(t,\omega)$ $\forall\ t \in J$ and $x(0,\omega) = q(\omega)$ for some measurable functions $v_1, v_2 : \Omega \to L^1(J, \mathbb{R})$ with $v_1(t,\omega) \in F(t, x(t,\omega), \omega)$ and $v_2(t,\omega) \in G(t, x(t,\omega), \omega)$ a.e. $t \in J$, where $AC(J, \mathbb{R})$ is the space of absolutely continuous real-valued functions on J.

To the best of our knowledge, the RDI (3.1) has not been discussed earlier in the literature. But the special case, when the random parameter is absent from the RDI (3.1), we obtain a classical perturbed differential inclusion

$$\left.\begin{aligned} x'(t) &\in F(t, x(t)) + G(t, x(t)) \text{ a.e. } t \in J, \\ x(0) &= q, \end{aligned}\right\} \tag{3.2}$$

is studied for the existence of solutions under certain mixed Lipschitz and compactness type conditions (see Dhage [7] and the references therein). Most of these results involve the hypothesis that the multi-function F has convex values on the domain of definition. In the present approach, we do not require any convexity assumption on the values of the multi-functions F and G involved in the random differential inclusion, instead, we assume certain monotonicity condition for proving the existence results. In the special case where $F(t, x, \omega) = \{f(t, x, \omega)\}$ and $G(t, x, \omega) = \{g(t, x, \omega)\}$, we obtain the random differential equation,

$$\left.\begin{aligned} x'(t,\omega) &= f(t, x(t,\omega), \omega) + g(t, x(t,\omega), \omega) \text{ a.e. } t \in J, \\ x(0,\omega) &= q(\omega), \end{aligned}\right\} \tag{3.3}$$

which has been studied for the existence results in Itoh [12] and for existence of the extremal random solutions in Dhage [6] via fixed point techniques. Therefore, our results include the results of Itoh [12] and Dhage [6] as special cases. We claim that our results as well as our approach is new to the theory of random differential inclusions.

We will obtain the solutions of RDI (3.1) in the function space $C(J, \mathbb{R})$ of continuous real-valued functions on J. Define a norm $\|\cdot\|$ in $C(J, \mathbb{R})$ by

$$\|x\| = \sup_{t \in J} |x(t)| \tag{3.4}$$

and the order relation $\leq$ in $C(J, \mathbb{R})$ by

$$x \leq y \iff y - x \in K, \tag{3.5}$$

where the cone K in $C(J, \mathbb{R})$ is defined by

$$K = \{x \in C(J, \mathbb{R}) \mid x(t) \geq 0 \text{ for all } t \in J\}. \tag{3.6}$$

Clearly, $C(J, \mathbb{R})$ becomes an ordered separable Banach space with the cone K which is normal in it. For any measurable function $x : \Omega \to C(J, \mathbb{R})$, let

$$S^1_{F(\omega)}(x) = \{v \in \mathcal{M}(\Omega, L^1(J, \mathbb{R})) \mid v(t, \omega) \in F(t, x(t, \omega), \omega) \text{ a.e. } t \in J\}. \tag{3.7}$$

This is our set of *selection functions*. The integral of the random multi-valued function F is defined as

$$\int_0^t F(s, x(s, \omega), \omega)\, ds = \left\{ \int_0^t v(s, \omega)\, ds : v \in S^1_{F(\omega)}(x) \right\}.$$

We need the following definitions in the sequel.

Definition 3.1 *A multi-valued mapping $F : J \times \Omega \to \mathcal{P}_{cp}(\mathbb{R})$ is said to be measurable if for any $y \in X$, the function $(t, \omega) \mapsto d(y, F(t, \omega)) = \inf\{|y - x| : x \in F(t, \omega)\}$ is measurable.*

Definition 3.2 *A multi-valued mapping $F : J \times \mathbb{R} \times \Omega \to \mathcal{P}_{cp}(\mathbb{R})$ is said to be integrably bounded if there exists a function $h \in \mathcal{M}(\Omega, L^1(J, \mathbb{R}))$ such that*

$$\|F(t, x, \omega)\|_{\mathcal{P}} = \sup\{|u| : u \in F(t, x, \omega)\} \leq h(t, \omega) \quad a.e. \quad t \in J$$

for all $\omega \in \Omega$ and $x \in \mathbb{R}$.

Remark 3.1 It is known that if $F : J \times \mathbb{R} \times \Omega \to \mathcal{P}_{cp}(\mathbb{R})$ is integerably bounded, then $S^1_{F(\omega)}(x) \neq \emptyset$ for each $x \in \mathbb{R}$.

Definition 3.3 *A multi-valued function $F : J \times \mathbb{R} \times \Omega \to \mathcal{P}_{cp}(\mathbb{R})$ is called Carathéodory if for each $\omega \in \Omega$,*

(i) $t \mapsto F(t, x, \omega)$ is measurable for each $x \in \mathbb{R}$, and

(ii) $x \mapsto F(t, x, \omega)$ is an upper semi-continuous almost everywhere for $t \in J$.

Again, a Carathéodory multi-valued function F is called L^1-Carathéodory if

(iii) for each real number $r > 0$ there exists a measurable function $h_r : \Omega \to L^1(J, \mathbb{R})$ such that for each $\omega \in \Omega$

$$\|F(t, x, \omega)\|_{\mathcal{P}} = \sup\{|u| : u \in F(t, x, \omega)\} \leq h_r(t, \omega) \qquad a.e. \quad t \in J$$

for all $x \in \mathbb{R}$ with $|x| \leq r$.

Furthermore, a Carathéodory multi-valued function F is called L^1_X-Carathéodory if *(iv) there exists a measurable function $h : \Omega \to L^1(J, \mathbb{R})$ such that*

$$\|F(t, x, \omega)\|_{\mathcal{P}} \le h(t, \omega) \quad a.e. \quad t \in J$$

for all $x \in \mathbb{R}$, and the function h is called a growth function of F on $J \times \mathbb{R} \times \Omega$.

Definition 3.4 *A multi-valued function $F : J \times \mathbb{R} \times \Omega \to \mathcal{P}_{cp}(\mathbb{R})$ is called s-Carathéodory if for each $\omega \in \Omega$,*

(i) $t \mapsto F(t, x, \omega)$ is measurable for each $x \in \mathbb{R}$, and

(ii) $x \mapsto F(t, x, \omega)$ is an Hausdorff continuous almost everywhere for $t \in J$.

Furthermore, a s-Carathéodory multi-valued function F is called s-L^1-Carathéodory if

(iii) for each real number $r > 0$ there exists a measurable function $h_r : \Omega \to L^1(J, \mathbb{R})$ such that for each $\omega \in \Omega$

$$\|F(t, x, \omega)\|_{\mathcal{P}} = \sup\{|u| : u \in F(t, x, \omega)\} \le h_r(t, \omega) \quad a.e. \quad t \in J$$

for all $x \in \mathbb{R}$ with $|x| \le r$.

Then we have the following lemmas which are well-known in the literature.

Lemma 3.1 (Lasota and Opial [13]) *Let E be a Banach space. If $dim(E) < \infty$ and $F : J \times E \times \Omega \to \mathcal{P}_{cp}(E)$ is L^1-Carathéodory, then $S^1_{F(\omega)}(x) \neq \emptyset$ for each $x \in E$.*

Lemma 3.2 (Lasota and Opial [13]) *Let E be a Banach space, F a Carathéodory multi-valued operator with $S^1_{F(\omega)} \neq \emptyset$, and $\mathcal{L} : L^1(J, E) \to C(J, E)$ be a continuous linear mapping. Then the composite operator*

$$\mathcal{L} \circ S^1_{F(\omega)} : C(J, E) \to \mathcal{P}_{bd,cl}(C(J, E))$$

is a closed graph operator on $C(J, E) \times C(J, E)$.

Lemma 3.3 (Hu and Papageorgiou [11]) *Let E be a Banach space. If $F : J \times E \to \mathcal{P}_p(E)$ is s-Caratheodory, then the multi-valued mapping $(t, x) \mapsto F(t, x)$ is jointly measurable.*

We need the following definition in the sequel.

Definition 3.5 *A measurable function $a : \Omega \to C(J, \mathbb{R})$ is a strict lower random solution for the RDI (3.1) if for all $v_1 \in S^1_{F(\omega)}(a)$, $v_2 \in S^1_{G(\omega)}(a)$, we have*

$$a'(t, \omega) \le v_1(t, \omega) + v_2(t, \omega), \quad and \quad a(0, \omega) \le q(\omega)$$

for all $t \in J$ and $\omega \in \Omega$. Similarly, a strict upper random solution for the RDI (3.1) on $J \times \Omega$ is defined.

We consider the following set of hypotheses in the sequel.

(A_1) The multi-valued mapping $(t, x, \omega) \mapsto F(t, x, \omega)$ is jointly measurable.

(A_2) $F(t, x, \omega)$ is closed and bounded for each $(t, \omega) \in J \times \Omega$ and $x \in \mathbb{R}$.

(A_3) F is integrably bounded on $J \times \Omega \times \mathbb{R}$.

(A_4) There is a function $\ell \in \mathcal{M}(\Omega, L^1(J, \mathbb{R}))$ such that for each $\omega \in \Omega$,

$$d_H(F(t, x, \omega), F(t, y, \omega)) \leq \ell(t, \omega)|x - y| \quad \text{a.e.} \quad t \in J$$

for all $x, y \in \mathbb{R}$.

(A_5) The multi-valued mapping $x \mapsto S^1_{F(\omega)}(x)$ is right monotone increasing in $x \in C(J, \mathbb{R})$ almost everywhere for $t \in J$.

(B_1) The multi-valued mapping $(t, x, \omega) \mapsto G(t, x, \omega)$ is jointly measurable.

(B_2) $G(t, x, \omega)$ is closed and bounded for each $(t, \omega) \in J \times \Omega$ and $x \in \mathbb{R}$.

(B_3) G is L^1-Carathéodory.

(B_4) The multi-valued mapping $x \mapsto S^1_{F(\omega)}(x)$ is right monotone increasing in $x \in C(J, \mathbb{R})$ almost everywhere for $t \in J$.

(B_5) RDI (3.1) has a strict lower random solution a and a strict upper random solution b with $a \leq b$ on $J \times \Omega$.

Hypotheses $(A_1) - (A_4)$ and $(B_1) - (B_3)$ are common in the literature. Some nice sufficient conditions for guarantying $S^1_{F(\omega)} \neq \emptyset$ appear in Deimling [2], and Lasota and Opial [13]. Hypothesis (B_5) holds, in particular, if the multi-valued random function F is bounded on $J \times \mathbb{R} \times \Omega$. Hypotheses (A_5) and (B_4) are relatively new to the literature, but special forms have appeared in the works of several authors. Some details on theses hypotheses appear in Dhage [3, 4] and the references therein.

Theorem 3.1 *Assume that the hypotheses $(A_1) - (A_5)$ and $(B_1) - (B_5)$ hold. If $\|\ell(\omega)\|_{L^1} < 1$, then the RDI (3.1) has a random solution in $[a, b]$ defined on $J \times \Omega$.*

Proof : Let $X = C(J, \mathbb{R})$. Define a random order interval $[a, b]$ in X which is well defined in view of hypothesis (B_5). Now the RDI (3.1) is equivalent to the random integral inclusion

$$x(t, \omega) \in q(\omega) + \int_0^t F(s, x(s, \omega), \omega)\, ds + \int_0^t G(s, x(s, \omega), \omega)\, ds, \ t \in J. \tag{3.8}$$

for all $\omega \in \Omega$ (see Dhage [5] and the references therein). Define two multi-valued operators $A, B : \Omega \times [a, b] \to \mathcal{P}_p(X)$ by

$$\begin{aligned} A(\omega)x &= \left\{ u \in \mathcal{M}(\Omega, X) \,\middle|\, u(t,\omega) = \int_0^t v_1(s,\omega)\,ds,\ v_1 \in S^1_{F(\omega)}(x) \right\} \\ &= (\mathcal{K}_1 \circ S^1_{F(\omega)})(x) \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} B(\omega)x &= \left\{ u \in \mathcal{M}(\Omega, X) \,\middle|\, u(t,\omega) = q(\omega) + \int_0^t v_2(s,\omega)\,ds,\ v_2 \in S^1_{F(\omega)}(x) \right\} \\ &= (\mathcal{K}_2 \circ S^1_{G(\omega)})(x) \end{aligned} \tag{3.10}$$

where $\mathcal{K}_1, \mathcal{K}_2 : \mathcal{M}(\Omega, L^1(J, \mathbb{R}))) \to C(J, \mathbb{R})$ are continuous operators defined by

$$\mathcal{K}_1 v_1(t,\omega) = \int_0^t v_1(s,\omega)\,ds. \tag{3.11}$$

and

$$\mathcal{K}_2 v_2(t,\omega) = q(\omega) + \int_0^t v_2(s,\omega)\,ds. \tag{3.12}$$

Clearly, the operators $A(\omega)$ and $B(\omega)$ are well defined in view of hypotheses (A_3) and (B_3). We will show that $A(\omega)$ and $B(\omega)$ satisfy all the conditions of Corollary 2.2.

Step I : First, we show that A is a closed valued multi-valued random operator on $\Omega \times [a, b]$. Observe that the operator $A(\omega)$ is equivalent to the composition $\mathcal{K}_1 \circ S^1_{F(\omega)}$ of two operators on $L^1(J, \mathbb{R})$, where $\mathcal{K}_1 : \mathcal{M}(\Omega, L^1(J, \mathbb{R})) \to X$ is the continuous operator defined by (3.3). To show $A(\omega)$ has closed values, it then suffices to prove that the composition operator $\mathcal{K}_1 \circ S^1_{F(\omega)}$ has closed values on $[a, b]$. Let $x \in [a, b]$ be arbitrary and let $\{v_n\}$ be a sequence in $S^1_{F(\omega)}(x)$ converging to v in measure. Then, by the definition of $S^1_{F(\omega)}$, $v_n(t,\omega) \in F(t, x(t,\omega), \omega)$ a. e. for $t \in J$. Since $F(t, x(t,\omega), \omega)$ is closed, $v(t,\omega) \in F(t, x(t,\omega), \omega)$ a.e. for $t \in J$. Hence, $v \in S^1_{G(\omega)}(x)$. As a result, $S^1_{G(\omega)}(x)$ is a closed set in $L^1(J, \mathbb{R})$ for each $\omega \in \Omega$. From the continuity of $\mathcal{K}_1$, it follows that $(\mathcal{K}_1 \circ S^1_{F(\omega)})(x)$ is a closed set in X. Therefore, $A(\omega)$ is a closed-valued multi-valued operator on $[a, b]$ for each $\omega \in \Omega$. Next, we show that $A(\omega)$ is a multi-valued random operator on $[a, b]$. First, we show that the multi-valued mapping $(\omega, x) \mapsto S^1_{F(\omega)}(x)$ is measurable. Let $f \in \mathcal{M}(\Omega, L^1(J, \mathbb{R}))$ be arbitrary. Then we have

$$\begin{aligned} d(f, S^1_{F(\omega)}(x)) &= \inf\{\|f(\omega) - h(\omega)\|_{L^1} : h \in S_{F(\omega)}(x)\} \\ &= \inf\left\{ \int_0^T |f(t,\omega) - h(t,\omega)|\,dt : h \in S_{F(\omega)}(x) \right\} \\ &= \int_0^T \inf\{|f(t,\omega) - z| : z \in F(t, x(t,\omega), \omega)\}\,dt \end{aligned}$$

$$= \int_0^T d(f(t,\omega), F(t,x(t,\omega),\omega)\,dt.$$

But by hypothesis (A_0), the mapping $(t,x,\omega) \mapsto F(t,x,\omega)$ is measurable and it is known that the multi-valued mapping $z \mapsto d(z, F(t,x,\omega)$ is continuous. Hence the mapping multi-valued mapping $(t,x,\omega,z) \mapsto d(z, F(t,x,\omega))$ measurable. It is also known that the evaluation mapping $(t,x(\cdot)) = e_t(x(\cdot)) = x(t)$ is continuous from $J \times X$ into X. Hence we deduce that the mapping $(t,x,\omega,f) \mapsto d(f(t,\omega), F(t,x(t,\omega),\omega)$ is measurable from $J \times X \times \Omega \times L^1(J,\mathbb{R})$ into $\mathbb{R}^+$. Now the integral is the limit of the finite sum of measurable functions, and so, $d(f, S^1_{F(\omega)}(x))$ is measurable. As a result, the multi-valued mapping $(\cdot,\cdot) \to S^1_{F(\cdot)}(\cdot)$ is jointly measurable.

Define a function ϕ on $J \times X \times \Omega$ by

$$\phi(t,x,\omega) = \big(\mathcal{K}_1 S^1_{F(\omega)}\big)(x)(t) = \int_0^t F(s, x(s,\omega,\omega))\,ds.$$

We shall show that $\phi(t,x,\omega)$ is continuous in t in the Hausdorff metric on $\mathbb{R}$. Let $\{t_n\}$ be a sequence in J converging to $t \in J$. Then we have

$$\begin{aligned}
&d_H(\phi(t_n,x,\omega), \phi(t,x,\omega)) \\
&\qquad = d_H\left(\int_0^{t_n} F(s,x(s,\omega),\omega)\,ds, \int_0^t F(s,x(s,\omega),\omega)\,ds\right) \\
&\qquad = d_H\left(\int_J \chi_{[0,t_n]}(s) F(s,x(s,\omega),\omega)\,ds, \int_J \chi_{[0,t]}(s) F(s,x(s,\omega),\omega)\,ds\right) \\
&\qquad = d_H\left(\int_J \chi_{[0,t_n]}(s) F(s,x(s,\omega),\omega)\,ds, \int_J \chi_{[0,t]}(s) F(s,x(s,\omega),\omega)\,ds\right) \\
&\qquad = \int_J |\chi_{[0,t_n]}(s) - \chi_{[0,t]}(s)|\, \|F(s,x(s,\omega),\omega)\|_{\mathcal{P}}\,ds \\
&\qquad = \int_J |\chi_{[0,t_n]}(s) - \chi_{[0,t]}(s)|\, h_r(s,\omega)\,ds \\
&\qquad \to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}$$

Thus the multi-valued mapping $t \mapsto \phi(t,x,\omega)$ is continuous, and hence, and by Lemma 3.3, the mapping

$$(t,x,\omega) \mapsto \int_0^t F(s,x(s,\omega),\omega)\,ds$$

is measurable. Consequently, $A(\omega)$ is a random multi-valued operator on $[a,b]$. Similarly, it can be shown that $B(\omega)$ is a closed-valued multi-valued operator on $[a,b]$ and the mapping $(t,x,\omega) \mapsto \int_0^t G(s,x(s,\omega),\omega)\,ds$ is measurable. Again, since the sum of two measurable multi-valued functions is measurable, the mapping $(t,x,\omega) \mapsto q(\omega) + \int_0^t G(s,x(s,\omega),\omega)\,ds$ is measurable.

Step II : Next we show that $A(\omega)$ is a multi-valued contraction on X. Let $x, y \in X$ be any two element and let $u_1 \in A(\omega)(x)$. Then $u_1 \in X$ and

$$u_1(t,\omega) = \int_0^t v_1(s,\omega)\, ds$$

for some $v_1 \in S^1_{F(\omega)}(x)$. Since

$$d_H(F(t,x(t,\omega),\omega), F(t,y(t,\omega),\omega) \leq \ell(t,\omega)|x(t,\omega) - y(t,\omega)|,$$

we obtain that there exists a $w \in F(t,y(t,\omega),\omega)$ such that

$$|v_1(t,\omega) - w| \leq \ell(t,\omega)|x(t,\omega) - y(t,\omega)|.$$

Thus, the multi-valued operator U defined by

$$U(t,\omega) = S^1_{F(\omega)}(y)(t) \cap K(\omega)(t),$$

where

$$K(\omega)(t) = \{w|\ |v_1(t,\omega) - w| \leq \ell(t,\omega)\ |x(t,\omega) - y(t,\omega)|\}$$

has nonempty values and is measurable. Let v_2 be a measurable selection function for U (which exists by the Kuratowski-Ryll-Nardzewski's selection theorem (see [14]). Then there exists $v_2 \in F(t,y(t,\omega),\omega)$ with $|v_1(t,\omega) - v_2(t,\omega)| \leq \ell(t,\omega)\ |x(t,\omega) - y(t,\omega)|$, a.e. on J.

Define $u_2(t,\omega) = \int_0^t v_2(s,\omega)\, ds$. It follows that $u_2 \in A(\omega)(y)$ and

$$\begin{aligned}
|u_1(t,\omega) - u_2(t,\omega)| &\leq \left| \int_0^t v_1(s,\omega)\, ds - \int_0^t v_2(s,\omega)\, ds \right| \\
&\leq \int_0^t |v_1(s,\omega) - v_2(s,\omega)|\, ds \\
&\leq \int_0^t \ell(t,\omega)|x(s,\omega) - y(s,\omega)|\, ds \\
&\leq \|\ell(\omega)\|_{L^1} \|x(\omega) - y(\omega)\|.
\end{aligned}$$

Taking the supremum over t, we obtain

$$\|u_1(\omega) - u_2(\omega)\| \leq \|\ell(\omega)\|_{L^1} \|x(\omega) - y(\omega)\|.$$

From this and the analogous inequality obtained by interchanging the roles of x and y we obtain

$$d_H(A(\omega)(x), A(\omega)(y)) \leq \|\ell(\omega)\|_{L^1} \|x(\omega) - y(\omega)\|,$$

for all $x, y \in X$. This shows that $A(\omega)$ is a multi-valued random contraction on X, since $\|\ell(\omega)\|_{L^1} < 1$.

Step III : Next, we show that $B(\omega)$ is completely continuous for each $\omega \in \Omega$. First, we show that $B(\omega)([a,b])$ is compact for each $\omega \in \Omega$. Let $\{y_n(\omega)\}$ be a sequence in $B(\omega)([a,b])$ for some $\omega \in \Omega$. We will show that $\{y_n(\omega)\}$ has a cluster point. This is achieved by showing that $\{y_n(\omega)\}$ is uniformly bounded and equi-continuous sequence in X.

Case I : First, we show that $\{y_n(\omega)\}$ is uniformly bounded sequence. By the definition of $\{y_n(\omega)\}$, we have a $v_n(\omega) \in S^1_{G(\omega)}(x)$ for some $x \in [a,b]$ such that

$$y_n(t,\omega) = q(\omega) + \int_0^t v_n(s,\omega)\,ds,\ t \in J.$$

Therefore,

$$\begin{aligned}
|y_n(t,\omega)| &\le |q(\omega)| + \int_0^t |v_n(s,\omega)|\,ds \\
&\le |q(\omega)| + \int_0^t \|F(s, x_n(s,\omega),\omega)\|_{\mathcal{P}}\,ds \\
&\le |q(\omega)| + \int_0^T h_r(s,\omega)\,ds \\
&\le |q(\omega)| + \|h_r(\omega)\|_{L^1}
\end{aligned}$$

for all $t \in J$, where $r = \|a(\omega)\| + \|b(\omega)\|$. Taking the supremum over t in the above inequality yields,

$$\|y_n(\omega)\| \le |q(\omega)| + \|h_r(\omega)\|_{L^1}$$

which shows that $\{y_n(\omega)\}$ is a uniformly bounded sequence in $Q(\omega)([a,b])$.

Next we show that $\{y_n(\omega)\}$ is an equi-continuous sequence in $Q(\omega)([a,b])$. Let $t,\tau \in J$. Then we have

$$\begin{aligned}
|y_n(t,\omega) - y_n(\tau,\omega)| &\le \Big|\int_0^t v_n(s,\omega)\,ds - \int_0^\tau v_n(s,\omega)\,ds\Big| \\
&\le \Big|\int_\tau^t v_n(s,\omega)\,ds\Big| \\
&\le \Big|\int_\tau^t h_r(s,\omega)\,ds\Big| \\
&\le |p(t,\omega) - p(\tau,\omega)|,
\end{aligned}$$

where $p(t,\omega) = \displaystyle\int_0^t h_r(s,\omega)\,ds$. From the above inequality, it follows that

$$|y_n(t,\omega) - y_n(\tau,\omega)| \to 0 \quad \text{as } t \to \tau.$$

This shows that $\{y_n(\omega)\}$ is an equi-continuous sequence in $B(\omega)([a,b])$. Now $\{y_n(\omega)\}$ is uniformly bounded and equi-continuous for each $\omega \in \Omega$, so it has a cluster point in view of Arzelà-Ascoli theorem. As a result, $B(\omega)$ is a compact multi-valued random operator on $[a, b]$.

Next we show that $B(\omega)$ is a upper semi-continuous multi-valued random operator on $[a, b]$. Let $\{x_n(\omega)\}$ be a sequence in X such that $x_n(\omega) \to x_*(\omega)$. Let $\{y_n(\omega)\}$ be a sequence such that $y_n(\omega) \in B(\omega)x_n$ and $y_n(\omega) \to y_*(\omega)$. We will show that $y_*(\omega) \in B(\omega)x_*$. Since $y_n(\omega) \in B(\omega)x_n$, there exists a $v_n(\omega) \in S^1_{G(\omega)}(x_n)$ such that

$$y_n(t, \omega) = q(\omega) + \int_0^t v_n(s, \omega)\, ds, \;\; t \in J.$$

We must prove that there is a $v_*(\omega) \in S^1_{G(\omega)}(x_*)$ such that

$$y_*(t, \omega) = q(\omega) + \int_0^t v_*(s, \omega)\, ds, \;\; t \in J.$$

Consider the continuous linear operator $\mathcal{L} : \mathcal{M}(\Omega, L^1(J, \mathbb{R})) \to C(J, \mathbb{R})$ defined by

$$\mathcal{L}v(t, \omega) = \int_0^t v(s, \omega)\, ds, \;\; t \in J.$$

Now

$$\|(y_n(\omega) - q(\omega)) - (y_*(\omega) - q(\omega))\| \to 0 \;\text{ as } n \to \infty.$$

From lemma 3.2, it follows that $\mathcal{L} \circ S^1_{G(\omega)}$ is a closed graph operator. Also, from the definition of $\mathcal{L}$, we have

$$y_n(t, \omega) - q(\omega) \in \big(\mathcal{L} \circ S^1_{G(\omega)}\big)(x_n).$$

Since $y_n(\omega) \to y_*(\omega)$, there is a point $v_*(\omega) \in S^1_{F(\omega)}(x_*)$ such that

$$y_*(t, \omega) = q(\omega) + \int_0^t v_*(s, \omega)\, ds, \;\; t \in J.$$

This shows that $B(\omega)$ is a upper semi-continuous multi-valued random operator on $[a, b]$. Thus, $B(\omega)$ is upper semi-continuous and compact and hence a completely continuous multi-valued random operator on $[a, b]$.

Step VI : Next, we show that $A(\omega)$ is a right monotone increasing and multi-valued random operator on $[a, b]$ into itself for each $\omega \in \Omega$. Let $x, y \in [a, b]$ be such that $x \leq y$. Since (A_5) holds, we have that $S^1_{F(\omega)}(x) \overset{i}{\leq} S^1_{F(\omega)}(y)$. Hence $A(\omega)(x) \overset{i}{\leq} A(\omega)(y)$. Similarly, $B(\omega)(x) \overset{i}{\leq} B(\omega)(y)$. From (B_5), it follows that $a \leq A(\omega)a + B(\omega)a$ and $A(\omega)b + B(\omega)b \leq b$ for all $\omega \in \Omega$. Now $A(\omega)$ and $B(\omega)$ are right monotone increasing, so we have for each $\omega \in \Omega$,

$$a \leq A(\omega)a + B(\omega)a \overset{i}{\leq} A(\omega)x + B(\omega)x \overset{i}{\leq} A(\omega)b + B(\omega)b \leq b$$

for all $x \in [a, b]$. Hence, $A(\omega)x + B(\omega)x \in [a, b]$ for all $x \in [a, b]$).

Thus, the multi-valued random operators $A(\omega)$ and $B(\omega)$ satisfy all the conditions of Corollary 2.2 and hence the random operator inclusion $x \in A(\omega)x + B(\omega)x$ has a random solution.This implies that the RDI (3.1) has a random solution on $J \times \Omega$. This complete the proof. $\square$

Next, we prove a result concerning the extremal random solutions of the RDI (3.1) on $J \times \Omega$. We need the following hypothesis in the sequel.

(A_6) For each $\omega \in \Omega$, the multi-valued mapping $x \mapsto F(t, x, \omega)$ is strict monotone increasing almost everywhere for $t \in J$.

(B_6) For each $\omega \in \Omega$, the multi-valued mapping $x \mapsto F(t, x, \omega)$ is strict monotone increasing almost everywhere for $t \in J$.

Theorem 3.2 *Assume that* (A_1)*-*(A_4)*,* (A_6) *and* (B_1)*-*(B_4)*,* (B_6) *hold. Then the RDI (3.1) has a minimal random solution and a maximal random solution in* $[a, b]$ *defined on* $J \times \Omega$.

Proof : The proof is quite similar to that of Theorem 3.1. Here, $S^1_{F(\omega)}(x) \neq \emptyset$ and $S^1_{F(\omega)}(x) \neq \emptyset$ for each $x \in [a, b]$ in view of hypothesis (A_3) and (B_3). Also, the multi-valued mapping $x \mapsto S^1_{F(\omega)}(x)$ and $x \mapsto S^1_{G(\omega)}(x)$ are strict monotone increasing on $[a, b]$. Consequently, the multi-valued random operators $A(\omega)$ and $B(\omega)$ defined respectively by (3.2) and (3.2) are strict monotone increasing on $[a, b]$. Hence, the desired result follows by an application of Corollary 2.5. $\square$

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