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Null Controllability of Some Impulsive Evolution Equation in a Hilbert Space

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Abstract

We shall establish a necessary and sufficient condition under which we have the null controllability of some first order impulsive evolution equation in a Hilbert space.

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1 Introduction

The problem of exact controllability of linear systems represented by infinite conservative systems has been extensively studied by several authors A. Haraux [8], R.Triggiani [16], Z.H. Guan, T.H. Qian, and X.Yu [7], see also the references [1, 2, 6, 10,15]. In the sequel, we shall be concerned with the problem of null controllability of some first order evolution equation subject to impulsive conditions and so we shall derive a necessary and sufficient condition under which null controllability occurs. Actually, we shall establish an equivalence between the null-controllability and some "observability" inequality in somehow more general framework than that proposed by A Haraux [8]. Regarding the literature on the impulsive differential equations we refer the reader to the works of D.D. Bainov and P.S. Simeonov [3, 4] and the references [5, 9,11, 12, 13]. We are going to study the following problem

$$\begin{aligned} y'(t) + Ay(t) &= Bu(t), \quad t \in (0,T) \setminus \{t_k\}_{k\in\sigma_1^m}, & (1)\\ y(0) &= y^0, \\ \Delta y(t_k) &= I_k y(t_k) + D_k v_k, \quad k \in \sigma_1^m, & (1_k) \end{aligned}$$

where the final time T is a positive number, y^0 is an initial condition in a Hilbert space H endowed with an inner product $\langle .,. \rangle_H$, $y(t) : [0,T] \to H$ is a vector function, σ_1^m is a subset of $\mathbb{N}$ given by $\sigma_1^m = \{1, 2, ..., m\}$, and finally, $\{t_k\}_{k\in\sigma_1^m}$ is an increasing sequence of numbers in the open interval $(0,T)$, and $\Delta y(t_k)$ denotes the jump of $y(t)$ at $t = t_k$, *i.e.*,

$$\Delta y(t_k) = y\left(t_k^+\right) - y\left(t_k^-\right)$$

where $y\left(t_k^+\right)$ and $y\left(t_k^-\right)$ represent the right and left limits of $y(t)$ at $t = t_k$ respectively. On the other hand, the operators A, B, I_k, $D_k : H \to H$ are given linear bounded operators. Moreover, we set the following assumptions:

(H1) $A^* = -A$,

(H2) $I_k^* = -I_k$, for every $k \in \sigma_1^m$, and for each $k \in \sigma_1^m$, the operator $\mathcal{I}_k = I_k + I$ is invertible,

(H3) $B^* = B \geq 0$ and there is $d_0 > 0$ such that

$$(Bu, u)_H \leq d_0 \|u\|_H^2, \text{ for all } u \in H,$$

(H4) $D_k^* = D_k \geq 0$, for every $k \in \sigma_1^m$, and for each $k \in \sigma_1^m$ there is $d_k > 0$ such that

$$(D_k u, u)_H \leq d_k \|u\|_H^2, \text{ for all } u \in H.$$

In the sequel we shall designate by h the function

$$h(t) = \left(u(t), \{v_k\}_{k\in\sigma_1^m}\right),$$

where $u(t) \in L^2\left((0,T) \setminus \{t_k\}_{k\in\sigma_1^m}; H\right)$ and

$$\{v_k\}_{k\in\sigma_1^m} \in l^2\left(\sigma_1^m; H\right) \doteqdot \left\{\{v_k\}_{k\in\sigma_1^m}, v_k \in H\right\}.$$

We point out that the space $\mathcal{K}_m = L^2\left(\left(0,T\right)\setminus\left\{t_k\right\}_{k\in\sigma_1^m};H\right)\times l^2\left(\sigma_1^m;H\right)$ is a Hilbert space with respect to the inner product

$$\left(h,\widetilde{h}\right)_{\mathcal{K}_m} = \int_0^T \left(u\left(t\right),\widetilde{u}\left(t\right)\right)_H dt + \sum_{k=1}^m \left(v_k,\widetilde{v}_k\right)_H ,$$

defined for all $h = \left(u\left(t\right),\left\{v_k\right\}_{k=1}^m\right)$ and $\widetilde{h} = \left(\widetilde{u}\left(t\right),\left\{\widetilde{v}_k\right\}_{k=1}^m\right)\in\mathcal{K}_m$.
We shall denote by $\mathcal{B}$ the control operator given by

$$\mathcal{B} = \left(B,\left\{D_k\right\}_{k\in\sigma_1^m}\right)\in\mathcal{L}\left(L^2\left(\left(0,T\right)\setminus\left\{t_k\right\}_{k\in\sigma_1^m};H\right)\times l^2\left(\sigma_1^m;H\right)\right),$$

so that

$$\mathcal{B}h\left(t\right) = \left(Bu\left(t\right),\left\{D_k v_k\right\}_{k\in\sigma_1^m}\right).$$

We have for every $h = (u(t),\{v_k\}_{k=1}^m)\in\mathcal{K}_m$

$$\begin{aligned}
\left(\mathcal{B}h,h\right)_{\mathcal{K}_m} &= \int_0^T \left(Bu\left(t\right),u\left(t\right)\right)_H dt + \sum_{k=1}^m \left(D_k v_k, v_k\right)_H \\
&= \int_0^T \left(u\left(t\right),Bu\left(t\right)\right)_H dt + \sum_{k=1}^m \left(v_k, D_k v_k\right)_H \\
&= \left(h,\mathcal{B}h\right)_{\mathcal{K}_m},
\end{aligned}$$

which shows that $\mathcal{B}^* = \mathcal{B}$, that is, $\mathcal{B}$ is self-adjoint. On the other hand, we have

$$\begin{aligned}
\left(\mathcal{B}h,h\right)_{\mathcal{K}_m} &= \int_0^T \left(Bu\left(t\right),u\left(t\right)\right)_H dt + \sum_{k=1}^m \left(D_k v_k, v_k\right)_H \\
&\le d_0\int_0^T \left\|u\left(t\right)\right\|_H^2 dt + \sum_{k=1}^m d_k\left\|v_k\right\|_H^2 \\
&\le \delta\left\|h\right\|_{\mathcal{K}_m}^2,
\end{aligned}$$

where $\delta = \max\left\{d_0,d_1,...,d_m\right\}$. Thus, the operator is $\mathcal{B}$ bounded in $\mathcal{K}_m$.

Next, we consider the *homogeneous system* associated with (1) :

$$\begin{aligned}
\varphi'\left(t\right) + A\varphi\left(t\right) &= 0, \quad t\in\left(0,T\right)\setminus\left\{t_k\right\}_{k\in\sigma_1^m}, && (2)\\
\varphi\left(0\right) &= \varphi^0, \\
\Delta\varphi\left(t_k\right) &= I_k\varphi\left(t_k\right),\ k\in\sigma_1^m. && (2_k)
\end{aligned}$$

We point out that on each interval $[t_k, t_{k+1})$, for $k = 0, ..., m$, the solution φ is left continuous at each time t_k.
Consider the corresponding homogeneous backward problem :

$$\begin{aligned} -\tilde{\varphi}^{'}(t) + \mathbf{A}\tilde{\varphi}(t) &= 0,, \quad t \in (0,T) \setminus \{t_k\}_{k\in\sigma_1^m}, && (3) \\ \tilde{\varphi}(T) &= \varphi^0, \\ \Delta\tilde{\varphi}(t_{m-(k-1)}) &= -\tilde{I}_{m-(k-1)}\tilde{\varphi}(t^+_{m-(k-1)}), \ k \in \sigma_1^m, && (3_k) \end{aligned}$$

where

$$\mathbf{A} = A^* = -A, \quad \tilde{I}_{m-(k-1)} = I^*_{m-(k-1)} = -I_{m-(k-1)}, \ k \in \sigma_1^m.$$

We observe that the problem (3) on the interval $[t_m, T]$ is equivalent to the classical backward problem

$$\begin{aligned} &-\tilde{\varphi}^{'}(t) + \mathbf{A}\tilde{\varphi}(t) = 0, \, t \in [t_m, T], \\ &\tilde{\varphi}(T) = \varphi^0. \end{aligned}$$

We introduce the following space : $\mathcal{PC}([0,T];H) = \{y, y : [0,T] \to H$ *such that* $y(t)$ *is continuous at* $t \neq t_k$*, and has discontinuities of first kind at* $t = t_k$*, for every* $k \in \sigma_1^m\}$.
Evidently, $\mathcal{PC}([0,T];H)$ is a Banach space with respect to the norm

$$\|y\|_{\mathcal{PC}} = \sup_{t\in(0,T)} \|y(t)\|.$$

On the other hand, we define the subspaces $\mathcal{PLC}$, (respectively, $\mathcal{PRC}$)= $\{y, y \in \mathcal{PC}$ *such that* $y(t)$ *is left (*respectively, *right) continuous at* $t = t_k$, *for every* $k \in \sigma_1^m\}$.

Remark 1 *1) The space $\mathcal{PLC}$, (respectively, $\mathcal{PRC}$) can be identified to a subspace of $\mathcal{K}_m$. That is, to each $y \in \mathcal{PLC}$, (respectively, $\tilde{y} \in \mathcal{PRC}$) is assigned the function* h *(respectively,* $\widetilde{\mathrm{h}}$*) defined by*

$$\mathrm{h}(t) = \Big(y(t), \{y(t_k)\}_{k\in\sigma_1^m}\Big),$$

and

$$\widetilde{\mathrm{h}}(t) = \Big(\tilde{y}(t), \{\tilde{y}(t_k)\}_{k\in\sigma_1^m}\Big).$$

The mapping $y \mapsto \mathrm{h}(t)$ *(respectively,* $\tilde{y} \mapsto \widetilde{\mathrm{h}}$*) is a linear injection.*

2) Let $\widetilde{y} \in \mathcal{PRC}$, the function y can be written as :

$$\widetilde{y}(t) = \begin{cases} \widetilde{y}_{[0]}(t) & \text{if} \quad t \in [t_0, t_1) \\ \widetilde{y}_{[k]}(t) & \text{if} \quad t \in [t_k, t_{k+1}) \\ \widetilde{y}_{[m]}(t) & \text{if} \quad t \in [t_m, T]\,. \end{cases}$$

Next, let $\tau_k = t_k - t_{k-1}$, we define the operator $\mathcal{T} : D(\mathcal{T}) = \mathcal{PRC} \subset \mathcal{K}_m \to \mathcal{K}_m$ by

$$(\mathcal{T}\widetilde{y})(t) = \begin{cases} \widetilde{y}_{[0]}((T-t)\frac{\tau_1}{\tau_{m+1}} + t_0) & \text{if} \quad t \in [t_m, T]\,, \\ \widetilde{y}_{[k]}((t_{m-(k-1)} - t)\frac{\tau_{k+1}}{\tau_{m-(k-1)}} + t_k) & \text{if} \quad t \in \left[t_{m-k}, t_{m-(k-1)}\right), \quad k \in \sigma_1^{m-1}, \\ \widetilde{y}_{[m]}((t_1 - t)\frac{\tau_{m+1}}{\tau_1} + t_m) & \text{if} \quad t \in (0, t_1]\,. \end{cases} \tag{4}$$

We note that the range of $\mathcal{T}$ is exactly $\mathcal{PLC}$. The function $(\mathcal{T}\widetilde{y})(t)$ can be written as follows:

$$(\mathcal{T}\widetilde{y})(t) = \begin{cases} y_{[0]}(t) & \text{if} \quad t \in [t_0, t_1]\,, \\ y_{[k]}(t) & \text{if} \quad t \in (t_k, t_{k+1}]\,, \quad k \in \sigma_1^{m-1}, \\ y_{[m]}(t) & \text{if} \quad t \in (t_m, T]\,. \end{cases}$$

Let $X(t)$ be the resolvent solution of the operator system

$$\begin{aligned} &X'(t) + AX(t) = 0,\ 0 = t_0 < t < t_{m+1} = T,\ t \neq t_k,\ k = 1, 2, ..., m, \\ &X(0) = I, \\ &X(t_k + 0) - X(t_k - 0) = I_k X(t_k),\ k = 1, 2, ..., m, \end{aligned}$$

where $I : H \to H$ is the identity operator. We shall suppose that the operator $\mathcal{I}_k = I_k + I$ has a bounded inverse.

Definition 1 *A function $y \in \mathcal{PC}([0, T]; H)$ is a mild solution to the impulsive problem (1) if the impulsive conditions are satisfied and*

$$\begin{aligned} y(t) \quad = \quad & G(t, 0^+)y^0 + \int_0^t G(t, s)Bu(s)\, ds \\ & + \sum_{0 < t_k \leq t} G(t, t_k)(D_k v_k),\ \textit{for every } t \in (0, T), \end{aligned}$$

where the evolution operator $G(t, s)$ is given by

$$G(t, s) = X(t)X^{-1}(s).$$

It is not hard to check that the operator $G(t,t_k)$ satisfies the operator system

$$\begin{array}{l} G^{'}(t,t_k)+AG(t,t_k)=0,\ \ t\in[t_k,t_{k+1}),\ k\in\sigma_0^m, \\ G(t_k,t_k)=I, \\ G(t_{k+1}^{+},t_k)-G(t_{k+1}^{-},t_k)=I_{k+1}G(t_{k+1}^{-},t_k). \end{array}$$

It is well known that (1) has a unique solution y such that

$$y\in\mathcal{PLC}\left([0,T];H\right)\cap C^1\left([0,T]\setminus\{t_k\}_{k\in\sigma_1^m};H\right).$$

Now, we define the concept of mild solution for the backward impulsive system (3) associated with system (2).

Definition 2 *We say that $\tilde{\varphi}\in\mathcal{PRC}\left([0,T];H\right)$ is a mild solution for the backward impulsive system (3) if $\mathcal{T}\tilde{\varphi}$ is a mild solution for the homogeneous impulsive system (2).*

Let us introduce the notion of the null controllability of the initial state as follows:

Definition 3 *We say that the initial state $y^0\in H$ is null controllable at time T, if there is a control function $\mathrm{h}\in\mathcal{K}_m$ for which the solution y of system (1) satisfies $y\left(T\right)=0$.*

2 Main Results

First we begin by the following lemma.

Lemma 1 *Assume that $\xi\left(t\right),\zeta\left(t\right)\in L^1\left([0,T];H\right)$ and $\left\{\xi_k\right\}_{k=1}^m,\left\{\zeta_k\right\}_{k=1}^m\in l^1(\sigma_1^m,H)$. Then, for every vector functions*

$$\gamma\left(t\right)\in\mathcal{PLC}\left([0,T];H\right)\cap C^1\left([0,T]\setminus\{t_k\}_{k\in\sigma_1^m};H\right)$$

and

$$\eta\left(t\right)\in\mathcal{PRC}\left([0,T];H\right)\cap C^1\left([0,T]\setminus\{t_k\}_{k\in\sigma_1^m};H\right)$$

satisfying the problem

$$\begin{array}{rcl} \dfrac{d}{dt}\langle\gamma\left(t\right),\eta\left(t\right)\rangle & = & \langle\xi\left(t\right),\zeta\left(t\right)\rangle,\quad t\neq t_k,\ \textit{for}\ k\in\sigma_1^m, \\ \Delta\langle\gamma\left(t_k\right),\eta\left(t_k\right)\rangle & = & \langle\Delta\gamma\left(t_k\right),\eta\left(t_k\right)\rangle+\langle\gamma\left(t_k\right),\Delta\eta\left(t_k\right)\rangle=\langle\xi_k,\zeta_k\rangle,\ k\in\sigma_1^m, \end{array}$$

we have the following identity

$$\begin{aligned} \langle \gamma(t), \eta(t) \rangle |_0^T &= \langle \gamma(T), \eta(T) \rangle - \langle \gamma(0), \eta(0) \rangle \\ &= \int_0^T \langle \xi(t), \zeta(t) \rangle dt + \sum_{k=1}^{m} \langle \xi_k, \zeta_k \rangle . \end{aligned} \tag{5}$$

Proof. It is straightforward. $\square$

We also need the following Lemmas.

Lemma 2 [14] *If $\mathcal{B} \in \mathcal{L}(\mathcal{K}_m)$ is self-adjoint and nonnegative, then*

$$\|\mathcal{B}\mathrm{h}\| \leq \|\mathcal{B}\|^{1/2} (\mathcal{B}\mathrm{h}, \mathrm{h})_{\mathcal{K}_m}^{1/2}, \quad \mathrm{h} \in \mathcal{K}_m.$$

Lemma 3 *If $\tau_{k+1} = \tau_{m-(k-1)}$, $k \in \sigma_0^{m-1}$, then for the mild solution $\widetilde{\varphi}$ of (3), the identity holds :*

$$\int_0^T |B\widetilde{\varphi}|_H^2 \, dt + \sum_{k=1}^{m} \left| D_k \widetilde{\varphi}\left(t_k^+\right) \right|_H^2 = \int_0^T |B\varphi|_H^2 \, dt + \sum_{k=1}^{m} \left| D_k \varphi\left(t_{m-(k-1)}\right) \right|_H^2 . \tag{6}$$

Proof. For each $k \in \sigma_0^m$, using the change of variable $t \to (t_{m-(k-1)} - t)\frac{\tau_{k+1}}{\tau_{m-(k-1)}} + t_k$ we have

$$\begin{aligned} & \int_{t_{m-k}}^{t_{m-(k-1)}} (B\varphi_{[m-k]}(t), B\varphi_{[m-k]}(t)) dt \\ = & \int_{t_{m-k}}^{t_{m-(k-1)}} (B\widetilde{\varphi}_{[k]}((t_{m-(k-1)} - t)\frac{\tau_{k+1}}{\tau_{m-(k-1)}} + t_k), B\widetilde{\varphi}_{[k]}((t_{m-(k-1)} - t)\frac{\tau_{k+1}}{\tau_{m-(k-1)}} + t_k)) dt \\ = & \frac{-\tau_{m-(k-1)}}{\tau_{k+1}} \int_{t_{k+1}}^{t_k} (B\widetilde{\varphi}_{[k]}(s), B\widetilde{\varphi}_{[k]}(s)) ds \\ = & \int_{t_k}^{t_{k+1}} (B\widetilde{\varphi}_{[k]}(s), B\widetilde{\varphi}_{[k]}(s)) ds. \end{aligned}$$

Summing up with respect to k, we get

$$\sum_{k=0}^{m} \int_{t_{m-k}}^{t_{m-(k-1)}} (B\varphi_{[m-k]}((t)), B\varphi_{[m-k]}(t)) dt = \sum_{k=0}^{m} \int_{t_k}^{t_{k+1}} (B\widetilde{\varphi}_{[k]}(t), B\widetilde{\varphi}_{[k]}(t)) dt.$$

Thus, we obtain

$$\int_0^T |B\widetilde{\varphi}|_H^2 \, dt = \int_0^T |B\varphi|_H^2 \, dt.$$

On the other hand, by virtue of the definition of the function $\widetilde{\varphi}$ we get

$$\varphi\left(t_{m-k}\right) = \widetilde{\varphi}\left(t_{k+1}\right), \quad k \in \sigma_0^{m-1}.$$

Also, we have

$$\varphi\left(t_{m-(k-1)}\right) = \widetilde{\varphi}\left(t_k\right), \quad k \in \sigma_1^m,$$

and

$$\widetilde{\varphi}\left(t_{m-k}\right) = \varphi\left(t_{k+1}\right), \quad k \in \sigma_0^{m-1}.$$

This implies that

$$\begin{aligned}
\sum_{k=1}^{m} |D_k \widetilde{\varphi}\left(t_k\right)|_H^2 &= \sum_{k=0}^{m-1} \langle D_{m-k} \widetilde{\varphi}\left(t_{m-k}\right), D_{m-k} \widetilde{\varphi}\left(t_{m-k}\right)\rangle_H \\
&= \sum_{k=0}^{m-1} \langle D_{m-k} \varphi\left(t_{k+1}\right), D_{m-k} \varphi\left(t_{k+1}\right)\rangle_H \\
&= \sum_{l=1}^{m} \left\langle D_l \varphi\left(t_{m-(l-1)}\right), D_l \varphi\left(t_{m-(l-1)}\right)\right\rangle_H \\
&= \sum_{k=1}^{m} \left\langle D_k \varphi\left(t_{m-(k-1)}\right), D_k \varphi\left(t_{m-(k-1)}\right)\right\rangle_H \\
&= \sum_{k=1}^{m} \left| D_k \varphi\left(t_{m-(k-1)}\right)\right|_H^2,
\end{aligned}$$

which gives (6). $\square$

Corollary 1 *If $\tau_{k+1} = \tau_{m-(k-1)}$, for $k \in \sigma_0^{m-1}$, and B, D_k are nonnegative in H, then the following holds:*

$$\begin{aligned}
& \int_0^T \langle B\widetilde{\varphi}(t), \widetilde{\varphi}(t)\rangle dt + \sum_{k=1}^{k=m} \langle D_k \widetilde{\varphi}(t_k), \widetilde{\varphi}(t_k)\rangle \\
= & \int_0^T \langle B\varphi(t), \varphi(t)\rangle dt + \sum_{k=1}^{k=m} \langle D_k \varphi(t_{m-(k-1)}), \varphi(t_{m-(k-1)})\rangle.
\end{aligned}$$

Proof. This follows immediately from Lemma 3 if we substitute B by $B^{\frac{1}{2}}$, and D_k by $D_k^{\frac{1}{2}}$. $\square$

Now, we state and establish the following Theorem.

Theorem 1 *Let $y^0 \in H$ be a given initial state for the system (1), then y^0 is null controllable at time T if and only if there is a positive constant C such that*

$$\left|\langle y^0, \widetilde{\varphi}^0\rangle_H\right| \leq C\left\{\int_0^T |B\varphi|_H^2\,dt + \sum_{k=1}^m \left|D_k\varphi\left(t_{m-(k-1)}\right)\right|_H^2\right\}^{1/2},\ \forall \widetilde{\varphi}^0 \in H, \tag{7}$$

where $\varphi \in \mathcal{PLC}\left([0,T];H\right)$ is the unique mild solution to (2) *with* $\varphi\left(T\right) = \widetilde{\varphi}^0$.

Proof. It suffices to prove this Theorem for the special case $\tau_{k+1} = \tau_{m-(k-1)}$, for $k \in \sigma_0^{m-1}$, because the norm $|||.||| \doteqdot \left\{\sum_{k=0}^m \frac{\tau_{m-(k-1)}}{\tau_{k+1}} \int_{t_k}^{t_{k+1}} |.|_H^2\,dt\right\}^{1/2}$ is equivalent to the usual norm of $L^2\left([0,T];H\right)$.

We shall proceed in several steps.

Step 1: Let y and $\widetilde{\varphi}$ be strong solutions to (1) and (3), respectively. Then, for $t \neq t_k$, $k \in \sigma_1^m$, we have

$$\begin{aligned} \frac{d}{dt}\langle y\left(t\right), \widetilde{\varphi}\left(t\right)\rangle &= \langle y\left(t\right), \widetilde{\varphi}'\left(t\right)\rangle + \langle y'\left(t\right), \widetilde{\varphi}\left(t\right)\rangle \\ &= \langle y\left(t\right), -A\widetilde{\varphi}\left(t\right)\rangle + \langle -Ay\left(t\right) + Bu\left(t\right), \widetilde{\varphi}\left(t\right)\rangle \\ &= \langle y\left(t\right), -A\widetilde{\varphi}\left(t\right)\rangle + \langle -Ay\left(t\right), \widetilde{\varphi}\left(t\right)\rangle + \langle Bu\left(t\right), \widetilde{\varphi}\left(t\right)\rangle \\ &= \langle Bu\left(t\right), \widetilde{\varphi}\left(t\right)\rangle. \end{aligned} \tag{8}$$

Multiplying equation (3_k) in (3) from the left by $y\left(t_{m-(k-1)}\right)$ the solution of (1), and multiplying equation (1_k) in (1) from the right by $\widetilde{\varphi}\left(t_k\right)$ the solution of (3), and finally adding memberwise we get

$$\begin{aligned} \Delta\langle y(t), \widetilde{\varphi}(t)\rangle_{|t=t_k} &= \langle y(t_k), \Delta\widetilde{\varphi}(t_k)\rangle + \langle \Delta y(t_k), \widetilde{\varphi}(t_k)\rangle \\ &= \langle y(t_k), I_k\widetilde{\varphi}(t_k)\rangle + \langle I_k y(t_k) + D_k v_k, \widetilde{\varphi}(t_k)\rangle \\ &= \langle y(t_k), I_k\widetilde{\varphi}(t_k)\rangle + \langle I_k y(t_k), \widetilde{\varphi}(t_k)\rangle + \langle D_k v_k, \widetilde{\varphi}(t_k)\rangle \\ &= \langle D_k v_k, \widetilde{\varphi}(t_k)\rangle. \end{aligned} \tag{9}$$

Setting $\gamma(t) = y(t)$, $\eta(t) = \widetilde{\varphi}(t)$, $\xi(t) = Bu(t)$, $\zeta(t) = \widetilde{\varphi}(t)$, $\xi_k = D_k v_k$, $\zeta_k = \widetilde{\varphi}(t_k)$, then equations (5), (8) and (9) give

$$\langle y(T), \widetilde{\varphi}(T)\rangle - \langle y(0), \widetilde{\varphi}(0)\rangle = \int_0^T \langle Bu(t), \widetilde{\varphi}(t)\rangle dt + \sum_{k=1}^{k=m} \langle D_k v_k, \widetilde{\varphi}(t_k)\rangle. \quad (10)$$

Since $\mathcal{B}$ is bounded, self-adjoint and $\mathcal{B} \geq 0$, then by density the latter identity is still valid for mild solutions y of (1). Identity (10) can be written as follows

$$\langle y(T), \widetilde{\varphi}(T)\rangle - \langle y(0), \widetilde{\varphi}(0)\rangle = \int_0^T \langle u(t), B\widetilde{\varphi}(t)\rangle dt + \sum_{k=1}^{k=m} \langle v_k, D_k\widetilde{\varphi}(t_k)\rangle. \quad (11)$$

Next, if there is a certain $h(t) \in \mathcal{K}_m$ such that the mild solution of (1) with $y(0) = y^0$ satisfies $y(T) = 0$, then

$$-\langle y(0), \widetilde{\varphi}(0)\rangle = \int_0^T \langle u(t), B\widetilde{\varphi}(t)\rangle dt + \sum_{k=1}^{k=m} \langle v_k, D_k\widetilde{\varphi}(t_k)\rangle,$$

and so by Cauchy-Schwarz Inequality we obtain

$$\begin{aligned} |\langle y(0), \widetilde{\varphi}(0)\rangle_H| &\leq \left\{ \int_0^T \|u(t)\|_H^2 \, dt + \sum_{k=1}^{k=m} \|v_k\|_H^2 \right\}^{1/2} \quad (12) \\ &\times \left\{ \int_0^T \|B\widetilde{\varphi}(t)\|_H^2 \, dt + \sum_{k=1}^{k=m} \|D_k\widetilde{\varphi}((t_k)\|_H^2 \right\}^{1/2}. \end{aligned}$$

Using Lemma 3, and equation (12) we have

$$\begin{aligned} |\langle y(0), \widetilde{\varphi}(0)\rangle_H| &\leq \left\{ \int_0^T \|u(t)\|_H^2 \, dt + \sum_{k=1}^{k=m} \|v_k\|_H^2 \right\}^{1/2} \\ &\times \left\{ \int_0^T \|B\varphi(t)\|_H^2 \, dt + \sum_{k=1}^{k=m} \left\|D_k\varphi(t_{m-(k-1)})\right\|_H^2 \right\}^{1/2}. \end{aligned}$$

Setting

$$C = \|h(t)\|_{\mathcal{K}_m} = \left\{ \int_0^T \|u(t)\|_H^2 \, dt + \sum_{k=1}^{k=m} \|v_k\|_H^2 \right\}^{1/2}$$

we find that

$$|(\langle y(0),\widetilde{\varphi}(0)\rangle_H| \leq C\left\{\int_0^T \|B\varphi(t)\|_H^2\,dt+\sum_{k=1}^{k=m}\left\|D_k\varphi(t_{m-(k-1)})\right\|_H^2\right\}^{1/2}.$$

This shows the necessary condition of the Theorem.

Step 2: To prove the sufficiency we need the following result when $\mathcal{B}\geq\alpha>0$.

Claim 1 *Assume that there is $\alpha>0$ such that*

$$\left\{\int_0^T \|Bu(t)\|_H^2\,dt+\sum_{k=1}^{k=m}\|D_kv_k\|_H^2\right\}\geq\alpha\left\{\int_0^T \|u(t)\|_H^2\,dt+\sum_{k=1}^{k=m}\|v_k\|_H^2\right\}$$

then, for every $y^0\in H$ there is $\varphi^0\in H$ such that the mild solution of (1) with

$$\mathrm{h}(t)=(\widetilde{\varphi}(t),\widetilde{\varphi}(t_1),..,\widetilde{\varphi}(t_k)..,\widetilde{\varphi}(t_m))\in\mathcal{K}_m\ \textit{ and } y(0)=y^0$$

satisfies $y(T)=0$.

To prove this Claim, we consider for every $z\in H$ the solution φ of (2) satisfying $\varphi(T)=z$ and the unique mild solution y to the problem

$$\begin{aligned} y'(t)+Ay(t) &= B\widetilde{\varphi}(t),t\in(0,T)\Big\backslash\{t_k\}_{k\in\sigma_1^m}, \\ \Delta y(t_k) &= I_ky(t_k)+D_k\widetilde{\varphi}(t_k), \\ y(T) &= 0. \end{aligned}$$

Next, we introduce a bounded linear operator $\Lambda: H\to H$ defined by

$$\Lambda z=-y(0).$$

According to formula (11) and the Corollary 1 we have

$$\begin{aligned} |\langle\Lambda z,z\rangle| &= |-\langle y(0),\widetilde{\varphi}(0)\rangle|=\left|\int_0^T\langle B\widetilde{\varphi}(t),\widetilde{\varphi}(t)\rangle dt+\sum_{k=1}^{k=m}\langle D_k\widetilde{\varphi}(t_k),\widetilde{\varphi}(t_k)\rangle\right| \\ &= \left|\int_0^T\langle B\varphi(t),\varphi(t)\rangle dt+\sum_{k=1}^{k=m}\langle D_k\varphi(t_{m-(k-1)}),\varphi(t_{m-(k-1)})\rangle\right| \\ &\leq \varsigma\left\{\int_0^T\|\varphi(t)\|^2\,dt+\sum_{k=1}^{k=m}\|\varphi(t_k)\|^2\right\}, \end{aligned}$$

where

$$\varsigma = \sup_{k\in\sigma_0^m} \{d_k\} < \infty.$$

We have

$$\int_0^T \|\varphi(t)\|^2 dt = \int_0^{t_1} \|\varphi(t)\|^2 dt + \int_{t_1}^{t_2} \|\varphi(t)\|^2 dt + ... + \int_{t_m}^{T} \|\varphi(t)\|^2 dt.$$

Since there is no impulse in the interval $[t_k, t_{k+1})$ we have

$$\|\varphi(t)\| = \left\|\varphi(t_k^+)\right\|, \text{ for every } t \in [t_k, t_{k+1}), k \in \sigma_0^m,$$

$$\left\|\varphi(t_{k+1}^-)\right\| = \left\|\varphi(t_{k-1}^+)\right\|, \quad k \in \sigma_0^m. \tag{13}$$

Therefore, there are $\tau_{k+1} = t_{k+1} - t_k > 0$, $k \in \sigma_0^m$ such that

$$\textstyle\int_{t_k}^{t_{k+1}} \|\varphi(t)\|^2 dt \le \rho_k \left\|\varphi(t_k^+)\right\|^2 = \tau_{k+1} \left\|I_k\varphi(t_k^-) + \varphi(t_k^-)\right\|^2, \quad k \in \sigma_1^m. \tag{14}$$

On the other hand, the continuity of I_k implies that

$$\left\|\varphi(t_k^+)\right\|^2 = \left\|(I_k + I)\varphi(t_k^-)\right\|^2 \le (1 + L(I_k))^2 \left\|\varphi(t_k^-)\right\|^2, \quad k \in \sigma_1^m. \tag{15}$$

It follows from (14) and (15) that

$$\textstyle\int_{t_k}^{t_{k+1}} \|\varphi(t)\|^2 dt \le \tau_{k+1}(1 + L(I_k))^2 \left\|\varphi(t_k^-)\right\|^2, \quad k \in \sigma_1^m. \tag{16}$$

Since m is finite, and due to (13),(16), then there is a constant $0 < \mu < \infty$ such that $\langle \Lambda z, z\rangle \le \mu \|z\|^2$, and thus, Λ is bounded.

Now, as $\mathcal{B}$ is nonnegative in $\mathcal{K}_m$, we have

$$\|\mathcal{B}\xi(t)\| \ge \alpha \{(\xi(t), \xi(t))_{\mathcal{K}_m}\}^{1/2}$$

for all $\xi \in \mathcal{K}_m$; thus, by virtue of Lemma 2, we have

$$\begin{aligned} &\left\{\int_0^T (Bu(t), u(t))_H dt + \sum_{k=1}^{k=m} (D_k v_k, v_k)_H\right\} \\ \ge\ & \alpha \|\mathcal{B}\| \left\{\int_0^T \|u(t)\|_H^2 dt + \sum_{k=1}^{k=m} \|v_k\|_H^2\right\}. \end{aligned} \tag{17}$$

It follows from (11), (17) and Corollary 1 that

$$\begin{aligned}
\langle \Lambda z, z\rangle &= -\langle y(0), \widetilde{\varphi}(0)\rangle \\
&= \int_0^T \langle B\varphi(t), \varphi(t)\rangle dt + \sum_{k=1}^{k=m} \langle D_k \varphi(t_{m-(k-1)}), \varphi(t_{m-(k-1)})\rangle \\
&\geq \alpha \|\mathcal{B}\| \left\{ \int_0^T \|\varphi(t)\|^2 dt + \sum_{k=1}^{k=m} \|\varphi(t_k)\|^2 \right\} \\
&\geq \alpha \|\mathcal{B}\| \int_0^{t_1} \|\varphi(t)\|^2 dt = \|\mathcal{B}\| \alpha t_1 \|z\|^2 = \theta \|z\|^2 ,
\end{aligned}$$

because there is no impulse before time t_1. Therefore, Λ is coercive on H. To show that there is a bijection from H onto H, it suffices to prove that $\Lambda + I$ is a bijection from H onto H. Clearly, $\Lambda + I$ is injective since

$$\langle \Lambda z + z, z\rangle = \langle \Lambda z, z\rangle + \langle z, z\rangle \geq (\theta + 1) \|z\|^2 .$$

On the other hand, let $y^0 \in H$, as the form $a(f,g) + \langle f, g\rangle = \langle \Lambda f, g\rangle + \langle f, g\rangle$ is symmetric and coercive, then, by virtue of Lax-Milgram Theorem, there is an element $f \in H$ such that

$$a(f,g) + \langle f, g\rangle = \langle y^0, g\rangle, \text{ for all } g \in H.$$

This implies that $\Lambda(H) = H$. Thus, for every $y^0 \in H$, there is a unique $z \in H$ such that $\Lambda(z) = -y^0$, which completes the proof of Claim 1.

Step 3: Assume that $B, D_k \geq 0$, then $\mathcal{B} \geq 0$,

$$\widetilde{B}^2 = B, \widetilde{D}_k^2 = D_k.$$

We define for $\varepsilon > 0$,

$$\beta^\varepsilon \doteqdot \widetilde{B}^2 + \varepsilon I,$$

$$\delta_k^\varepsilon \doteqdot \widetilde{D}_k^2 + \varepsilon I,$$

and

$$\mathcal{B}^\varepsilon \doteqdot (\beta^\varepsilon; \delta_1^\varepsilon, .., \delta_m^\varepsilon) = (\widetilde{B}^2 + \varepsilon I; \widetilde{D}_1^2 + \varepsilon I, .., \widetilde{D}_m^2 + \varepsilon I).$$

According to Claim 1, there is $\tilde{\varphi}^{0,\varepsilon} \in H$ such that the mild solution y_ε of (1) with $y_\varepsilon(0) = y^0$ satisfies $y_\varepsilon(T) = 0$; where $\mathcal{B}(h)$ has been replaced by

$$\mathcal{B}^\varepsilon(\widetilde{\varphi}(t), \widetilde{\varphi}(t_1), .., \widetilde{\varphi}(t_k) .., \widetilde{\varphi}(t_m)) \in \mathcal{K}_m.$$

We obtain from (11) and Corollary 1

$$-\langle y(0),\widetilde{\varphi}_\varepsilon(0)\rangle=\int_0^T\langle\beta_\varepsilon^\varepsilon\widetilde{\varphi}(t),\widetilde{\varphi}_\varepsilon(t)\rangle dt+\sum_{k=1}^{k=m}\langle\delta_k^\varepsilon\widetilde{\varphi}_\varepsilon(t_k),\widetilde{\varphi}_\varepsilon(t_k)\rangle, \tag{18}$$

and (7) gives

$$-\langle y(0),\widetilde{\varphi}_\varepsilon(0)\rangle\leq C\left\{\int_0^T\langle\widetilde{B}^2\varphi_\varepsilon(t),\varphi_\varepsilon(t)\rangle dt+\sum_{k=1}^{k=m}\langle\widetilde{D}_k^2\varphi_\varepsilon(t_{m-(k-1)}),\varphi_\varepsilon(t_{m-(k-1)})\rangle\right\}^{1/2}. \tag{19}$$

Whence,

$$-\langle y(0),\widetilde{\varphi}_\varepsilon(0)\rangle\leq C\left\{\int_0^T\langle\beta^\varepsilon\varphi_\varepsilon(t),\varphi_\varepsilon(t)\rangle dt+\sum_{k=1}^{k=m}\langle\delta_k^\varepsilon\varphi_\varepsilon(t_{m-(k-1)}),\varphi_\varepsilon(t_{m-(k-1)})\rangle\right\}^{1/2}. \tag{20}$$

It follows at once from (18), (19) and (20) that

$$\begin{aligned}&\varepsilon\left\{\int_0^T\|\varphi_\varepsilon(t)\|^2\,dt+\sum_{k=1}^{k=m}\|\varphi_\varepsilon(t_k)\|^2\right\}\\&\qquad+\int_0^T\langle\widetilde{B}\varphi_\varepsilon(t),\widetilde{B}\varphi_\varepsilon(t)\rangle dt+\sum_{k=1}^{k=m}\langle\widetilde{D}_k\varphi_\varepsilon(t_{m-(k-1)}),\widetilde{D}_k\varphi_\varepsilon(t_{m-(k-1)})\rangle\\&=\int_0^T(\beta^\varepsilon\varphi_\varepsilon(t),\varphi_\varepsilon(t))dt+\sum_{k=1}^{k=m}\langle\delta_k^\varepsilon\varphi_\varepsilon(t_{m-(k-1)}),\varphi_\varepsilon(t_{m-(k-1)}))\leq C^2.\end{aligned} \tag{21}$$

Step 4: According to the estimate (20) the family

$$\begin{aligned}b_\varepsilon&=\mathcal{B}^\varepsilon(\widetilde{\varphi}_\varepsilon(t);\widetilde{\varphi}_\varepsilon(t_1)...,\widetilde{\varphi}_\varepsilon(t_m)\\&=(\widetilde{B}_\varepsilon^2\widetilde{\varphi}(t);\widetilde{D}_1^2\widetilde{\varphi}_\varepsilon(t_1)...,\widetilde{D}_m\widetilde{\varphi}_\varepsilon(t_m))+\varepsilon(\widetilde{\varphi}_\varepsilon(t);\widetilde{\varphi}_\varepsilon(t_1)...,\widetilde{\varphi}_\varepsilon(t_m))\end{aligned}$$

is contained in a bounded subset $\mathcal{K}_m$.
Thus, both of the families

$$\sqrt{\varepsilon}(\widetilde{\varphi}_\varepsilon(t);\widetilde{\varphi}_\varepsilon(t_1)...,\widetilde{\varphi}_\varepsilon(t_m))\text{ and }(B\widetilde{\varphi}_\varepsilon(t);D_1\widetilde{\varphi}_\varepsilon(t_1)...,D_m\widetilde{\varphi}_\varepsilon(t_m))$$

are bounded in $\mathcal{K}_m$. Therefore, we may extract a subfamily, say

$$(B\widetilde{\varphi}_\varepsilon(t);D_1\widetilde{\varphi}_\varepsilon(t_1)...,D_m\widetilde{\varphi}_\varepsilon(t_m))\rightharpoonup h,\ \text{weakly in }\mathcal{K}_m.$$

Then clearly

$$(\widetilde{B}^2\widetilde{\varphi}_\varepsilon(t);\widetilde{D}_1^2\widetilde{\varphi}_\varepsilon(t_1)...,\widetilde{D}_m^2\widetilde{\varphi}_\varepsilon(t_m))+\varepsilon(\widetilde{\varphi}_\varepsilon(t);\widetilde{\varphi}_\varepsilon(t_1)...,\widetilde{\varphi}_\varepsilon(t_m))\rightharpoonup \mathcal{B}h, \text{ weakly in } \mathcal{K}_m.$$

Step 5: Taking the limit as $\varepsilon \to 0$, we see that the solution y of (1) with initial condition $y(0) = y^0$, h being as in **step 4** satisfies $y(T) = 0$. This completes the proof of Theorem 1. □

As an immediate application of the foregoing Theorem we give the following example.

Example. One dimensional impulsive Schrödinger equation :
We consider the problem

$$\begin{aligned} \frac{\partial y\,(t,x)}{\partial t} + i\frac{\partial^2 y}{\partial x^2}(t,x) &= \chi_{\omega_0}u\,(t,x)\,, \quad t\in(0,T)\setminus\{t_k\}_{k\in\sigma_1^m}\,, x\in\Omega=(0,2\pi), \\ y(t,0) &= y(t,2\pi)=0, \qquad\qquad (22) \\ y\,(0,x) &= y^0, \\ \Delta y\,(t_k,x) &= i\alpha_k y\,(t_k,x)+\chi_{\omega_k}v_k(x),\ \ k\in\sigma_1^m, \end{aligned}$$

where

$$t_{k+1}-t_k>2\pi,\quad \omega_k=(a_1^k,a_2^k)\subset\Omega, k\in\sigma_0^m,\quad \{\alpha_k\}_{k\in\sigma_1^m}\subset\mathbb{R}^+.$$

Let

$$H=L^2(\Omega,\mathbb{C}), Aw(x)=i\tfrac{\partial^2 w}{\partial x^2}(x),\quad D(A)=\left\{w\in H,\tfrac{\partial^2 w}{\partial x^2}\in H, w(0)=w(\pi)=0\right\},$$

and $I_k w(x) = i\alpha_k w(x)$ and the control operator is given by $B = \chi_{\omega_0}$, $D_k = \chi_{\omega_k}$, then the system (22) becomes an abstract formulation of (1). As a consequence of Theorem 1, the initial state $y^0 \in L^2(\Omega,\mathbb{C}) = H$ of the solution of (22) is null-controllable at $t = T$, if and only if, there is $C > 0$ such that

$$\begin{aligned} &\left|\int_\Omega y^0(x)\widetilde{\varphi}^0(x)dx\right| \qquad (23) \\ \leq\ & C\left\{\int_0^T\int_{\omega_0}|\varphi|^2(t,x)dxdt+\sum_{k=1}^m\int_{\omega_k}|\varphi|^2(t_{m-(k-1)},x)\right\}^{\frac{1}{2}},\ \forall\widetilde{\varphi}^0\in L^2(\Omega,\mathbb{C}), \end{aligned}$$

where $\widetilde{\varphi}^0(x) = \varphi(T,x)$ and φ is the mild solution of

$$\begin{aligned}
\frac{\partial\varphi(t,x)}{\partial t} + i\frac{\partial^2\varphi(t,x)}{\partial x^2} &= 0, \quad t \in (0,T) \setminus \{t_k\}_{k\in\sigma_1^m}, \ x \in \Omega, \\
\varphi(t,0) &= \varphi(t,2\pi) = 0, \\
\varphi(0,x) &= \varphi^0(x), \ x \in \Omega, \\
\Delta\varphi(t_k,x) &= i\alpha_k\varphi(t_k,x), \ x \in \Omega, \ k \in \sigma_1^m.
\end{aligned}$$

Here φ is given by

$$\varphi(t) = \begin{cases} \varphi_{[0]}(t) & , \text{if} \quad t \in [t_0, t_1) \\ \varphi_{[k]}(t) & , \text{if} \quad t \in \left[t_k, t_{k+1)}\right) \\ \varphi_{[m]}(t) & , \text{if} \quad t \in [t_m, T], \end{cases}$$

where $\varphi_{[k]}(t)$ is a solution of the classical Schrödinger equation

$$\begin{aligned}
\frac{\partial\varphi_{[k]}(t,x)}{\partial t} + i\frac{\partial^2\varphi_{[k]}}{\partial x^2}(t,x) &= \chi_{\omega_0}u(t,x), \quad t \in (t_0,t_1), \ x \in \Omega = (0,2\pi), \\
\varphi_{[k]}(t,0) &= \varphi_{[k]}(t,2\pi) = 0, \\
\varphi_{[0]}(t_0,x) &= \varphi^0(x), \ x \in \Omega,
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial\varphi_{[k]}(t,x)}{\partial t} + i\frac{\partial^2\varphi_{[k]}}{\partial x^2}(t,x) &= \chi_{\omega_0}u(t,x), \quad t \in (t_k,t_{k+1}), \mathrm{x} \in \Omega = (0,2\pi), \\
\varphi_{[k]}(t,0) &= \varphi_{[k]}(t,2\pi) = 0, \\
\varphi_{[k]}(t_k,x) &= (1+i\alpha_k)\varphi_{[k-1]}(t_k,x), x \in \Omega, k \in \sigma_1^m.
\end{aligned}$$

Then a standard application of a variant of Ingham's Inequality [8] shows that

$$\int_{t_k}^{t_{k+1}}\int_{w_0}\left|\varphi_{[k]}\right|(t,x)dtdx \geq c(\tau_k,w_0)\int_\Omega\left|\varphi_{[k]}\right|(t_k^+,x)dx,$$

for some positive constants $c(\tau_k,w_0) > 0$. Summing up we get

$$\begin{aligned}
\sum_{k=0}^{m}\int_{t_k}^{t_{k+1}}\int_{w_0}\left|\varphi_{[k]}\right|(t,x)dtdx &= \int_0^T\int_{\omega_0}|\varphi|^2(t,x)dxdt \\
&\geq c_1\sum_{k=1}^{m}\int_\Omega\left|\varphi_{[k]}\right|(t_k^+,x)dx,
\end{aligned}$$

where $c_1 = \min\limits_{k \in \sigma_0^m} c(\tau_k, w_0) > 0$.

On the other hand, there is a positive constant $c_2 > 0$ such that

$$\sum_{k=1}^{m} \int_{\omega_k} |\varphi|^2 (t_{m-(k-1)}, x) \geq c_2 \sum_{k=1}^{m} \int_{\Omega} \left|\varphi_{[k]}\right|^2 (t_k^+, x) dx.$$

It follows that

$$\begin{aligned}
\int_0^T \int_{\omega_0} |\varphi|^2 (t, x) dx dt & \\
&+ \sum_{k=1}^{m} \int_{\omega_k} |\varphi|^2 (t_{m-(k-1)}, x) \\
&\geq (c_1 + c_2) \sum_{k=1}^{m} \int_{\Omega} \left|\varphi_{[k]}\right|^2 (t_k^+, x) dx \\
&\geq (c_1 + c_2) \int_{\Omega} \left|\varphi_{[m]}\right|^2 (t_m^+, x) dx \\
&= (c_1 + c_2) \int_{\Omega} |\varphi|^2 (T, x) dx.
\end{aligned}$$

Now, since $\widetilde{\varphi}^0(x) = \widetilde{\varphi}(0, x) = \varphi(T, x)$, then,

$$\int_0^T \int_{\omega_0} |\varphi|^2 (t, x) dx dt + \sum_{k=1}^{m} \int_{\omega_k} |\varphi|^2 (t_{m-(k-1)}, x) \geq m(c_1 + c_2) \int_{\Omega} \left|\widetilde{\varphi}^0\right|^2 (x) dx,$$

from which we get

$$\int_{\Omega} \left|\widetilde{\varphi}^0\right|^2 (x) dx \leq \frac{1}{m(c_1 + c_2)} \left(\int_0^T \int_{\omega_0} |\varphi|^2 (t, x) dx dt + \sum_{k=1}^{m} \int_{\omega_k} |\varphi|^2 (t_{m-(k-1)}, x) \right).$$

We conclude by Cauchy-Schwarz inequality that

$$\begin{aligned}
\left| \int_{\Omega} y^0(x) \widetilde{\varphi}^0(x) dx \right| &\leq \left\{ \int_{\Omega} \left|y^0\right|^2 (x) dx \int_{\Omega} \left|\widetilde{\varphi}^0\right|^2 (x) dx \right\}^{1/2} \\
&\leq \left\{ \frac{\int_{\Omega} \left|y^0\right|^2 (x) dx}{m(c_1 + c_2)} \right\}^{1/2} \left(\int_0^T \int_{\omega_0} |\varphi|^2 (t, x) dx dt \right. \\
&\quad \left. + \sum_{k=1}^{m} \int_{\omega_k} |\varphi|^2 (t_{m-(k-1)}, x) dx \right)^{1/2},
\end{aligned}$$

which establishes the necessary and sufficient condition of null controllability stated in Theorem 1.

We conclude our paper by a special case when our initial state is an eigensolution of the following linear operator $\Gamma : H \to H$ defined by

$$\Gamma(\psi) = \int_0^T X^{-1}(s)B^2X(s)\psi ds + \sum_{k=1}^{k=m} X^{-1}(t_k)D_k^2X(t_k)\psi.$$

We have the following result of null-controllability.

Proposition 1 *Let $\lambda > 0$ be an eigenvalue of Γ with eigenvector $\psi \in H$. Then, the solution y to the problem*

$$\begin{cases} y^{'}(t) + Ay(t) = -\frac{1}{\lambda}B^2(X(t)\psi), & t \in (0,T) \setminus \{t_k\}_{k\in\sigma_1^m}, \\ \Delta y(t_k) = I_k y(t_k) - \frac{1}{\lambda}D_k^2(X(t_k)\psi), & k \in \sigma_1^m \\ y(0) = \psi, \end{cases} \tag{24}$$

satisfies

$$y(T) = 0.$$

Proof.

Write system (24) into the form

$$\begin{cases} y^{'}(t) + Ay(t) = -\frac{1}{\lambda}B^2(X(t)\psi), & t \in (0,T) \setminus \{t_k\}_{k\in\sigma_1^m}, \\ y(t_k^+) = \mathcal{I}_k y(t_k) - \frac{1}{\lambda}D_k^2(X(t_k)\psi), & k \in \sigma_1^m \\ y(0) = \psi. \end{cases}$$

Therefore, this impulsive problem has a solution which can be represented explicitly as follows

$$y(t) = X(t)\psi + \int_0^t G(t,s)\left[-\frac{1}{\lambda}B^2(X(s)\psi\right]ds + \sum_{0<t_k\leq t} G(t,t_k)\left[-\frac{1}{\lambda}D_k^2X(t_k)\psi\right],$$

where the evolution operator $G(t,s)$ is given by

$$G(t,s) = X(t)X^{-1}(s).$$

On the other hand, the system (24) yields

$$
\begin{aligned}
y(T) &= X(T)\psi + \int_0^T G(T,s)\left\{-\frac{1}{\lambda}B^2(X(s)\psi\right\} ds \\
&\quad + \sum_{0<t_k\leq T} G(T,t_k)\left\{-\frac{1}{\lambda}D_k^2 X(t_k)\psi\right\} \\
&= X(T)\left[\psi + \int_0^T X^{-1}(T)G(T,s)\left\{-\frac{1}{\lambda}B^2(X(s)\psi\right\} ds\right. \\
&\quad \left. -\frac{1}{\lambda}\sum_{0<t_k\leq T} X^{-1}(T)G(T,t_k)\left\{D_k^2 X(t_k)\psi\right\}\right] \\
&= X(T)\left[\psi + \int_0^T X^{-1}(s)\left\{-\frac{1}{\lambda}B^2(X(s)\psi\right\} ds\right. \\
&\quad \left. -\frac{1}{\lambda}\sum_{0<t_k\leq T} X^{-1}(t_k)\left\{D_k^2 X(t_k)\psi\right\}\right] \\
&= X(T))\left[\psi - \frac{1}{\lambda}\Gamma(\psi)\right] = 0.
\end{aligned}
$$

This shows that the initial state ψ is null-controllable at time T with control

$$
h(t) = \left(u(t), \{v_k\}_{k\in\sigma_1^m}\right) = \left(-\frac{1}{\lambda}X(t)\psi, \left\{-\frac{1}{\lambda}X(t_k)\psi\right\}_{k\in\sigma_1^m}\right),
$$

which completes the proof of the Proposition. □

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