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A NOTE ON THE EXISTENCE OF SOLUTIONS TO SOME NONLINEAR FUNCTIONAL INTEGRAL EQUATIONS

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ABSTRACT. Substituting the usual growth condition by an assumption that a specific initial value problem has a maximal solution, we obtain existence results for functional nonlinear integral equations with variable delay. Application of the technique to initial value problems for differential equations as well as to integrodifferential equations are given.

1. INTRODUCTION

Nonlinear integral equations and nonlinear functional integral equations have been some topics of great interest in the field of nonlinear analysis for a long time. Since the pioneering work of Volterra up to our days, integral equations have attracted the interest of scientists not only because of their mathematical context but also because of their miscellaneous applications in various fields of science and technology. In particular, existence theory for nonlinear integral equations, strongly related with the evolution on fixed point theory, has been boosted ahead after the remarkable work of Krasnoselskii [6] which signaled a new era in the research of the subject.

The present note is motivated by a recent paper by Dhage and Ntouyas [3] presenting some results on the existence of solutions to the nonlinear functional integral equation

(E)

$$x(t) = q(t) + \int_0^{\mu(t)} k(t,s)f\left(s, x(\theta(s))\right) ds + \int_0^{\sigma(t)} v(t,s)g\left(s, x(\eta(s))\right) ds, \quad t \in J,$$

where $J = [0,1]$, $q : J \to \mathbb{R}$, $k, v : J \times J \to \mathbb{R}$, $f, g : J \times \mathbb{R} \to \mathbb{R}$ and $\mu, \theta, \sigma, \eta : J \to J$. We note that equation (E) is general in the sense that it includes the well-known Volterra integral equation and the Hammerstein integral equation, as well. These equations have been extensively studied for various aspects of their solutions.

One of our purposes here is to show that existence results for the functional integral equation (E) can be obtained under some slightly different assumptions than those appearing in [3]. To be more specific, in Theorem 1 we substitute a usual growth condition by the assumption that an appropriate initial value problem has a maximal solution. Moreover, it is shown that the techniques developed may also be used to yield existence results for the solutions of some other (more general) functional integral equations and for some classes of differential equations, as well.

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An advantage of the fixed point results of Krasnoselskii's type such as the result by Burton and Kirk employed in the proof of Theorem 1 is, above all, that they combine a contraction operator with a completely continuous operator at the left hand side of the operator equation $Ax + Bx = x$. On the basis of this observation, it is not difficult to see that application of the technique employed for the proof of Theorem 1 of this note can yield similar type results for integral equations as well as for differential equations that possess the same character as (E): their left hand side can be written as a sum of a contraction and a completely continuous operator. In Theorem 2, attempting to relax the Lipschitz type condition posed on the function f in Theorem 1, we rely on the well-known Leray-Schauder Nonlinear Alternative to obtain some existence results in a case where assumptions do not ensure the presence of a contraction mapping.

Since the proofs of these results share a large part with the proof of Theorem 1, we avoid stating them in detail preferring to outline the common steps and show only the parts that differ than the proof of Theorem 1.

The paper is organized in four sections. In Section 2, some preliminaries and notation are presented. Section 3 contains the main results of the paper (Theorems 1 and 2) along with their proofs and a remark leading to a corollary (Corollary 1). Finally, in Section 4 we show that the technique used in obtaining the main results of the paper, can be applied to yield existence results for some integral equations closely related to (E) and for differential equations, as well.

2. PRELIMINARIES

Let $BM(J, \mathbb{R})$ denote the space of bounded, measurable real-valued functions defined on J. Clearly, $BM(J, \mathbb{R})$ equipped with the norm

$$\|x\| = \max_{t \in J} |x(t)|$$

becomes a Banach space. Also, by $L^1(J, \mathbb{R})$ we denote the Banach space of all Lebesque measurable functions on J with the usual norm $\|\cdot\|_{L^1}$ defined by

$$\|x\|_{L^1} = \int_0^1 |x(s)|\, ds.$$

If X is a normed space, then an operator $T : X \to X$ is called *totally bounded* if T maps bounded subsets of X into relatively compact subsets of X. An operator $T : X \to X$ which is totally bounded and continuous is called *completely continuous*. Finally, a mapping $T : X \to X$ is called a *contraction on* X if there exists a real constant $\alpha \in (0, 1)$ such that

$$\|T(x) - T(y)\| \leq \alpha \|x - y\| \quad \text{for all } x, y \in X.$$

In the present note we seek solutions of the integral equation (E) that belong to $BM(J, \mathbb{R})$. By a *solution* of (E), we mean a bounded, measurable real-valued function x defined on J which satisfies (E) for all $t \in J$.

In order to state our results, we need the following definition.

Definition: *A mapping $\beta : J \times \mathbb{R} \to \mathbb{R}$ is said to satisfy Carathéodory's conditions if*

(i) $t \to \beta(t,x)$ *is measurable for each* $x \in \mathbb{R}$,

(ii) $t \to \beta(t,x)$ *is continuous almost everywhere for each* $t \in J$.

Moreover, β is called L^1-Carathéodory, if, in addition,

(iii) for each real number $r > 0$, there exists a function $h_r \in L^1(J,\mathbb{R})$ such that

$$|\beta(t,x)| \le h_r(t), \quad \textit{almost everywhere in } J$$

for all $x \in \mathbb{R}$ with $\|x\| \le r$.

The main results in the present note and the main result in [3] share a common ground consisting of the following assumptions (H_0)-(H_4):

(H_0) *The functions $\mu, \theta, \sigma, \eta : J \to J$ are continuous and $\mu(t) \le t$, $\theta(t) \le t$, $\sigma(t) \le t$, $\eta(t) \le t$ for all $t \in J$.*

(H_1) *The function $q : J \to \mathbb{R}$ is bounded and measurable.*

(H_2) *The functions $k, v : J \times J \to \mathbb{R}$ are continuous.*

(H_3) *The function $f : J \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists a function $a \in L^1(J,\mathbb{R})$ such that $a(t) > 0$ a.e. in J and*

$$|f(t,x) - f(t,y)| \le a(t)\,|x-y| \quad \textit{a.e. in } J$$

for all $x, y \in \mathbb{R}$, and $K\,\|a\|_{L^1} < 1$ where $K = \sup\limits_{(t,s)\in J\times J} |k(t,s)|$.

(H_4) *The function g is L^1-Carathéodory.*

On this basis, the main results of this note can be considered as some different versions of the main result in [3], which is stated as Theorem DN, below.

Theorem DN. (Dhage and Ntouyas [3]) *Assume that the conditions* (H_0)-(H_4) *are satisfied. Moreover, suppose that:*

(H_5) *There exists a nondecreasing function $\psi : [0,\infty) \to [0,\infty)$ and a function $\phi \in L^1(J,\mathbb{R})$ such that $\phi(t) > 0$ a.e. in J and*

$$|g(t,x)| \le \phi(t)\psi(|x|) \quad \textit{a.e. in } J$$

for all $x \in \mathbb{R}$ with

$$\int_C^\infty \frac{ds}{s + \psi(s)} > \|\gamma\|_{L^1},$$

where $C = q\int_0^1 |f(s,0)|\,ds$, $\gamma(t) = \max\{Ka(t), V\phi(t)\}$, $t \in J$, $V = \sup\limits_{(t,s)\in J\times J} |v(t,s)|$.

Then the functional integral equation (E) *has a solution on J.*

Our aim, here, is to show that condition (H_5) in Theorem DN can be replaced by some other conditions still yielding existence of solutions to the nonlinear functional integral equation (E). Towards this direction, we consider an initial value problem of the form

$$\begin{aligned} v'(t) &= f(t,v), \quad t \in J \\ v(0) &= v_0 \end{aligned}$$

and recall that a *maximal* solution $\overline{v}$ of the above inital value problem is a solution $\overline{v}$ such that for any other solution v of this initial problem it holds $\overline{v}(t) \geq v(t)$, for $t \in J$. For existence results on maximal solutions we refer to the book by Lakshmikantham and Leela [7].

As conditions (H_0)-(H_5) appear in some of the most commonly considered frames in studying the existence of solutions for integral equations of the type considered here, modifying one of them (namely, (H_5)) might cast some light on a different view of the subject.

Existence of solutions to equations like (E) is often studied by using Krasnoselskii's type fixed point theorems. Theorem DN as well as Theorem 1 are obtained via the following theorem given by Burton and Kirk [2].

Theorem BK. (Burton and Kirk [2]) *Let X be a Banach space and let A, $B : X \to X$ be two operators satisfying:*
(a) A is a contraction, and
(b) B is completely continuous.
Then, either
(i) the operator equation $Ax + Bx = x$ has a solution, or
(ii) the set $\Sigma = \left\{u \in X : \lambda A\left(\frac{u}{\lambda}\right) + \lambda Bu = u,\, 0 < \lambda < 1\right\}$ is unbounded.

Furthermore, for the proof of Theorem 2, we rely on the well-known Leray-Schauder Nonlinear Alternative (see, Granas and Dugundji [5]).

Theorem LSNA. (Granas-Dugundji [5]) *Let E be a Banach space, C a closed convex subset of E and U an open subset of C with $0 \in U$. Suppose that $\overline{F} : \overline{U} \to C$ is a continuous, compact (that is $F(\overline{U})$ is a relatively compact subset of C) map. Then either*
(i) F has a fixed point in $\overline{U}$, or
(ii) there exists a $u \in \partial U$ (the boundary of u in C) and a $\lambda \in (0,1)$ with $u = \lambda F(u)$.

3. MAIN RESULTS

In this section we present two theorems and a corollary, which consist the main results of this note.

Theorem 1. *Assume that hypotheses* (H_0)-(H_4) *hold. In addition, suppose that the following assumption* (H_6) *is satisfied:*

(H_6) (i) *there exists an L^1-Carathéodory function $\psi : J \times \mathbb{R} \to [0, \infty)$ such that*

$$|g(t,x)| \leq \psi(t,|x|), \quad \text{for almost all } t \in J \text{ and all } x \in \mathbb{R},$$

(ii) *$\psi(t,x)$ is nondecreasing in x for almost all $t \in J$.*
(iii) *the problem*

$$\begin{aligned} v'(t) &= \gamma(t)\left[v(t) + \psi(t,x)\right], \quad a.e. \text{ in } J \\ v(0) &= C \end{aligned}$$

where $\gamma(t) = \max\{Ka(t), V\phi(t)\}$*,* $t \in J$*,* $V = \sup\limits_{(t,s)\in J\times J} |v(t,s)|$*, and* $C = \|q\| + K\int_0^1 |f(s,0)|\,ds$*, has a maximal solution on* J*.*

Then the functional integral equation (E) *has a solution on* J*.*

Proof. Let A, $B : BM(J,\mathbb{R}) \to BM(J,\mathbb{R})$ be two mappings defined by

$$(Ax)(t) = q(t) + \int_0^{\mu(t)} k(t,s) f\left(s, x(\theta(s))\right) ds, \quad t \in J,$$

and

$$(Bx)(t) = \int_0^{\sigma(t)} v(t,s) g\left(s, x(\eta(s))\right) ds, \quad t \in J,$$

respectively. In view of the assumptions (H_0)-(H_4), we can easily see that A and B are well defined operators on the Banach space $BM(J,\mathbb{R})$.

We first show that A is a contraction on $BM(J,\mathbb{R})$. To this end, let $x, y \in BM(J,\mathbb{R})$. Then we have for $t \in J$

$$\begin{aligned}
|(Ax)(t) - (Ay)(t)| &= \left| \int_0^{\mu(t)} k(t,s) f\left(s, x(\theta(s))\right) ds - \int_0^{\mu(t)} k(t,s) f\left(s, y(\theta(s))\right) ds \right| \\
&\leq \int_0^{\mu(t)} |k(t,s)| \left| f\left(s, x(\theta(s))\right) ds - f\left(s, y(\theta(s))\right) \right| ds \\
&\leq K \int_0^{\mu(t)} \alpha(s) \left\| x(\theta(s)) - y(\theta(s)) \right\| ds \\
&= K \int_0^{t} \alpha(s) ds \left\| x - y \right\| \\
&\leq K \left\| \alpha \right\|_{L^1} \left\| x - y \right\|
\end{aligned}$$

i.e.

$$|(Ax)(t) - (Ay)(t)| \leq K \left\| \alpha \right\|_{L^1} \left\| x - y \right\|, \quad \text{for all } t \in J.$$

Taking the maximum over t, from the last inequality we obtain

$$\left\| Ax - Ay \right\| \leq K \left\| \alpha \right\|_{L^1} \left\| x - y \right\|.$$

Since from (H_3) we have $K \left\| \alpha \right\|_{L^1} < 1$, it follows that A is a contraction on $BM(J,\mathbb{R})$.

Next we show that B is completely continuous on $BM(J,\mathbb{R})$.

To this end, let S be a bounded subset of $BM(J,\mathbb{R})$ and $(x_n)_{n\in\mathbb{N}}$ be a sequence in S. Then there exists a constant $r > 0$ such that $\|x_n\| \leq r$ for all $n \in \mathbb{N}$. In view of (H_4), for any $n \in \mathbb{N}$ and any $t \in J$, we have

$$\begin{aligned}
|(Bx_n)(t)| &= \left| \int_0^{\sigma(t)} v(t,s) g\left(s, x_n(\eta(s))\right) ds \right| \\
&\leq \int_0^{\sigma(t)} |v(t,s)| \left|g\left(s, x_n(\eta(s))\right)\right| ds \\
&\leq V \int_0^t h_r(s) ds \\
&\leq V \|h_r\|_{L^1}
\end{aligned}$$

which implies that $(Bx_n)_{n\in\mathbb{N}}$ is a uniformly bounded sequence in $BM(J,\mathbb{R})$ with a uniform bound $V\|h_r\|_{L^1}$. Also, for $n \in \mathbb{N}$ and $t, \tau \in J$ we have

$$\begin{aligned}
& |(Bx_n)(t) - (Bx_n)(\tau)| \\
= & \left| \int_0^{\sigma(t)} v(t,s) g\left(s, x_n(\eta(s))\right) ds - \int_0^{\sigma(\tau)} v(\tau,s) g\left(s, x_n(\eta(s))\right) ds \right| \\
\leq & \left| \int_0^{\sigma(t)} v(t,s) g\left(s, x_n(\eta(s))\right) ds - \int_0^{\sigma(t)} v(\tau,s) g\left(s, x_n(\eta(s))\right) ds \right| \\
& + \left| \int_0^{\sigma(t)} v(\tau,s) g\left(s, x_n(\eta(s))\right) ds - \int_0^{\sigma(\tau)} v(\tau,s) g\left(s, x_n(\eta(s))\right) ds \right| \\
\leq & \int_0^{\sigma(t)} |v(t,s) - v(\tau,s)| \left|g\left(s, x_n(\eta(s))\right)\right| ds \\
& + \left| \int_{\sigma(\tau)}^{\sigma(t)} v(\tau,s) g\left(s, x_n(\eta(s))\right) ds \right| \\
\leq & \int_0^{\sigma(t)} |v(t,s) - v(\tau,s)| h_r(s) ds + V \left| \int_{\sigma(\tau)}^{\sigma(t)} \left|g\left(s, x_n(\eta(s))\right)\right| ds \right| \\
\leq & \|h_r\|_{L^1} \int_0^{\sigma(t)} |v(t,s) - v(\tau,s)| ds + V \left| \int_{\sigma(\tau)}^{\sigma(t)} h_r(s) ds \right|
\end{aligned}$$

and

$$\begin{aligned}
(1) \quad & |(Bx_n)(t) - (Bx_n)(\tau)| \\
& \leq \|h_r\|_{L^1} \int_0^{\sigma(t)} |v(t,s) - v(\tau,s)| ds + V |H(t) - H(\tau)|, \quad \text{for } \ t \in J, \ \ n \in \mathbb{N},
\end{aligned}$$

where $H(t) = \int_0^{\sigma(t)} h_r(s) ds$, $t \in J$. In view of the continuity of σ, v and H and the fact that the right hand side of (1) is independent of n, it follows by the Arzelà-Ascoli theorem that $B(S)$ is relatively compact. Hence, B is a completely continuous operator on $BM(J,\mathbb{R})$.

As it has been proved that A is a contraction while B is a completely continuous operator, from Theorem BK it follows that either the operator equation $Ax+Bx =$

x has a solution (i.e., (E) has a solution) or the set

$$\Sigma := \left\{ u \in BM(J,\mathbb{R}) : \lambda A\left(\frac{u}{\lambda}\right) + \lambda Bu = u,\ 0 < \lambda < 1 \right\}$$

is unbounded. We claim that the set Σ is bounded. Towards this direction, we consider an element $u \in \Sigma$. Then for any $\lambda \in (0,1)$ and $t \in J$ we find

$$\begin{aligned} u(t) &= \lambda A\left(\frac{u}{\lambda}\right)(t) + \lambda (Bu)(t) \\ &= \lambda \left[q(t) + \int_0^{\mu(t)} k(t,s) f\left(s, \frac{1}{\lambda} u(\theta(s))\right) ds \right] + \lambda \left[\int_0^{\sigma(t)} v(t,s) g\left(s, u(\eta(s))\right) ds \right] \\ &= \lambda q(t) + \lambda \int_0^{\mu(t)} k(t,s) \left[f\left(s, \frac{1}{\lambda} u(\theta(s))\right) - f(s,0) \right] ds \\ &\quad + \lambda \int_0^{\mu(t)} k(t,s) f(s,0)\, ds + \lambda \int_0^{\sigma(t)} v(t,s) g\left(s, u(\eta(s))\right) ds. \end{aligned}$$

Thus, for any $\lambda \in (0,1)$ we have

$$\begin{aligned} |u(t)| &\le \lambda |q(t)| + \lambda \int_0^{\mu(t)} |k(t,s)| \left| f\left(s, \frac{1}{\lambda} u(\theta(s))\right) - f(s,0) \right| ds \\ &\quad + \lambda \int_0^{\mu(t)} |k(t,s)|\, |f(s,0)|\, ds + \lambda \int_0^{\sigma(t)} |v(t,s)|\, |g(s, u(\eta(s)))|\, ds \\ &\le \|q\| + K \int_0^{\mu(t)} \alpha(s) |u(\theta(s))|\, ds + K \int_0^1 |f(s,0)|\, ds + V \int_0^{\sigma(t)} \psi(s, |u(\eta(s))|) ds \end{aligned}$$

and

$$|u(t)| \le C + K \int_0^t \alpha(s) |u(\theta(s))|\, ds + V \int_0^t \psi(s, |u(\eta(s))|) ds, \quad t \in J, \tag{2}$$

where we have set $C = \|q\| + K \int_0^1 |f(s,0)|\, ds$.

Now let

$$m(t) = \max_{s \in [0,t]} |u(s)|, \quad t \in J.$$

As an immediate consequence of the definition of the function $m : J \to \mathbb{R}$, we note that m is continuous and nondecreasing, and it holds

$$|u(t)| \le m(t), \quad \text{for all } t \in J. \tag{3}$$

By continuity of u it follows that for any $t \in J$ there exists a $t^* \in [0,t]$ such that $m(t) = |u(t^*)|$. Employing (3) and the nondecreasing character of m and ψ, from

(2) we obtain

$$\begin{aligned} m(t) &= |u(t^*)| \le C + K\int_0^{t^*} \alpha(s)\, |u(\theta(s))|\, ds + V\int_0^{t^*} \psi(s, |u(\eta(s))|)ds \\ &\le C + K\int_0^{t} \alpha(s)\, |u(\theta(s))|\, ds + V\int_0^{t} \psi(s, |u(\eta(s))|)ds \\ &\le C + K\int_0^{t} \alpha(s) m(\theta(s))ds + V\int_0^{t} \psi(s, m(\eta(s)))ds \\ &\le C + K\int_0^{t} \alpha(s) m(s)ds + V\int_0^{t} \psi(s, m(s))ds. \end{aligned}$$

i.e., m satisfies the inequality

$$m(t) \le C + K\int_0^{t} \alpha(s) m(s)ds + V\int_0^{t} \psi(s, m(s))ds, \quad t \in J. \tag{4}$$

Let $w : J \to \mathbb{R}$ be defined by

$$w(t) = C + K\int_0^{t} \alpha(s) m(s)ds + V\int_0^{t} \psi(s, m(s))ds, \quad t \in J.$$

By the definition of w we have $w(0) = C$ and

$$m(t) \le w(t), \quad \text{for } t \in J. \tag{5}$$

Differentiating w, in view of (5) and the nondecreasing character of ψ, we find

$$w'(t) = K\alpha(t)m(t) + V\psi(t, m(t)) \le K\alpha(t)w(t) + V\psi(t, w(t)), \quad t \in J.$$

Employing the definitions of γ and C, we see that w satisfies

$$\begin{aligned} w'(t) &\le \gamma(t)\left[w(t) + \psi(t, w(t))\right], \quad t \in J, \\ w(0) &= C \end{aligned}$$

where $\gamma(t) = \max\{K\alpha(t), V\}$, $t \in J$. If $\overline{r}$ is the maximal solution of the intitial value problem

$$\begin{aligned} v'(t) &= \gamma(t)\left[v(t) + \psi(t, x)\right], \quad t \in J \\ v(0) &= C, \end{aligned}$$

from (5) it follows ([7], Theorem 1.10.2) that

$$w(t) \le \overline{r}(t), \quad t \in J.$$

In view of (5), from the last inequality we obtain

$$m(t) \le \overline{r}(t), \quad t \in J.$$

Consequently, by (3) it follows that

$$|u(t)| \le m(t) \le \overline{r}(t) \le \sup_{s \in J} \overline{r}(s) = b, \quad t \in J,$$

which, in turn, implies that $\|u\| \le b$. Since b is independent of u (b depends only on the maximal solution of the initial value problem in (H_6)-(iii)), it follows that the set Σ is bounded. The conclusion of our theorem is straightforward from Theorem BK. ■

It is not difficult to observe that dropping assumption (H_3) results that the operator A in the proof of Theorem 1 fails to be a contraction. In Theorem 2, below, we relax the Lipschitz condition in (H_3) asking for f to be Lipschitzian only at the set $\{(s,0), s \in J\}$ but, as a makeweight, we take f to be L^1-Carathèodory, however, still avoiding (H_5) in Theorem DN by replacing it by the demand for a maximal solution to an initial value problem. For this case, we obtain existence results for (E) by employing the Leray-Schauder NonlinearAlternative.

Theorem 2. *Assume that hypotheses* (H_0)-(H_2) hold. *In addition, suppose that assumptions* $(\widetilde{\mathrm{H}}_3)$,$(\widetilde{\mathrm{H}}_4)$ *and* (H_7) *below are satisfied:*

$(\widetilde{\mathrm{H}}_3)$ *The function f is L^1-Carathèodory and there exists a function* $a : J \to \mathbb{R}^+$ such that

$$|f(t,u) - f(t,0)| \leq \alpha(t)\,|u|\,, \quad \textit{for all } t \in J,\ u \in \mathbb{R}.$$

$(\widetilde{\mathrm{H}}_4)$ *The function g is L^1-Carathèodory.*

(H_7) *There exist a continuous non-decreasing function* $\overline{\psi} : [0,\infty) \to [0,\infty)$ *and a function* $p \in L^1(J, \mathbb{R}^+)$ *such that*

(i) $$|g(t,u)| \leq p(t)\overline{\psi}(|u|) \quad \textit{for all } (t,u) \in J \times \mathbb{R},$$

as well as a constant M^* such that

(ii) $$\frac{M^*}{C + KM^* \int_0^1 a(s)ds + V\overline{\psi}(M^*) \int_0^1 p(s)ds} > 1,$$

where $C = \|q\| + K\int_0^1 |f(s,0)|\, ds$.

Then the functional integral equation (E) *has at least one solution on J.*

Proof. Let $N : BM(J,\mathbb{R}) \to BM(J,\mathbb{R})$ be defined by

$$N(x)(t) = q(t) + \int_0^{\mu(t)} k(t,s) f(s, x(\theta(s)))\, ds + \int_0^{\sigma(t)} v(t,s) g(s, x(\eta(s)))\, ds, \quad t \in J.$$

Here we note that in proving that the oprerator B is completely continuous in the proof of Theorem 1, apart from the fact that f is L^1-Carathèodory, we used only the fact that f satisfies the Lipschitz condition in $(\widetilde{\mathrm{H}}_3)$ than that f satisfies the condition (H_3). (The assumption that f satisfies the Lipschitz condition (H_3) is used only in showing that the operator A is a contraction.) Therefore, as the proof that the operator N is completely continuous follows the same lines as in the proof that the operator B in the proof of Theorem 1 possesses the same property, we skip this part of the proof and concentrate on showing that assumption (H_7) suffices to deactivate the second alternative in Theorem LSNA. In other words, we show that there exists an open set $U \subseteq C(J,E)$ such that for any $\lambda \in (0,1)$ and any x in boundary U it follows that $x \neq \lambda N(x)$.

Assume that for some $\lambda \in (0,1)$ and some $x \in BM(J,\mathbb{R})$, it holds $x = \lambda N(x)$. Then for $t \in J$, we have

$$
\begin{aligned}
|x(t)| &= |\lambda|\,|N(x)(t)| \\
&< |N(x)(t)| \\
&= \left| q(t) + \int_0^{\mu(t)} k(t,s) f\left(s, x(\theta(s))\right) ds + \int_0^{\sigma(t)} v(t,s) g\left(s, x(\eta(s))\right) ds \right| \\
&\leq \|q\| + K \int_0^{\mu(t)} |f(s, x(\theta(s))| \, ds + V \int_0^{\sigma(t)} |g(s, x(\eta(s))| \, ds \\
&\leq \|q\| + K \int_0^{t} |f(s, x(\theta(s)) - f(s,0)| \, ds + K \int_0^{t} |f(s,0)| \, ds \\
&\quad + V \int_0^{t} |g(s, x(\eta(s))| \, ds
\end{aligned}
$$

and

$$
|x(t)| < C + K \int_0^t \alpha(s)\, |x(\theta(s))| \, ds + V \int_0^t p(s) \overline{\psi}(|x(\eta(s)|) ds, \quad t \in J.
$$

Let $m(t) = \max\limits_{s \in [0,t]} |x(s)|$, $t \in J$. Clearly, $|x(t)| \leq m(t)$ holds for any $t \in J$. Due to the continuity of x, there always exists a $t^* \in [0,t]$ such that $m(t) = x(t^*)$. Then from the last inequality we have, for $t \in J$

$$
\begin{aligned}
m(t) &= |x(t^*)| \\
&< C + K \int_0^{t^*} \alpha(s)\, |x(\theta(s))| \, ds + V \int_0^{t^*} p(s) \overline{\psi}(|x(\eta(s)|) ds \\
&\leq C + K \int_0^{t^*} \alpha(s) m(s) ds + V \int_0^{t^*} p(s) \overline{\psi}(m(s)) ds \\
&\leq C + K \int_0^{t} \alpha(s) m(s) ds + V \int_0^{t} p(s) \overline{\psi}(m(s)) ds
\end{aligned}
$$

and

$$
m(t) < C + K \int_0^1 \alpha(s)\, \|m\| \, ds + V \int_0^1 p(s) \overline{\psi}(\|m\|) ds, \quad t \in J.
$$

From the last inequality it follows

$$
\|m\| \leq C + K \int_0^1 \alpha(s)\, \|m\| \, ds + V \int_0^1 p(s) \overline{\psi}(\|m\|) ds
$$

i.e.,

$$
\frac{\|m\|}{C + K \|m\| \int_0^1 \alpha(s) ds + V \overline{\psi}(\|m\|) \int_0^1 p(s) ds} \leq 1.
$$

In view of assumption (H_7) and the last inequality, there exists a constant $M^* > 0$ such that $\|x\| \neq M^*$ for any $x \in BM(J, \mathbb{R})$ with $x = \lambda N(x)$, $\lambda \in (0,1)$. Thus, taking

$$
U = \{x \in BM(J, \mathbb{R}) : \|x\| < M^*\},
$$

it follows that there is no $x \in \partial U$ (i.e., $\|x\| = M^*$) such that $x = \lambda N(x)$ for some $\lambda \in (0,1)$.

The conclusion of our theorem follows immediately by the Leray-Schauder Nonlinear Alternative. ■

It is worth noticing here that the need for $K\|a\|_{L^1} < 1$ is still present in (H_7). For if $K\|a\|_{L^1} \geq 1$ then we can immediately see that the fraction in (H_7)-(ii) is always less than unity.

Looking for the existence of $M^* > 0$ such that the inequality in (H_7)-(ii) is satisfied, it suffices to ask for ψ to be $\lambda-$sublinear with $\lambda < \frac{1-K\|a\|_{L^1}}{V\|p\|_{L^1}}$. Then there exists an $\varepsilon > 0$ such that

$$\lambda < \frac{(1 - K\|a\|_{L^1}) - \varepsilon}{V\|p\|_{L^1}}.$$

Set

$$h(t) = (K\|a\|_{L^1} - 1)\,t + V\|p\|_{L^1}\,\overline{\psi}(t) + C, \quad t \geq 0.$$

Then, as $\overline{\psi}$ is (eventually) $\lambda-$sublinear on $[0,\infty)$, there exists some $t_0 \geq 0$ such that we have $\overline{\psi}(t) \leq \lambda t$ for all $t \geq t_0$. In view of the choice of ε, we obtain for $t \geq t_0$

$$\begin{aligned} h(t) &= (K\|a\|_{L^1} - 1)\,t + V\|p\|_{L^1}\,\overline{\psi}(t) + C \\ &\leq (K\|a\|_{L^1} - 1)\,t + V\|p\|_{L^1}\,\lambda t + C \\ &< (K\|a\|_{L^1} - 1)\,t + V\|p\|_{L^1}\,\frac{(1 - K\|a\|_{L^1}) - \varepsilon}{V\|p\|_{L^1}}t + C \\ &= -\varepsilon t + C \end{aligned}$$

i.e.

$$h(t) < -\varepsilon t + C, \quad \text{for all } t \geq t_0.$$

Thus, for $t > \max\left\{t_0, \frac{C}{\varepsilon}\right\}$, we see that $h(t) < 0$, i.e.,

$$h(t) = (K\|a\|_{L^1} - 1)\,t + (V\|p\|_{L^1})\,\overline{\psi}(t) + C < 0$$

or

$$K\|a\|_{L^1}\,t + V\|p\|_{L^1}\,\overline{\psi}(t) + C < t.$$

This means that we have

$$1 < \frac{t}{C + K\|a\|_{L^1}\,t + V\|p\|_{L^1}\,\overline{\psi}(t)} \quad \text{for all } t > \max\left\{t_0, \frac{C}{\varepsilon}\right\}.$$

Therefore, if ψ is (eventually) sublinear with $\lambda < \frac{1-K\|a\|_{L^1}}{V\|p\|_{L^1}}$, then we can always find a positive real constant M^* such that the fraction in condition (H_7)-(ii) is greater than unity. Hence, we have the following corollary.

Corollary 1. *Assume that hypotheses* (H_0)-(H_2) *and* ($\widetilde{\mathrm{H}}_3$), ($\widetilde{\mathrm{H}}_4$) *hold. In addition, assume that the following assumption* ($\overline{\mathrm{H}}_7$) *is satisfied.*

($\overline{\mathrm{H}}_7$) *There exist a continuous non-decreasing function* $\overline{\psi} : [0,\infty) \to [0,\infty)$ *which is (eventually) sublinear [i.e., satisfies* $\overline{\psi}(s) \leq \lambda s$ *(eventually)*on $[0,\infty)$*] with* $\lambda < \frac{1-K\|a\|_{L^1}}{V\|p\|_{L^1}}$ *and a function* $\overline{p} \in L^1(J,\mathbb{R}^+)$ *such that*

$$|g(t,u)| \leq \overline{p}(t)\overline{\psi}(|u|) \quad \textit{for } (t,u) \in J \times \mathbb{R}.$$

Then the functional integral equation (E) *has a solution on* J.

4. APPLICATIONS

Our aim in this section is to show that the technique employed for the proof of the main results of this note turns out to be effective in obtaining existence results for some more general equations than (E) as well as some other type of equations.

4.1. **A generalization.** There is no difficulty in extending the existence results of Section 2 to the, somehow, more general than (E), integral equation
(E_g)

$$x(t) = q(t) + \int_{\alpha(t)}^{\mu(t)} k(t,s) f\left(s, x(\theta(s))\right) ds + \int_{\beta(t)}^{\sigma(t)} v(t,s) g\left(s, x(\eta(s))\right) ds, \quad t \in J,$$

where $\alpha,\ \beta : J \to J$ are continuous functions.

More precisely, we have the following corollary.

Corollary 2. *Assume that hypotheses* (H_0)-(H_2) *hold. Moreover, suppose that the functions* α, β *are such that*

($\widetilde{\mathrm{H}}_0$) $$0 \leq \alpha(t),\ \beta(t) \leq t \quad \textit{for all } t \in J.$$

(i) *If assumptions* (H_3), (H_4) *and* (H_6) *are satisfied, then the functional integral equation* (E_g) *has a solution on* J.

(ii) *If assumptions* ($\widetilde{\mathrm{H}}_3$), ($\widetilde{\mathrm{H}}_4$) *and* (H_7) *are satisfied, then the functional integral equation* (E_g) *has a solution on* J.

It is worth noticing here, there is no need to impose some kind of ordering on the delays $\alpha, \mu, \beta, \sigma$ (e.g., $\beta(t) \leq \sigma(t)$, e.t.c.).

If the delays α, μ or the delays β, σ do not overspend J, then condition (H_7)-(ii) in Theorem 2 can be refined. More precisely we have the following corollary.

Corollary 3. *Assume that assumptions* (H_0)-(H_2), ($\widetilde{\mathrm{H}}_0$), ($\widetilde{\mathrm{H}}_3$), ($\widetilde{\mathrm{H}}_4$) *and* (H_7)-(i) *hold and that there exist constants* $\widetilde{\mu}$, $\widetilde{\sigma} \in [0,1]$ *such that*

$$\alpha(t),\ \mu(t) \leq \widetilde{\mu},\ \beta(t),\ \sigma(t) \leq \widetilde{\sigma}, \quad \textit{for } t \in J.$$

If there exists a constant $\widetilde{M}$ *such that*

($\widetilde{\mathrm{I}}$) $$\frac{\widetilde{M}}{C^* + K(\widetilde{\mu})\widetilde{M}\int_0^{\widetilde{\mu}} \alpha(s)ds + V(\widetilde{\sigma})\overline{\psi}(\widetilde{M})\int_0^{\widetilde{\sigma}} p(s)ds} > 1,$$

where $C^* = \|q\| + K(\widetilde{\mu})\int_0^{\widetilde{\mu}} |f(s,0)|\, ds$, $K(\widetilde{\mu}) = \sup\limits_{(t,s)\in J\times[0,\widetilde{\mu}]} |k(t,s)|$, and $V(\widetilde{\sigma}) = \sup\limits_{(t,s)\in J\times[0,\widetilde{\sigma}]} |v(t,s)|$, *then* ($\mathrm{E}_g$) *has at least one solution on* J.

Proof. As the proof of this corollary follows exactly the same lines as the proof of Theorem 2, we restrict ourselves here to giving only the inequalities that are different than the corresponding ones in the proof of Theorem 2. Let $\widetilde{N} : BM(J, \mathbb{R}) \to BM(J, \mathbb{R})$ be defined by

$$\widetilde{N}(x)(t) = q(t) + \int_{\alpha(t)}^{\mu(t)} k(t,s) f\left(s, x(\theta(s))\right) ds + \int_{\beta(t)}^{\sigma(t)} v(t,s) g\left(s, x(\eta(s))\right) ds, \quad t \in J.$$

Assume that for some $\lambda \in (0,1)$ and some $x \in BM(J, \mathbb{R})$, it holds $x = \lambda \widetilde{N}(x)$. Then for $t \in J$, we have

$$\begin{aligned}
|x(t)| &= |\lambda| \left| \widetilde{N}(x)(t) \right| \\
&< \left| q(t) + \int_{\alpha(t)}^{\mu(t)} k(t,s) f\left(s, x(\theta(s))\right) ds + \int_{\beta(t)}^{\sigma(t)} v(t,s) g\left(s, x(\eta(s))\right) ds \right| \\
&\le |q(t)| + \left| \int_{\alpha(t)}^{\mu(t)} |k(t,s) f(s, x(\theta(s))| \, ds \right| + \left| \int_{\beta(t)}^{\sigma(t)} |v(t,s) g(s, x(\eta(s))| \, ds \right| \\
&\le \|q\| + \int_0^{\widetilde{\mu}} |k(t,s)| \, |f(s, x(\theta(s))| \, ds + \int_0^{\widetilde{\sigma}} |v(t,s)| \, |g(s, x(\eta(s))| \, ds \\
&\le \|q\| + K(\widetilde{\mu}) \int_0^{\widetilde{\mu}} |f(s, x(\theta(s))| \, ds + V(\widetilde{\sigma}) \int_0^{\widetilde{\sigma}} |g(s, x(\eta(s))| \, ds \\
&\le \|q\| + K(\widetilde{\mu}) \int_0^{\widetilde{\mu}} |f(s, x(\theta(s)) - f(s,0)| \, ds + K(\widetilde{\mu}) \int_0^{\widetilde{\mu}} |f(s,0)| \, ds \\
&\quad + V(\widetilde{\sigma}) \int_0^{\widetilde{\sigma}} |g(s, x(\eta(s))| \, ds
\end{aligned}$$

and

$$|x(t)| < C^* + K(\widetilde{\mu}) \int_0^{\widetilde{\mu}} \alpha(s) \, |x(\theta(s))| \, ds + V(\widetilde{\sigma}) \int_0^{\widetilde{\sigma}} p(s) \overline{\psi}(|x(\eta(s)|) ds, \quad t \in J,$$

Let $m = \max_{t \in [0,1]} |x(t)|$. Clearly, $|x(t)| \le m$ for any $t \in J$. As the last inequality holds for all $t \in J$, we obtain

$$\begin{aligned}
m &< C^* + K(\widetilde{\mu}) \int_0^{\widetilde{\mu}} \alpha(s) \, |x(\theta(s))| \, ds + V(\widetilde{\sigma}) \int_0^{\widetilde{\sigma}} p(s) \overline{\psi}(|x(\eta(s)|) ds \\
&\le C^* + K(\widetilde{\mu}) \int_0^{\widetilde{\mu}} \alpha(s) m ds + V(\widetilde{\sigma}) \int_0^{\widetilde{\sigma}} p(s) \overline{\psi}(m) ds
\end{aligned}$$

and

$$m \le C^* + K(\widetilde{\mu}) m \int_0^{\widetilde{\mu}} \alpha(s) ds + V(\widetilde{\sigma}) \overline{\psi}(m) \int_0^{\widetilde{\sigma}} p(s) ds$$

from which it follows that

$$\frac{m}{C^* + K(\widetilde{\mu}) m \int_0^{\widetilde{\mu}} \alpha(s) ds + V(\widetilde{\sigma}) \overline{\psi}(m) \int_0^{\widetilde{\sigma}} p(s) ds} \le 1.$$

The rest of the proof follows the same lines as that of Theorem 2. ■

The following example illustrates Corollary 3.

Example. Let us consider the integral equation (E_g) with

$$\mu(t)=\frac{t^2}{t+1},\quad \sigma(t)=\frac{t^3}{3(t+1)},\quad t\in J,$$

and

$$\begin{aligned} k(t,s) &= pts=v(t,s),\quad (t,s)\in J^2,\\ f(t,u) &= r(\sin^2 u+1)t\quad (t,u)\in J\times\mathbb{R},\\ g(t,u) &= r(u^2+1)t,\quad (t,u)\in J\times\mathbb{R}, \end{aligned}$$

where p and r are nonnegative real numbers, i.e., consider the nonlinear functional integral equation

$$(\mathrm{E}_0)\quad x(t)=q(t)+\int_{\alpha(t)}^{\frac{t^2}{t+1}} pts\cdot r\left[\sin^2 x(\theta(s))+1\right]sds$$
$$+\int_{\beta(t)}^{\frac{t^3}{3(t+1)}} pts\cdot r\left\{[x(\eta(s))]^2+1\right\}sds,\quad t\in J,$$

where $q:J\to\mathbb{R}$ is a continuous function and the delays $\alpha,\ \beta,\ \theta,\eta:J\to J$ are continuous functions with $\alpha(t),\ \beta(t),\ \theta(t),\eta(t)\le t$ for $t\in J$.

Clearly conditions (H_0)-(H_2), $(\widetilde{\mathrm{H}}_0)$ and $(\widetilde{\mathrm{H}}_4)$ are satisfied. Furthermore, as $u^2\le u$ on $[-1,1]$, we can see that for all $(t,u)\in J\times\mathbb{R}$ we have

$$|f(t,u)-f(t,0)|=\left|r(\sin^2 u+1)t-rt\right|=rt\,|\sin u|^2\le rt\cdot|\sin u|\le rt\cdot|u|$$

i.e., $(\widetilde{\mathrm{H}}_3)$ *holds with* $a(t)=rt,\quad t\in J$. Also, by the definition of g it follows immediately that (H_7)-(i) *is satisfied with* $p(t)=rt,\ \ t\in J$ and $\overline{\psi}(u)=u^2+1$, $u\in\mathbb{R}$. Hence, in order to apply Corollary 3 to the integral equation (E_0), it suffices to establish the existence of a positive constant $\widetilde{M}$ which satisfies the inequality $(\widetilde{\mathrm{I}})$ in Corollary 3.

To this end, first we note that as the functions μ and σ are nondecreasing on J, we find

$$\widetilde{\mu}=\max_{t\in J}\mu(t)=\mu(1)=\frac{1}{2}\quad\text{and}\quad\widetilde{\sigma}=\max_{t\in J}\sigma(t)=\sigma(1)=\frac{1}{6}.$$

Then

$$K(\widetilde{\mu})=\sup_{(t,s)\in J\times[0,\widetilde{\mu}]}|k(t,s)|=\sup_{(t,s)\in J\times[0,\frac{1}{2}]}|pts|=\frac{1}{2}p,$$

$$V(\widetilde{\sigma})=\sup_{(t,s)\in J\times[0,\widetilde{\sigma}]}|v(t,s)|=\sup_{(t,s)\in J\times[0,\frac{1}{6}]}|pts|=\frac{1}{6}p,$$

$$C^*=\|q\|+K(\widetilde{\mu})\int_0^{\widetilde{\mu}}|f(s,0)|\,ds=\|q\|+\frac{1}{2}p\int_0^{\frac{1}{2}}rsds=\|q\|+\frac{1}{2}pr\cdot\frac{1}{8}.$$

i.e.,

$$C^*=\|q\|+\frac{1}{16}pr$$

and

$$K(\widetilde{\mu})\int_0^{\widetilde{\mu}} a(s)ds = \frac{1}{2}p\int_0^{\frac{1}{2}} rsds = \frac{1}{2}pr\cdot\frac{1}{8} = \frac{1}{16}pr$$

$$V(\widetilde{\sigma})\int_0^{\widetilde{\sigma}} p(s)ds = \frac{1}{6}p\int_0^{\frac{1}{6}} rsds = \frac{1}{6}pr\int_0^{\frac{1}{6}} sds = \frac{1}{6}pr\cdot\frac{(1/6)^2}{2} = \frac{1}{2\cdot 6^3}pr.$$

Now let us consider the inequality

$$\frac{x}{C^* + \left[K(\widetilde{\mu})\int_0^{\widetilde{\mu}}\alpha(s)ds\right]x + \left[V(\widetilde{\sigma})\int_0^{\widetilde{\sigma}} p(s)ds\right]\overline{\psi}(x)} > 1, \quad x > 0.$$

Looking for positive solutions of the last inequality, we may study the second order inequality in x

$$rpx^2 - 3^3(2^4 - pr)x + 3^3pr + pr + 2^4\cdot 3^3\,\|q\| < 0, \quad x \in \mathbb{R}.$$

Observe that some neccessary and sufficient conditions so that the last inequality has positive real solutions for x, is that the nonnegative numbers p, r and $\|q\|$ are such that

$$\left[3^3(2^4 - pr)\right]^2 - 4\cdot rp\left(3^3pr + pr + 2^4\cdot 3^3\,\|q\|\right) > 0 \quad \text{and} \quad pr < 2^4.$$

In order that the left of the last two inequalities hold, we find that we must have

$$\begin{aligned} 3^3\left|2^4 - pr\right| &> 2rp\sqrt{3^3+1} \\ \left|2^4 - pr\right| &> \frac{2\sqrt{3^3+1}}{3^3}rp \end{aligned}$$

and so

$$rp < \frac{2^4 3^3}{3^3 + 2\sqrt{3^3+1}} \quad \text{or} \quad rp > \frac{2^4 3^3}{3^3 - 2\sqrt{3^3+1}}.$$

As $\frac{2^4 3^3}{3^3 - 2\sqrt{3^3+1}} > 2^4 > \frac{2^4 3^3}{3^3 + 2\sqrt{3^3+1}}$, we conclude that *if the nonnegative real numbers r, and p are such that $rp < \frac{2^4 3^3}{3^3+2\sqrt{3^3+1}}$, then for any function q with*

$$\|q\| < \frac{\left[3^3(2^4 - pr)\right]^2 - 4r^2p^2\left(3^3+1\right)}{2^6 3^3 rp},$$

the equation (E_0) *has a solution on J.*

It is worth noticing here that, in case that we attempt to apply Theorem 2 to the integral equation (E_0) (i.e., if the upper bounds of the delays μ and σ are not be taken into consideration), then we have $K = p$. Furthermore, as $\frac{2^4 3^3}{3^3+2\sqrt{3^3+1}} > 2^3$, if r and p are chosen so that $rp \in \left(2^3, \frac{2^4 3^3}{3^3+2\sqrt{3^3+1}}\right)$, we find

$$K\int_0^1 a(s)ds = p\int_0^{\frac{1}{2}} rsds = pr\frac{1}{8} > 1,$$

which implies that the fraction in the inequality (H_7)-(ii) is always less than 1. Clearly, this means that Theorem 2 cannot be applied while Corollary 3 yields the existence of a solution of (E_0) on J.

4.2. **A differential equation.** We consider the differential equation

$$x'(t) = p(t) + f\left(t, x(\theta(t))\right) + g\left(t, x(\eta(t))\right), \quad t \in J, \tag{d}$$

where $p : J \to \mathbb{R}$ and $f, g : J \times \mathbb{R} \to \mathbb{R}$ are continuous and the delays $\theta, \eta : J \to J$ are continuous functions with $\theta(t)$, $\eta(t) \le t$, and we seek solutions to the equation (d) on J that satisfy the initial condition

$$x(0) = x_0 \in \mathbb{R}. \tag{c}$$

Integrating (d) on $[0, t]$ for $t \in J$, we see that the inital value problem (d)-(c) is equivalent to the integral equation

$$x(t) = x_0 + \int_0^t p(s)ds + \int_0^t f\left(s, x(\theta(s))\right) ds + \int_0^t g\left(s, x(\eta(s))\right) ds, \quad t \in J. \tag{E_d}$$

It is clear that equation (E_d) is equation (E) for $\mu(t) = t = \sigma(t)$, $k(t, s) = 1 = v(t, s)$ and $q(t) = x_0 + \int_0^t p(s)ds$. Furthermore, it is not difficult to verify that conditions (H_1), (H_3)-(H_6) are satisfied. Finally, as we have $K = 1$, $V = 1$, it follows that condition (H_2) is satisfied, too.

Applying Theorem 1 to the integral equation (E_d), we obtain the following result for the initial problem (d)-(c).

Proposition 1. *Assume that the following conditions are satisfied:*

(I) *The functions $\theta, \eta : J \to J$ are continuous and $\theta(t)$, $\eta(t) \le t$, for $t \in J$.*

(II) *The function $p : J \to \mathbb{R}$ is integrable.*

(III) *The function $f : J \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists a function $a \in L^1(J, \mathbb{R})$ such that $a(t) > 0$ a.e. in J and*

$$|f(t, x) - f(t, y)| \le a(t)\,|x - y|, \quad t \in J$$

for all $x, y \in \mathbb{R}$, and $\|a\|_{L^1} < 1$.

(IV) *The function g satisfies Carathèodory conditions.*

(V) (i) *there exists an L^1-Carathèodory function $\psi : J \times \mathbb{R} \to [0, \infty)$ which is nondecreasing in x for almost all $t \in J$ and such that*

$$|g(t, x)| \le \psi(t, |x|), \quad \textit{for almost all } t \in J \textit{ and all } x \in \mathbb{R}$$

(ii) *the problem*

$$\begin{aligned} v'(t) &= \gamma(t)\left[v(t) + \psi(t, v)\right], \quad t \in J \\ v(0) &= v_0 \end{aligned}$$

has a maximal solution on J, where we have set, $\gamma(t) = \max\{a(t), 1\}$, $t \in J$ and $v_0 = \left\| x_0 + \int_0^t p(s)ds \right\| + \|f(t, 0)\|_{L^1}$.

Then the initial value problem (d)-(c) *has a solution on J.*

In the special case where g is sublinear with $\lambda < \frac{1 - \|a\|_{L^1}}{\|p\|_{L^1}}$, by Corollary 1 we obtain the following proposition.

Proposition 2. *Assume that conditions* (I), (II), *and* (IV) *of Propostion* 1 *are satisfied. If there exist a continuous non-decreasing function $\psi : [0, \infty) \to [0, \infty)$*

which is (eventually) sublinear with $\lambda < \frac{1-\|a\|_{L^1}}{\|p\|_{L^1}}$ *and a function* $\overline{p} \in L^1(J, \mathbb{R}^+)$ *such that*

$$|g(t,u)| \leq \overline{p}(t)\overline{\psi}(|u|) \quad \text{for any } (t,u) \in J \times \mathbb{R},$$

then the initial value problem (d)-(c) *has a solution on* J.

4.3. **A more general equation.** Now let us consider the nonlinear functional integral equation

$$(\mathrm{E}_F) \quad x(t) = q(t) + \int_0^{\mu(t)} k(t,s) f\left(s, x(\theta(s))\right) ds$$
$$+ \int_0^{\lambda(t)} \widehat{k}(t,s) F\left(s, x(\nu(s)), \int_0^{\sigma(s)} k_0(s,v,x(\eta(v)))dv\right) ds, \quad t \in J,$$

where q, μ, k, f, θ, v, σ, *and* η *are as in the previous section and the functions* λ, $\nu : J \to J$, $\widehat{k} : J \times J \to \mathbb{R}$, $k_0 : J \times J \times \mathbb{R} \to \mathbb{R}$ *are continuous. Moreover, we assume that the function* $F : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is* L^1-*Carathéodory, that is,*

(i) $t \to F(t,x,y)$ *is measurable for each* $(x,y) \in \mathbb{R} \times \mathbb{R}$.

(ii) $t \to F(t,x,y)$ *is continuous almost everywhere for each* $t \in J$.

(iii) for each real number $r > 0$, *there exists a function* $\widehat{h}_r \in L^1(J, \mathbb{R})$ *such that*

$$|F(t,x,y)| \leq \widehat{h}_r(t), \quad \text{a. e. in } J \text{ for all } (x,y) \in \mathbb{R} \times \mathbb{R} \text{ with } \|x\|, \|y\| \leq r.$$

We have the following result.

Theorem 3. *Suppose that hypotheses* (H_0), (H_1) and (H_3) *hold along with the hypotheses:*

($\widehat{\mathrm{H}}_0$) *The functions* $\lambda, \nu : J \to J$ *are continuous and* $\lambda(t), \nu(t) \leq t$ *for all* $t \in J$.

($\widehat{\mathrm{H}}_2$) *The functions* k, $\widehat{k}$, $k_0 : J \times J \to \mathbb{R}$ *are continuous.*

($\widehat{\mathrm{H}}_4$) *The function* F *is* L^1-*Carathéodory.*

Furthermore, we assume that

(H_8) *there exist a nondecreasing function* $\widehat{\Omega} : \mathbb{R}^+ \to \mathbb{R}^+$ *and a function* $\widehat{\gamma} : J \to \mathbb{R}^+$ *such that*

$$|F(t,x,y)| \leq \widehat{\gamma}(t)\widehat{\Omega}(|x| + |y|), \quad t \in J, \, (x,y) \in \mathbb{R} \times \mathbb{R}$$

(H_9) *there exist a function* $\widehat{a} : J \to \mathbb{R}^+$ *such that*

$$|k_0(t,s,x)| \leq \widehat{a}(s) \, |x| \, , \quad t,s \in J, \, x \in \mathbb{R}.$$

If the problem

$$\begin{aligned} z'(t) &= \zeta(t)\left[z(t) + \widehat{\Omega}\left(z(t)\right)\right], \quad t \in J \\ w(0) &= C \end{aligned}$$

where $\zeta(t) = \max\left\{K\alpha(t), \widehat{K}\widehat{\gamma}(t)\right\}$ *for* $t \in J$, $\widehat{K} = \sup\limits_{(t,s)\in J\times J}\left|\widehat{k}(t,s)\right|$,*and* $C = \|q\| + K\,\|f(t,0)\|_{L^1}$ *has a maximal solution on* J, *then the functional integral equation* (E_F) *has a solution on* J.

Proof. Following the proof of Theorem 1, we define the operator $A : BM(J,\mathbb{R}) \to BM(J,\mathbb{R})$ by

$$A(x)(t) = q(t) + \int_0^{\mu(t)} k(t,s)f\left(s, x(\theta(s))\right) ds,$$

and the operator $\widehat{B} : BM(J,\mathbb{R}) \to BM(J,\mathbb{R})$ by

$$\widehat{B}(x)(t) = \int_0^{\lambda(t)} \widehat{k}(t,s)F\left(s, x(\nu(s)), \int_0^{\sigma(s)} k_0(s,v,x(\eta(v)))dv\right) ds.$$

The fact that A is a contraction has been proved in Theorem 1. So let us concentrate on showing that $\widehat{B}$ is completely continuous on $BM(J,\mathbb{R})$. As the continuity of $\widehat{B}$ follows easily by the continuity of the delays λ, ν, σ and η, the kernels k_0, $\widehat{k}$ and the function F, what is left to prove is to show that $\widehat{B}$ maps bounded sets of $BM(J,\mathbb{R})$ on relatively compact sets of $BM(J,\mathbb{R})$. In view of the Arzelà-Ascoli theorem, it suffices to prove that $\widehat{B}$ is uniformly bounded and equicontinuous on bounded subsets of $BM(J,\mathbb{R})$.

To this end, let S be a bounded subset of $BM(J,\mathbb{R})$ and $(x_n)_{n\in\mathbb{N}}$ be a sequence in S.

As S is assumed to be bounded, there exists a constant $r > 0$ such that $\|x_n\| \leq r$ for all $n \in \mathbb{N}$. In view of the assumption that F is L^1-Carathéodory, there exists a a function $\widehat{h}_r \in L^1(J,\mathbb{R})$ such that

$$|F(t,x,y)| \leq \widehat{h}_r(t), \quad \text{a. e. in } J \text{ for all } (x,y) \in \mathbb{R}\times\mathbb{R} \text{ with } \|x\|, \|y\| \leq r.$$

Then, for any $t \in J$ and $n \in \mathbb{N}$, it holds

$$\begin{aligned}
\left|\widehat{B}(x_n)(t)\right| &= \left|\int_0^{\lambda(t)} \widehat{k}(t,s)F\left(s, x_n(\nu(s)), \int_0^{\sigma(s)} k_0(s,v,x_n(\eta(v)))dv\right) ds\right| \\
&\leq \int_0^{\lambda(t)} \left|\widehat{k}(t,s)\right| \left|F\left(s, x_n(\nu(s)), \int_0^{\sigma(s)} k_0(s,v,x_n(\eta(v)))dv\right)\right| ds \\
&\leq \widehat{K}\int_0^{\lambda(t)} \widehat{h}_r(s)ds \\
&\leq \widehat{K}\int_0^{t} \widehat{h}_r(s)ds \\
&\leq \widehat{K}\int_0^{1} \widehat{h}_r(s)ds
\end{aligned}$$

i.e.

$$\left|\widehat{B}(x_n)(t)\right| \leq \widehat{K}\left\|\widehat{h}_r\right\|_{L^1}, \quad t \in J, \qquad (n = 1,2,...).$$

This means that $\left(\widehat{B}x_n\right)_{n\in\mathbb{N}}$ is a uniformly bounded sequence in $BM(J,\mathbb{R})$ (with a uniform bound $\widehat{K}\left\|\widehat{h}_r\right\|_{L^1}$), hence the set $\widehat{B}(S)$ is bounded.

Now, we show that the sequence $(\widehat{B}x_n)_{n\in\mathbb{N}}$ is equicontinuous on J. For $n\in\mathbb{N}$ and t, $\tau\in J$ we have

$$
\begin{aligned}
& \left|(\widehat{B}x_n)(t)-(\widehat{B}x_n)(\tau)\right| \\
= & \left|\int_0^{\lambda(t)}\widehat{k}(t,s)F\left(s,x_n(\nu(s)),\int_0^{\sigma(s)}k_0(s,v,x_n(\eta(v)))dv\right)ds\right. \\
& \left.-\int_0^{\lambda(\tau)}\widehat{k}(\tau,s)F\left(s,x_n(\nu(s)),\int_0^{\sigma(s)}k_0(s,v,x_n(\eta(v)))dv\right)ds\right| \\
\leq & \left|\int_0^{\lambda(t)}\left[\widehat{k}(t,s)-\widehat{k}(\tau,s)\right]F\left(s,x_n(\nu(s)),\int_0^{\sigma(s)}k_0(s,v,x_n(\eta(v)))dv\right)ds\right| \\
& +\left|\int_{\lambda(\tau)}^{\lambda(t)}\widehat{k}(\tau,s)F\left(s,x_n(\nu(s)),\int_0^{\sigma(s)}k_0(s,v,x_n(\eta(v)))dv\right)ds\right| \\
\leq & \int_0^{\lambda(t)}\left|\widehat{k}(t,s)-\widehat{k}(\tau,s)\right|\left|F\left(s,x_n(\nu(s)),\int_0^{\sigma(s)}k_0(s,v,x_n(\eta(v)))dv\right)\right|ds \\
& +\left|\int_{\lambda(\tau)}^{\lambda(t)}\left|\widehat{k}(\tau,s)\right|\left|F\left(s,x_n(\nu(s)),\int_0^{\sigma(s)}k_0(s,v,x_n(\eta(v)))dv\right)\right|ds\right| \\
\leq & \int_0^{\lambda(t)}\left|\widehat{k}(t,s)-\widehat{k}(\tau,s)\right|\widehat{h}_r(s)ds+\widehat{K}\left|\int_{\lambda(\tau)}^{\lambda(t)}\widehat{h}_r(s)ds\right|,
\end{aligned}
$$

from which equicontinuity follows.

Finally, we prove that the set

$$
\Sigma=\left\{u\in BM(J,\mathbb{R}):\lambda A\left(\frac{u}{\lambda}\right)+\lambda\widehat{B}(u)=u,\ 0<\lambda<1\right\}
$$

is bounded. To this end, let $u\in\Sigma$. Then for any $\lambda\in(0,1)$ and $t\in J$ we have

$$
\begin{aligned}
u(t) &= \lambda A\left(\frac{u}{\lambda}\right)(t)+\lambda\widehat{B}(u)(t) \\
&= \lambda\left[q(t)+\int_0^{\mu(t)}k(t,s)f\left(s,\frac{1}{\lambda}u(\theta(s))\right)ds\right] \\
&\quad +\lambda\int_0^{\lambda(t)}\widehat{k}(t,s)F\left(s,u(\mu(s)),\int_0^{\sigma(s)}k_0(t,v,u(\eta(v)))dv\right)ds.
\end{aligned}
$$

Since $\lambda \in (0,1)$, in view of (H_8)and (H_9) we have for $t \in J$

$$
\begin{aligned}
& |u(t)| \\
\leq\ & \lambda |q(t)| + \lambda \int_0^{\mu(t)} |k(t,s)| \left| f\left(s, \frac{1}{\lambda} u(\theta(s))\right) - f(s,0)\right| ds + \lambda \int_0^{\mu(t)} |k(t,s)| \, |f(s,0)| \, ds \\
& + \lambda \left| \int_0^{\lambda(t)} \widehat{k}(t,s) F\left(s, u(\nu(s)), \int_0^{\sigma(s)} k_0(s,v,u(\eta(v)))dv\right) ds \right| \\
<\ & \|q\| + K \int_0^{\mu(t)} \alpha(s) |u(\theta(s))| \, ds + K \int_0^1 |f(s,0)| \, ds \\
& + \widehat{K} \int_0^{\lambda(t)} \left| F\left(s, u(\nu(s)), \int_0^{\sigma(s)} k_0(s,v,u(\eta(v)))dv\right)\right| ds \\
\leq\ & C + K \int_0^{\mu(t)} \alpha(s) |u(\theta(s))| \, ds \\
& + \widehat{K} \int_0^{\lambda(t)} \widehat{\gamma}(s) \widehat{\Omega} \left(|u(\nu(s))| + \left| \int_0^{\sigma(s)} k_0(s,v,u(\eta(v)))dv \right| \right) ds \\
\leq\ & C + K \int_0^{\mu(t)} \alpha(s) |u(\theta(s))| \, ds \\
& + \widehat{K} \int_0^{\lambda(t)} \widehat{\gamma}(s) \widehat{\Omega} \left(|u(\nu(s))| + \int_0^{\sigma(s)} |k_0(s,v,u(\eta(v)))| \, dv \right) ds \\
\leq\ & C + K \int_0^{\mu(t)} \alpha(s) |u(\theta(s))| \, ds \\
& + \widehat{K} \int_0^{\lambda(t)} \widehat{\gamma}(s) \widehat{\Omega} \left(|u(\nu(s))| + \int_0^{\sigma(s)} \widehat{a}(v) \, |u(\eta(v))| \, dv \right) ds
\end{aligned}
$$

hence, we take

$$
\begin{aligned}
|u(t)| \leq\ & C + K \int_0^t \alpha(s) |u(\theta(s))| \, ds \\
& + \widehat{K} \int_0^t \widehat{\gamma}(s) \widehat{\Omega} \left(|u(\nu(s))| + \int_0^s \widehat{a}(v) \, |u(\eta(v))| \, dv \right) ds, \quad t \in J.
\end{aligned}
$$

Set

$$
m(t) = \max_{s \in [0,t]} |u(s)|, \quad t \in J.
$$

Then from the last inequality we see that m satisfies

$$
m(t) \leq C + K \int_0^t \alpha(s) m(\theta(s)) ds + \widehat{K} \int_0^t \widehat{\gamma}(s) \widehat{\Omega} \left(m(\nu(s)) + \int_0^s \widehat{a}(v) m(\eta(v)) dv \right) ds
$$

or

$$
m(t) \leq C + K \int_0^t \alpha(s) m(s) ds + \widehat{K} \int_0^t \widehat{\gamma}(s) \widehat{\Omega} \left(m(s) + \int_0^s \widehat{a}(v) m(v) dv \right) ds, \quad t \in J.
$$

Let

$$w(t)=C+K\int_0^t \alpha(s)m(s)ds+\widehat{K}\int_0^t \widehat{\gamma}(s)\widehat{\Omega}\left(m(s)+\int_0^s \widehat{a}(v)m(v)dv\right)ds,\quad t\in J.$$

Then

$$m(0)=C \quad \text{and} \quad m(t)\le w(t),\quad t\in J.$$

Differentiating w we find for $t\in J$

$$\begin{aligned} w'(t) &= K\alpha(t)m(t)+\widehat{K}\widehat{\gamma}(t)\widehat{\Omega}\left(m(t)+\int_0^t \widehat{a}(s)m(s)ds\right) \\ &\le K\alpha(t)m(t)+\widehat{K}\widehat{\gamma}(t)\widehat{\Omega}\left(w(t)+\int_0^t \widehat{a}(s)w(s)ds\right). \end{aligned}$$

Put $\phi(t)=w(t)+\int_0^t \widehat{a}(s)w(s)ds$, $t\in J$. Then we have

$$\phi(0)=w(0)=C,\quad w(t)\le z(t),\quad t\in J,$$

and we obtain for $t\in J$

$$\begin{aligned} \phi'(t) &= w'(t)+\widehat{a}(t)w(t) \\ &\le K\alpha(t)w(t)+\widehat{K}\widehat{\gamma}(t)\widehat{\Omega}\left(\phi(t)\right) \\ &\le K\alpha(t)z(t)+\widehat{K}\widehat{\gamma}(t)\widehat{\Omega}\left(\phi(t)\right) \\ &\le \zeta(t)\left[\phi(t)+\widehat{\Omega}\left(\phi(t)\right)\right]. \end{aligned}$$

Consequently, if $\overline{r}$ is the maximal solution of the intitial value problem

$$\begin{aligned} r'(t) &= \zeta(t)\left[r(t)+\widehat{\Omega}\left(r(t)\right)\right],\quad t\in J \\ r(0) &= C \end{aligned}$$

it follows ([7], Theorem 1.10.2) that

$$\phi(t)\le \overline{r}(t),\quad t\in J.$$

and the conlusion follows by the same arguments as in Theorem 1. ■

Finally, employing the technique engaged in the proof of Theorem 2 we obtain the following result

Theorem 4. *Assume that hypotheses* (H_0), (H_1), ($\widehat{H}_2$), ($\widetilde{H}_3$), (H_8) *and* (H_9) *hold. Moreover, suppose that the function* $\widehat{\Omega}$ *is submultiplicative, i.e., there exists some* $\mu>0$ *such that it holds*

$$\widehat{\Omega}(\mu x)\le \mu\widehat{\Omega}(x),\quad x\in\mathbb{R}^+.$$

If there exists a positive constant $\widehat{M}$ *such that*

$$\frac{\widehat{M}}{C+K\widehat{M}\int_0^1 a(s)ds+K\widehat{\Omega}(\widehat{M})\left[1+\int_0^1 \widehat{\alpha}(\tau)d\tau\right]\int_0^1 \gamma(s)ds}>1,$$

then the functional integral equation (E_F) *has a solution on* J.

Proof. Let $\widehat{N} : BM(J,\mathbb{R}) \to BM(J,\mathbb{R})$ be defined by

$$\widehat{N}(x)(t) = q(t) + \int_0^{\mu(t)} k(t,s) f\left(s, x(\theta(s))\right) ds$$
$$+ \int_0^{\lambda(t)} \widehat{k}(t,s) F\left(s, x(\nu(s)), \int_0^{\sigma(s)} k_0(s,v,x(\eta(v)))dv\right) ds, \quad t \in J,$$

As the proof follows the same arguments as the proof of Theorem 2, we restrict ourselves only to showing that if we assume that for some $\lambda \in (0,1)$ and some $x \in BM(J,\mathbb{R})$, it holds $x = \lambda \widehat{N}(x)$, then we have

$$\frac{\|m\|}{C + K\|m\| \int_0^1 a(s)ds + K\widehat{\Omega}(\|m\|)\left[1 + \int_0^1 \widehat{\alpha}(\tau)d\tau\right] \int_0^1 \gamma(s)ds} \le 1,$$

where $m(t) = \max_{s\in[0,t]} |x(s)|$, $t \in J$.

Indeed, as $\lambda \in (0,1)$ we find for $t \in J$

$$\begin{aligned}
|x(t)| &= \lambda \left|\widehat{N}(x)(t)\right| \\
&< \left|\widehat{N}(x)(t)\right| \\
&= |q(t)| + \int_0^{\mu(t)} |k(t,s)| \left|f\left(s, x(\theta(s))\right)\right| ds \\
&\quad + \int_0^{\lambda(t)} \left|\widehat{k}(t,s)\right| \left|F\left(s, x(\nu(s)), \int_0^{\sigma(s)} k_0(s,v,x(\eta(v)))dv\right)\right| ds \\
&\le \|q\| + K\int_0^{\mu(t)} |f(s,x(\theta(s))| ds \\
&\quad + \widehat{K}\int_0^{\sigma(t)} \left|F\left(s, x(\nu(s)), \int_0^{\sigma(s)} k_0(s,v,x(\eta(v)))dv\right)\right| ds \\
&\le \|q\| + K\int_0^t |f(s,x(\theta(s)) - f(s,0)| ds + K\int_0^t |f(s,0)| ds \\
&\quad + \widehat{K}\int_0^t \left|F\left(s, x(\nu(s)), \int_0^{\sigma(s)} k_0(s,v,x(\eta(v)))dv\right)\right| ds
\end{aligned}$$

and

$$|x(t)| < C + K\int_0^t \alpha(s)\,|x(\theta(s))|\, ds$$
$$+ \widehat{K}\int_0^t \widehat{\gamma}(s)\widehat{\Omega}\left(|x(\mu(s))| + \left|\int_0^{\sigma(s)} k_0(s,v,x(\eta(v)))dv\right|\right) ds, \quad t \in J.$$

Clearly, by the definition of m it holds $|x(t)| \le m(t)$ for any $t \in J$. Due to the continuity of x, there always exists a $t^* \in [0,t]$ such that $m(t) = x(t^*)$. Then from the last inequality we have, for $t \in J$

$$
\begin{aligned}
m(t) &= |x(t^*)| \\
&< C + K\int_0^{t^*} \alpha(s)\,|x(\theta(s))|\,ds \\
&\quad + \widehat{K}\int_0^{t^*} \widehat{\gamma}(s)\widehat{\Omega}\left(|x(\nu(s))| + \left|\int_0^{\sigma(s)} k_0(s,v,x(\eta(v)))dv\right|\right)ds \\
&\leq C + K\int_0^{t} \alpha(s)\,|x(\theta(s))|\,ds \\
&\quad + \widehat{K}\int_0^{t} \widehat{\gamma}(s)\widehat{\Omega}\left(|x(\nu(s))| + \left|\int_0^{\sigma(s)} k_0(s,v,x(\eta(v)))dv\right|\right)ds \\
&\leq C + K\int_0^t \alpha(s)m(s)ds + \widehat{K}\int_0^t \widehat{\gamma}(s)\widehat{\Omega}\left(m(s) + \int_0^t |k_0(s,v,x(\eta(v)))|\,dv\right)ds \\
&\leq C + K\int_0^t \alpha(s)m(s)ds + \widehat{K}\int_0^t \widehat{\gamma}(s)\widehat{\Omega}\left(m(s) + \int_0^t \widehat{a}(v)\,|x(\eta(v))|\,dv\right)ds \\
&\leq C + K\int_0^t \alpha(s)m(s)ds + \widehat{K}\int_0^t \widehat{\gamma}(s)\widehat{\Omega}\left(m(s) + \int_0^t \widehat{a}(v)m(v)dv\right)ds \\
&\leq C + K\int_0^t \alpha(s)\,\|m\|\,ds + \widehat{K}\int_0^t \widehat{\gamma}(s)\widehat{\Omega}\left(\|m\| + \int_0^1 \widehat{a}(v)\,\|m\|\,dv\right)ds \\
&\leq C + K\,\|m\|\int_0^t \alpha(s)ds + \widehat{K}\int_0^t \widehat{\gamma}(s)\widehat{\Omega}\left(\left[1 + \int_0^1 \widehat{a}(v)dv\right]\|m\|\right)ds \\
&\leq C + K\,\|m\|\int_0^t \alpha(s)ds + \widehat{K}\int_0^t \widehat{\gamma}(s)\left[1 + \int_0^1 \widehat{a}(v)dv\right]\widehat{\Omega}\left(\|m\|\right)ds
\end{aligned}
$$

and

$$
m(t) \leq C + K\,\|m\|\int_0^1 \alpha(s)ds + \widehat{K}\widehat{\Omega}\left(\|m\|\right)\left[1 + \int_0^1 \widehat{a}(v)dv\right]\int_0^t \widehat{\gamma}(s)ds, \quad t \in J.
$$

Consequently,

$$
\|m\| \leq C + K\,\|m\|\int_0^1 \alpha(s)ds + \widehat{K}\widehat{\Omega}\left(\|m\|\right)\left[1 + \int_0^t \widehat{a}(v)dv\right]\int_0^t \gamma(s)ds
$$

from which the desired inequality follows. ■

It is not difficult to obtain results similar to those in Theorems 3 and 4 for the equation

$$
\begin{aligned}
x(t) = q(t) &+ \int_{\alpha(t)}^{\mu(t)} k(t,s)f\left(s, x(\theta(s))\right)ds \\
&+ \int_{\beta(t)}^{\lambda(t)} \widehat{k}(t,s)F\left(s, x(\nu(s)), \int_0^{\sigma(s)} k_0(s,v,x(\eta(v)))dv\right)ds, \quad t \in J,
\end{aligned}
$$

where α, β are as in Corollary 2, as well as a result similar to that in Corollary 3 concerning the equation (E_g).

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