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Infinitely many solutions for a class of $p(x)$-Laplacian equations in $\mathbb{R}^N$

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Abstract. In this paper, we study the existence of infinitely many solutions for a class of $p(x)$-Laplacian equations in $\mathbb{R}^N$, where the nonlinearity is sublinear. The main tool used here is a variational method combined with the theory of variable exponent Sobolev spaces. Recent results from the literature are extended.

Keywords: $p(x)$-Laplacian equation, sublinear, variational method, variant fountain theorem.

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1 Introduction

In this paper, we consider the following $p(x)$-Laplacian equation in $\mathbb{R}^N$

$$\begin{cases} -\Delta_{p(x)}u + V(x)|u|^{p(x)-2}u = f(x,u) \quad \text{in} \quad \mathbb{R}^N, \\ u \in W^{1,p(x)}(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where the $p(x)$-Laplacian operator is defined by $\Delta_{p(x)}u = \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$, $p\colon \mathbb{R}^N \to \mathbb{R}$ is Lipschitz continuous and $1 < p^- := \inf_{\mathbb{R}^N} p(x) \le \sup_{\mathbb{R}^N} p(x) := p^+ < N$, V is the new potential function, f obeys some conditions which will be stated later and $W^{1,p(x)}(\mathbb{R}^N)$ is the variable exponent Sobolev space.

In recent years, the study of various mathematical problems with $p(x)$-growth condition has attracted more and more attention because these problems possess a solid background in physics and originate from the study on electrorheological fluids (see [1]) and elastic mechanics (see [2]). They also have wide applications in different research fields (see e.g. [3–5] and the references therein) and raise many difficult mathematical problems. In particular, the presence of the $p(x)$-Laplacian operator together with the appearance of the potential function V make its mathematical analysis more difficult than the corresponding p-Laplacian

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equations. Therefore, the mathematical results on the $p(x)$-Laplacian equations are far from being perfect.

To go directly to the theme of the present paper, we only review some former results which are closely related to our main results (a complete literature on $p(x)$-Laplacian equation is beyond the scope of this paper, interested authors are referred to [1,6–21], and the references therein. When $V(x)$ is radial (for example $V(x) \equiv 1$), Dai studied the following problem in [9]:

$$\begin{cases} -\Delta_{p(x)}u + |u|^{p(x)-2}u = f(x,u) \quad \text{in} \quad \mathbb{R}^N, \\ u \in W^{1,p(x)}(\mathbb{R}^N), \end{cases} \tag{1.2}$$

by means of a direct variational approach and the theory of variable exponent Sobolev spaces, sufficient conditions ensuring the existence of infinitely many distinct homoclinic radially symmetric solutions are established. Based on the theory of variable exponent Sobolev spaces, Avci in [8] studied the existence of infinitely many solutions of problem (1.2) with Dirichlet boundary condition in a bounded domain. Fan and Han in [11] discussed the existence and multiplicity of solutions to problem (1.2). Fu and Zhang in [13] also obtained that problem (1.2) possesses at least two nontrivial weak solutions.

For $p(x) = p$, problem (1.1) reduces to

$$\begin{cases} -\Delta_{p}u + V(x)|u|^{p-2}u = f(x,u) \quad \text{in} \quad \mathbb{R}^N, \\ u \in W^{1,p(x)}(\mathbb{R}^N). \end{cases} \tag{1.3}$$

The existence of ground states of problem (1.3) with a potential which is periodic or has a bounded potential well is studied in [21] by Liu. Liu and Zheng in [22] studied problem (1.3) with sign-changing potential and subcritical p-superlinear nonlinearity, by using the cohomological linking method for cones, an existence result of nontrivial solution is obtained. Li and Wang in [23] proved that problem (1.3) has at least a nontrivial solution by using variational methods combined with perturbation arguments.

Recently, Alves and Liu in [7] established the existence of ground state solution for problem (1.1) via modern variational methods under some hypotheses on the potential V and the nonlinear term f, particularly, the nonlinearity is superlinear. However, one of the remaining cases is that V is nonradial potential and $f(x,u)$ is sublinear at infinity in u and to the best of our knowledge, no results on this case have been obtained up to now. Based on the above fact and motivated by techniques used in [24,25], the main purpose of this paper is devoted to investigate the existence of infinitely many solutions for problem (1.1) when the nonlinearity is sublinear in u at infinity. Our analysis is based on the variable exponent Lebesgue–Sobolev space theory and variational methods.

We are now in a position to state our main results.

Theorem 1.1. *Suppose that the following conditions are satisfied.*

(H_1) $V \in C(\mathbb{R}^N)$ *satisfies* $\inf_{x\in\mathbb{R}^N} V(x) > 0$ *and for all* $M > 0$, $\mu\big(V^{-1}(-\infty, M]\big) < \infty$, *where* μ *denotes the Lebesgue measure on* $\mathbb{R}^N$.

(H_2) $F(x,u) = b(x)|u|^{q(x)}$, *where* $F(x,u) = \int_0^u f(x,t)dt$, $b : \mathbb{R}^N \to \mathbb{R}^+$ *is a positive continuous function such that* $b \in L^{\frac{s(x)}{s(x)-q(x)}}(\mathbb{R}^N)$ *and* $1 < q^- \le q^+ < p^-$, *where* $p(x) \le s(x) \ll p^*(x)$, $p^*(x) = \frac{Np(x)}{N-p(x)}$, *and* $s(x) \ll p^*(x)$ *means that* $\operatorname{ess\,inf}_{x\in\mathbb{R}^N}(p^*(x) - s(x)) > 0$.

Then problem (1.1) *possesses infinitely many solutions* $\{u_k\}$ *satisfying*

$$\int_{\mathbb{R}^N} \frac{1}{p(x)} \left(|\nabla u_k|^{p(x)} + V(x)|u_k|^{p(x)}\right) dx - \int_{\mathbb{R}^N} F(x,u_k)\, dx \to 0^-, \quad as \quad k \to \infty.$$

Remark 1.2. From the variational viewpoint, the main difficulty in treating problem (1.1) in $\mathbb{R}^N$ arises from the lack of compactness of the Sobolev embeddings which prevents from checking directly that the energy functional associated with problem (1.1) satisfies the Palais–Smale condition. To overcome this difficulty, we use a Bartsch–Wang type compact embedding theorem for variable exponent spaces established by Alves and Liu in [7].

Remark 1.3. In this paper, we consider the case that the nonlinearity is sublinear and obtain infinitely many small negative-energy solutions of problem (1.1), which complement and extend previously known results in [7,8,11,13,21,22].

The structure of this paper is outlined as follows. In Section 2, some preliminary results and the variational tools we used are presented. In Section 3, the proof of the main result is given.

Notations: Throughout this paper, we denote a generic positive constant by C which may vary from line to line. If the dependence needs to be explicitly pointed out, then the notations C_i ($i \in \mathbb{Z}^+$) are used.

2 Preliminaries

In this section, we first recall some preliminary results about Lebesgue and Sobolev variable exponent spaces, which are useful for discussing problem (1.1). We refer the reader to [26–29] and the references therein for a more detailed account on this topic.

Set

$$C_+(\mathbb{R}^N) = \left\{p \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) : p(x) > 1 \quad \text{for all} \quad x \in \mathbb{R}^N\right\}.$$

In this paper, for any $p \in C_+(\mathbb{R}^N)$, we will denote

$$p^- = \operatorname*{ess\,inf}_{x\in\mathbb{R}^N} p(x), \quad p^+ = \operatorname*{ess\,sup}_{x\in\mathbb{R}^N} p(x)$$

and denote by $p_1 \ll p_2$ the fact that $\operatorname{ess\,inf}_{x\in\mathbb{R}^N}(p_2(x) - p_1(x)) > 0$.

Denote by $S(\mathbb{R}^N)$ the set of all measurable real-valued functions defined on $\mathbb{R}^N$. Note that two measurable functions in $S(\mathbb{R}^N)$ are considered as the same element of $S(\mathbb{R}^N)$ when they are equal almost everywhere.

Let $p \in C_+(\mathbb{R}^N)$, the variable exponent Lebesgue space is defined by

$$L^{p(x)}(\mathbb{R}^N) = \left\{u \in S(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^{p(x)} < \infty\right\}$$

furnished with the Luxemburg norm

$$|u|_{L^{p(x)}(\mathbb{R}^N)} = |u|_{p(x)} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^N} \left|\frac{u}{\lambda}\right|^{p(x)} dx \le 1\right\},$$

and the variable exponent Sobolev space is defined by

$$W^{1,p(x)}(\mathbb{R}^N) = \left\{u \in L^{p(x)}(\mathbb{R}^N) : |\nabla u| \in L^{p(x)}(\mathbb{R}^N)\right\}$$

equipped with the norm

$$\|u\|_{1,p(x)} = \|u\|_{W^{1,p(x)}(\mathbb{R}^N)} = |u|_{p(x)} + |\nabla u|_{p(x)}.$$

Proposition 2.1 ([27]). *The spaces $L^{p(x)}(\mathbb{R}^N)$ and $W^{1,p(x)}(\mathbb{R}^N)$ are separable and reflexive Banach spaces.*

Now, let us introduce the modular of the space $L^{p(x)}(\mathbb{R}^N)$ as the functional $\rho_{p(x)}(u) : L^{p(x)}(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$\rho_{p(x)}(u) = \int_{\mathbb{R}^N} |u|^{p(x)} dx$$

for all $u \in L^{p(x)}(\mathbb{R}^N)$. The relation between modular and Luxemburg norm is clarified by the following propositions.

Proposition 2.2 ([12]). *Let $u \in L^{p(x)}(\mathbb{R}^N)$ and let $\{u_m\}$ be a sequence in $L^{p(x)}(\mathbb{R}^N)$, then*

(1) For $u \neq 0$, $|u|_{p(x)} = \lambda \Leftrightarrow \rho_{p(x)}(\frac{u}{\lambda}) = 1$;

(2) $|u|_{p(x)} < 1\ (= 1; > 1) \Leftrightarrow \rho_{p(x)}(u) < 1\ (= 1; > 1)$;

(3) If $|u|_{p(x)} > 1$, then $|u|_{p(x)}^{p^-} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p^+}$;

(4) If $|u|_{p(x)} < 1$, then $|u|_{p(x)}^{p^+} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p^-}$;

(5) $\lim_{m\to\infty} |u_m - u|_{p(x)} \Leftrightarrow \rho_{p(x)}(u_m - u) = 0$.

Let

$$E = \left\{ u \in W^{1,p(x)}(\mathbb{R}^N) : \int_{\mathbb{R}^N} \left(|\nabla u|^{p(x)} + V(x)|u|^{p(x)}\right) dx < \infty \right\},$$

we equip it with the norm

$$\|u\| = \|u\|_E = \inf\left\{ \lambda > 0 : \int_{\mathbb{R}^N} \left(|\frac{\nabla u}{\lambda}|^{p(x)} + V(x)|\frac{u}{\lambda}|^{p(x)} \right) dx \leq 1 \right\}.$$

Then $(E, \|\cdot\|)$ is continuously embedded into $W^{1,p(x)}(\mathbb{R}^N)$ as a closed subspace. Therefore, $(E, \|u\|)$ is also a separable reflexive Banach space. In addition, defining the modular $\rho_{p(x),V}(u) \colon E \to R$ associated with E as

$$\rho_{p(x),V}(u) = \int_{\mathbb{R}^N} \left(|\nabla u|^{p(x)} + V(x)|u|^{p(x)} \right) dx$$

for all $u \in E$, in a similar way to Proposition 2.2, the following proposition holds.

Proposition 2.3. *Let $u \in E$ and let $\{u_m\}$ be a sequence in E, then*

(1) For $u \neq 0, ||u|| = \lambda \Leftrightarrow \rho_{p(x),V}(\frac{u}{\lambda}) = 1$;

(2) $||u|| < 1(= 1; > 1) \Leftrightarrow \rho_{p(x),V}(u) < 1(= 1; > 1)$;

(3) If $||u|| > 1$, then $||u||^{p^-} \leq \rho_{p(x),V}(u) \leq ||u||^{p^+}$;

(4) If $||u|| < 1$, then $||u||^{p^+} \leq \rho_{p(x),V}(u) \leq ||u||^{p^-}$;

(5) $\lim_{m\to\infty} \|u_m - u\| \Leftrightarrow \rho_{p(x),V}(u_m - u) = 0.$

Lemma 2.4 (Hölder-type inequality [12]). *The conjugate space of $L^{p(x)}(\mathbb{R}^N)$ is $L^{q(x)}(\mathbb{R}^N)$, where $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$. For any $u \in L^{p(x)}(\mathbb{R}^N)$ and $v \in L^{q(x)}(\mathbb{R}^N)$, we have*

$$\left| \int_{\mathbb{R}^N} uv\,dx \right| \le \left(\frac{1}{p^-} + \frac{1}{q^-} \right) |u|_{p(x)} |v|_{q(x)} \le 2|u|_{p(x)}|v|_{q(x)}.$$

Remark 2.5. Likewise, if $\frac{1}{p(x)} + \frac{1}{q(x)} + \frac{1}{r(x)} = 1$, then for any $u \in L^{p(x)}(\mathbb{R}^N)$, $v \in L^{q(x)}(\mathbb{R}^N)$, $w \in L^{r(x)}(\mathbb{R}^N)$, we have

$$\left| \int_{\mathbb{R}^N} uvw\,dx \right| \le \left(\frac{1}{p^-} + \frac{1}{q^-} + \frac{1}{r^-} \right) |u|_{p(x)} |v|_{q(x)} |w|_{r(x)} \le 3|u|_{p(x)}|v|_{q(x)}|w|_{r(x)}.$$

Lemma 2.6 ([6,11]). *Let $q, s \in C_+(\mathbb{R}^N)$ with $q(x) \le s(x)$ for all $x \in \mathbb{R}^N$ and $u \in L^{s(x)}(\mathbb{R}^N)$. Then, $|u(x)|^{q(x)} \in L^{\frac{s(x)}{q(x)}}(\mathbb{R}^N)$ and*

$$||u|^{q(x)}|_{\frac{s(x)}{q(x)}} \le |u|^{q^+}_{s(x)} + |u|^{q^-}_{s(x)}, \tag{2.1}$$

or there exists a number $\tilde{q} \in [q^-, q^+]$ such that

$$||u|^{q(x)}|_{\frac{s(x)}{q(x)}} = |u|^{\tilde{q}}_{s(x)}. \tag{2.2}$$

The following Bartsch–Wang type compact embedding will play a crucial role in our subsequent arguments.

Lemma 2.7 ([7, Lemma 2.6]). *If V satisfies (H_1), then*

(i) we have a compact embedding $E \hookrightarrow L^{p(x)}(\mathbb{R}^N)$, $1 < p_- \le p_+ < N$;

(ii) for any measurable function $s(x) : \mathbb{R}^N \to \mathbb{R}$ with $p < s \ll p^*$, we have a compact embedding $E \hookrightarrow L^{s(x)}(\mathbb{R}^N)$.

Remark 2.8. By virtue of Lemma 2.7, we know that there exists a constant $C_1 > 0$ such that

$$|u|_{p(x)} \le C_1\|u\| \quad \text{for any} \quad u \in E. \tag{2.3}$$

Remark 2.9. The case $p(x) = 2$ is due to Bartsch and Wang [30]. If V satisfies

(H_1') $V \in C(\mathbb{R}^N)$ satisfies $\inf_{x\in\mathbb{R}^N} V(x) > 0$ and there exists $r > 0$ such that for all $M > 0$,

$$\mu\Big(\{x \in \mathbb{R}^N : V(x) \le M\} \bigcap B_r(y)\Big) = 0,$$

where μ denotes the Lebesgue measure on $\mathbb{R}^N$,

then a similar compact embedding has been established by Ge et al. in [14].

In the following, we present the variational tools named the variant fountain theorem established by Zou [31], which will be used to get our result.

Let E be a Banach space with the norm $\|\cdot\|$ and $E = \overline{\oplus_{j\in\mathbb{N}} X_j}$ with $\dim X_j < \infty$ for any $j \in \mathbb{N}$. Set

$$Y_k = \oplus_{j=0}^{k} X_j, \qquad Z_k = \overline{\oplus_{j=k+1}^{\infty} X_j}. \tag{2.4}$$

Consider the following C^1 functional $I_\lambda \colon E \to \mathbb{R}$ defined by

$$I_\lambda(u) = A(u) - \lambda B(u), \quad \lambda \in [1,2],$$

where $A, B \colon E \to \mathbb{R}$ are two functionals.

Theorem 2.10 ([31, Theorem 2.2]). *Suppose that the functional I_λ defined above satisfies the following conditions:*

(C_1) *I_λ maps bounded sets to bounded sets uniformly for $\lambda \in [1,2]$. Furthermore, $I_\lambda(-u) = I_\lambda(u)$ for all $(\lambda, u) \in [1,2] \times E$.*

(C_2) *$B(u) \geq 0$; $B(u) \to \infty$ as $\|u\| \to \infty$ on any finite dimensional subspace of E.*

(C_3) *There exist $\rho_k > r_k > 0$ such that $a_k(\lambda) := \inf_{u \in Z_k, \|u\| = \rho_k} I_\lambda(u) \geq 0 > b_k(\lambda) := \max_{u \in Y_k, \|u\| = r_k} I_\lambda(u)$ for $\lambda \in [1,2]$, $d_k(\lambda) := \inf_{u \in Z_k, \|u\| \leq \rho_k} I_\lambda(u) \to 0$ as $k \to \infty$ uniformly for $\lambda \in [1,2]$.*

Then there exist $\lambda_n \to 1$, $u(\lambda_n) \in Y_n$ such that $I'_{\lambda_n}|_{Y_n}(u(\lambda_n)) = 0$, $I_{\lambda_n}(u(\lambda_n)) \to c_k \in [d_k(2), d_k(1)]$ as $n \to \infty$. In particular, if $\{u(\lambda_n)\}$ has a convergent subsequence for every k, then I_1 has infinitely many nontrivial critical points $\{u_k\} \subset E \setminus \{0\}$ satisfying $I_1(u_k) \to 0^-$ as $k \to \infty$.

In order to discuss the problem 1.1, we need to consider the energy functional $I \colon E \to \mathbb{R}$ defined by

$$I(u) = \int_{\mathbb{R}^N} \frac{1}{p(x)} \left(|\nabla u|^{p(x)} + V(x)|u|^{p(x)} \right) dx - \int_{\mathbb{R}^N} F(x,u) dx.$$

Under our conditions, it follows from Hölder-type inequality and Sobolev embedding theorem that the energy functional I is well-defined. It is well known that $I \in C^1(E, \mathbb{R})$ and its derivative is given by

$$\langle I'(u), v \rangle = \int_{\mathbb{R}^N} \left(|\nabla u|^{p(x)-2} \nabla u \cdot \nabla v + V(x)|u(x)|^{p(x)-2} uv - f(x,u)v \right) dx \tag{2.5}$$

for each $u \in E$. It is standard to verify that the weak solutions of problem (1.1) correspond to the critical points of the functional I.

3 Proof of main result

In order to apply Theorem 2.10, we define the functionals A, B and I_λ on the working space E by

$$A(u) = \int_{\mathbb{R}^N} \frac{1}{p(x)} \left(|\nabla u|^{p(x)} + V(x)|u|^{p(x)} \right) dx, \quad B(u) = \int_{\mathbb{R}^N} F(x,u)\, dx,$$

and

$$I_\lambda(u) = \int_{\mathbb{R}^N} \frac{1}{p(x)} \left(|\nabla u|^{p(x)} + V(x)|u|^{p(x)} \right) dx - \lambda \int_{\mathbb{R}^N} F(x,u)\, dx$$

for all $u \in E$ and $\lambda \in [1,2]$. Clearly, $I_\lambda(u) \in C^1(E, \mathbb{R})$ for all $\lambda \in [1,2]$. We choose a completely orthogonal basis $\{e_j\}$ of E and define $X_j := \mathbb{R} e_j$, and Z_k, Y_k defined as (2.4).

Now, we show that I_λ has the geometric property needed by Theorem 2.10.

Lemma 3.1. *Under the assumptions of Theorem 1.1, then $B(u) \geq 0$. Moreover, $B(u) \to \infty$ as $\|u\| \to \infty$ on any finite dimensional subspace of E.*

Proof. It is obvious that $B(u) \geq 0$ from the definition of the functional B and (H_2).

Next, we claim that

$$B(u) \to \infty \quad \text{as} \quad \|u\| \to \infty$$

on any finite dimensional subspace of E. First, for any finite dimensional subspace $F \subset E$, there exists $\delta > 0$ such that

$$\mu\Big\{x \in \mathbb{R}^N : b(x)|u(x)|^{q(x)} \geq \delta \|u\|^{q(x)}\Big\} \geq \delta \quad \text{for all} \quad u \in F \setminus \{0\}. \tag{3.1}$$

Otherwise, for any positive integer n, there exists $u_n \in F \setminus \{0\}$ such that

$$\mu\left\{x \in \mathbb{R}^N : b(x)|u_n(x)|^{q(x)} \geq \frac{1}{n}\|u_n\|^{q(x)}\right\} < \frac{1}{n}.$$

Set

$$v_n(x) := \frac{u_n(x)}{\|u_n\|} \in F \setminus \{0\},$$

then

$$\|v_n\| = 1 \quad \text{for all} \quad n \in \mathbb{N}$$

and

$$\mu\left\{x \in \mathbb{R}^N : b(x)|v_n(x)|^{q(x)} \geq \frac{1}{n}\right\} < \frac{1}{n}. \tag{3.2}$$

Since $\dim F < \infty$, we know from the compactness of the unit sphere of F that there exists a subsequence, say $\{v_n\}$, such that

$$v_n \to v_0 \quad \text{in} \quad F,$$

and hence

$$\|v_0\| = 1. \tag{3.3}$$

In view of the equivalence of the norms on the finite dimensional space F, we obtain

$$v_n \to v_0 \quad \text{in} \quad L^{s(x)}(\mathbb{R}^N), \quad p(x) \leq s(x) \ll p^*(x)$$

that is

$$|v_n - v_0|_{s(x)} \to 0 \quad \text{as} \quad n \to \infty. \tag{3.4}$$

By Lemma 2.4, (2.1) and (3.4), we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} b(x)|v_n - v_0|^{q(x)}\,dx \\
&\quad \leq 2|b(x)|_{\frac{s(x)}{s(x)-q(x)}} \left||v_n - v_0|^{q(x)}\right|_{\frac{s(x)}{q(x)}} \\
&\quad \leq 2|b(x)|_{\frac{s(x)}{s(x)-q(x)}} \left(|v_n - v_0|_{s(x)}^{q^+} + |v_n - v_0|_{s(x)}^{q^-}\right) \\
&\quad \to 0 \quad \text{as} \quad n \to \infty.
\end{aligned} \tag{3.5}$$

Then there exist $\alpha_1, \alpha_2 > 0$ such that

$$\mu\{x \in \mathbb{R}^N : b(x)|v_0(x)|^{q(x)} \geq \alpha_1\} \geq \alpha_2. \tag{3.6}$$

If this is not true, then, for all positive integer n, one has

$$\mu\left\{x \in \mathbb{R}^N : b(x)|v_0(x)|^{q(x)} \geq \frac{1}{n}\right\} = 0,$$

which, together with (2.3), implies that

$$\begin{aligned} 0 &\leq \int_{\mathbb{R}^N} b(x)|v_0|^{q(x)+s(x)}dx < \frac{1}{n}\int_{\mathbb{R}^N} |v_0|^{s(x)}dx \\ &\leq \frac{1}{n}\left(|v_0|^{s^+}_{s(x)} + |v_0|^{s^-}_{s(x)}\right) \leq \frac{C}{n}\left(\|v_0\|^{s^+} + \|v_0\|^{s^-}\right) \to 0 \quad \text{as} \quad n \to \infty, \end{aligned}$$

and hence one easily checks that $\|v_0\| = 0$. This is a contradiction with (3.3) and therefore (3.6) holds.

Now let

$$\Omega_0 = \left\{x \in \mathbb{R}^n : b(x)|v_0(x)|^{q(x)} \geq \alpha_1\right\}, \quad \Omega_n = \left\{x \in \mathbb{R}^n : b(x)|v_0(x)|^{q(x)} < \frac{1}{n}\right\}$$

and

$$\Omega_n^c = \mathbb{R}^N \setminus \Omega_n = \left\{x \in \mathbb{R}^n : b(x)|v_0(x)|^{q(x)} \geq \frac{1}{n}\right\}.$$

From (3.2) and (3.6), we have

$$\begin{aligned} \mu(\Omega_n \cap \Omega_0) &= \mu(\Omega_0 \setminus (\Omega_n^c \cap \Omega_0)) \\ &\geq \mu(\Omega_0) - \mu(\Omega_n^c \cap \Omega_0) \\ &\geq \alpha_2 - \frac{1}{n} \end{aligned}$$

for all positive integer n. Let n be large enough such that

$$\alpha_2 - \frac{1}{n} \geq \frac{1}{2}\alpha_2$$

and

$$\frac{1}{2^{(q^+-1)}}\alpha_1 - \frac{1}{n} \geq \frac{1}{2^{q^+}}\alpha_1.$$

Then we have

$$\begin{aligned} \int_{\mathbb{R}^N} b(x)|v_n - v_0|^{q(x)}\,dx &\geq \int_{\Omega_n\cap\Omega_0} b(x)|v_n - v_0|^{q(x)}\,dx \\ &\geq \frac{1}{2^{(q^+-1)}}\int_{\Omega_n\cap\Omega_0} b(x)|v_0|^{q(x)}\,dx - \int_{\Omega_n\cap\Omega_0} b(x)|v_n|^{q(x)}\,dx \\ &\geq \left(\frac{1}{2^{(q^+-1)}}\alpha_1 - \frac{1}{n}\right)\mu(\Omega_n \cap \Omega_0) \\ &\geq \frac{1}{2^{q^+}}\alpha_1 \cdot \frac{1}{2}\alpha_2 \\ &= \frac{\alpha_1\alpha_2}{2^{(q^++1)}} > 0 \end{aligned}$$

for all sufficiently large n, which is a contradiction to (3.5). Therefore, (3.1) holds. Second, for the δ given in (3.1), let

$$\Omega_u = \{x \in \mathbb{R}^N : b(x)|u(x)|^{q(x)} \geq \delta\|u\|^{q(x)}\} \quad \text{for all} \quad u \in F \setminus \{0\}. \tag{3.7}$$

Then by (3.1),

$$\mu(\Omega_u) \geq \delta \quad \text{for all} \quad u \in F \setminus \{0\}. \tag{3.8}$$

Combining (H_2) and (3.8), for any $u \in F \setminus \{0\}$, we have

$$\begin{aligned} B(u) &= \int_{\mathbb{R}^N} F(x,u)\,dx = \int_{\mathbb{R}^N} b(x)|u(x)|^{q(x)}\,dx \\ &\geq \int_{\Omega_u} b(x)|u(x)|^{q(x)}\,dx \geq \delta\|u\|^{q(x)}\mu(\Omega_u) \\ &\geq \delta^2\|u\|^{q(x)}, \end{aligned}$$

which implies that

$$B(u) \to \infty \quad \text{as} \quad \|u\| \to \infty$$

on any finite dimensional subspace of E. The proof is completed. □

Lemma 3.2. *Under the assumptions of Theorem 1.1, there exists a sequence $\rho_k \to 0^+$ as $k \to \infty$ such that*

$$a_k(\lambda) := \inf_{u\in Z_k, \|u\|=\rho_k} I_\lambda(u) \geq 0,$$

and

$$d_k(\lambda) := \inf_{u\in Z_k, \|u\|\leq\rho_k} I_\lambda(u) \to 0$$

as $k \to \infty$ uniformly for $\lambda \in [1,2]$, where $Z_k = \overline{\oplus_{j=k}^{\infty} X_j} = \overline{\operatorname{span}\{e_k, \dots\}}$ for all $k \in \mathbb{N}$.

Proof. Set $\beta_k := \sup\limits_{u\in Z_k, \|u\|=1} |u|_{s(x)}$, then $\beta_k \to 0$ as $k \to \infty$ (see [11]). By (H_2), Proposition 2.3, Lemma 2.4 and Lemma 2.6, we have

$$\begin{aligned} I_\lambda(u) &= \int_{\mathbb{R}^N} \frac{1}{p(x)}(|\nabla u|^{p(x)} + V(x)|u|^{p(x)})\,dx - \lambda\int_{\mathbb{R}^N} F(x,u)\,dx \\ &\geq \frac{1}{p^+}\min\left\{\|u\|^{p^+}, \|u\|^{p^-}\right\} - 2\int_{\mathbb{R}^N} b(x)|u|^{q(x)}\,dx \\ &\geq \frac{1}{p^+}\min\left\{\|u\|^{p^+}, \|u\|^{p^-}\right\} - 4|b|_{\frac{s(x)}{s(x)-q(x)}}\left||u|^{q(x)}\right|_{\frac{s(x)}{q(x)}} \\ &= \frac{1}{p^+}\min\left\{\|u\|^{p^+}, \|u\|^{p^-}\right\} - 4|b|_{\frac{s(x)}{s(x)-q(x)}}|u|_{s(x)}^{\tilde{q}(\|u\|)} \\ &\geq \frac{1}{p^+}\min\left\{\|u\|^{p^+}, \|u\|^{p^-}\right\} - 4\beta_k^{\tilde{q}}|b|_{\frac{s(x)}{s(x)-q(x)}}\|u\|^{\tilde{q}(\|u\|)}, \end{aligned} \tag{3.9}$$

where $\tilde{q}(\|u\|) \in [q^-, q^+]$, and $\tilde{q}(\|u\|)$ is a constant which is dependent on $\|u\|$.

Let

$$\rho_k = \min\left\{\left(8p^+\beta_k^{\tilde{q}(\|u\|)}|b|_{\frac{s(x)}{s(x)-q(x)}}\right)^{\frac{1}{p^+-\tilde{q}(\|u\|)}}, \left(8p^+\beta_k^{\tilde{q}(\|u\|)}|b|_{\frac{s(x)}{s(x)-q(x)}}\right)^{\frac{1}{p^--\tilde{q}(\|u\|)}}\right\}.$$

Obviously, $\rho_k \to 0$ as $k \to \infty$. Combining this with (3.9), straightforward computation shows that

$$a_k(\lambda) := \inf_{u\in Z_k, \|u\|=\rho_k} I_\lambda(u) \geq \frac{1}{2p^+}\min\left\{\rho_k^{p^+}, \rho_k^{p^-}\right\} > 0.$$

Furthermore, by (3.9), for any $u \in Z_k$ with $\|u\| \leq \rho_k$, we have

$$I_\lambda(u) \geq -4\beta_k^{\tilde{q}(\|u\|)} |b|_{\frac{p(x)}{p(x)-q(x)}} \|u\|^{\tilde{q}(\|u\|)},$$

and therefore

$$0 \geq \inf_{u \in Z_k, \|u\| \leq \rho_k} I_\lambda(u) \geq -4\beta_k^{\tilde{q}(\|u\|)} |b|_{\frac{s(x)}{s(x)-q(x)}} \|u\|^{\tilde{q}(\|u\|)}. \tag{3.10}$$

Since $\beta_k, \rho_k \to 0$, $k \to \infty$, we derive from (3.10) that

$$d_k(\lambda) := \inf_{u \in Z_k, \|u\| \leq \rho_k} I_\lambda(u) \to 0 \quad \text{as} \quad k \to \infty \quad \text{uniformly for} \quad \lambda \in [1,2].$$

The proof is completed. $\square$

Lemma 3.3. *Under the assumptions of Theorem 1.1, for the sequence $\{\rho_k\}_{k \in \mathbb{N}}$ obtained in Lemma 3.2, there exist $0 < r_k < \rho_k$ for all $k \in \mathbb{N}$ such that*

$$b_k(\lambda) := \max_{u \in Y_k, \|u\| = r_k} I_\lambda(u) < 0, \quad \text{for all} \quad \lambda \in [1,2],$$

where $Y_k = \overline{\oplus_{j=1}^k X_j} = \overline{\operatorname{span}\{e_1, \ldots, e_k\}}$ for all $k \in \mathbb{N}$.

Proof. For any $u \in Y_k$ and $\lambda \in [1,2]$, one can deduce from (H_2), Proposition 2.3, (3.7) and (3.8) that

$$\begin{aligned} I_\lambda(u) &= \int_{\mathbb{R}^N} \frac{1}{p(x)} (|\nabla u|^{p(x)} + V(x)|u|^{p(x)}) dx - \lambda \int_{\mathbb{R}^N} F(x,u)\, dx \\ &\leq \frac{1}{p^-} \max\left\{\|u\|^{p^+}, \|u\|^{p^-}\right\} - \int_{\Omega_u} b(x)|u(x)|^{q(x)}\, dx \\ &\leq \frac{1}{p^-} \max\left\{\|u\|^{p^+}, \|u\|^{p^-}\right\} - \delta^2 \min\left\{\|u\|^{q^+}, \|u\|^{q^-}\right\}, \end{aligned}$$

which, together with $1 < q^- \leq q^+ < p^-$, leads to

$$b_k(\lambda) := \max_{u \in Y_k, \|u\| = r_k} I_\lambda(u) < 0, \quad \text{for all} \quad k \in \mathbb{N},$$

for $\|u\| = r_k < \rho_k$ sufficiently small. The proof is completed. $\square$

Now we are in a position to prove Theorem 1.1. In our proof of Theorem 1.1, we will consider A as a functional on $(E, \|\cdot\|)$. We say that an operator $L \colon E \to E^*$ is of (S_+) type if $u_n \rightharpoonup u$ and

$$\overline{\lim_{n \to \infty}} \langle L(u_n) - L(u), u_n - u \rangle \leq 0$$

imply $u_n \to u$ in E.

Proof of Theorem 1.1. Obviously, condition (C_1) in Theorem 2.10 holds. By Lemmas 3.1, 3.2 and 3.3, conditions (C_2) and (C_3) in Theorem 2.10 are also satisfied. Therefore, we know from Theorem 2.10 that there exist $\lambda_n \to 1, u(\lambda_n) \in Y_n$ such that

$$I'_{\lambda_n}|_{Y_n}(u(\lambda_n)) = 0, \quad I_{\lambda_n}(u(\lambda_n)) \to c_k \in [d_k(2), b_k(1)] \quad \text{as} \quad n \to \infty. \tag{3.11}$$

For simplicity, we denote $u(\lambda_n)$ by u_n for all $n \in \mathbb{N}$. We will show that $\{u_n\}$ is bounded in E. To verify this, thanks to (H_2) and Lemmas 2.4, 2.6 and (2.3), one has

$$\begin{aligned}\frac{1}{p^+}\min\left\{\|u\|^{p^+},\|u\|^{p^-}\right\} &\le I_{\lambda_n}(u_n)+\lambda_n\int_{\mathbb{R}^N} b(x)|u_n(x)|^{q(x)}\,dx\\ &\le M_1+4|b|_{\frac{s(x)}{s(x)-q(x)}}\left||u_n|^{q(x)}\right|_{\frac{s(x)}{q(x)}}\\ &\le M_1+4|b|_{\frac{s(x)}{s(x)-q(x)}}\left(|u_n|^{q^+}_{s(x)}+|u_n|^{q^-}_{s(x)}\right)\\ &\le M_1+4C|b|_{\frac{s(x)}{s(x)-q(x)}}\left(\|u_n\|^{q^+}+\|u_n\|^{q^-}\right)\end{aligned} \tag{3.12}$$

for some $M_1>0$. Since $1<q^-\le q^+<p^-$, (3.12) implies that $\{u_n\}$ is bounded in E.

Finally, we show that there is a strongly convergent subsequence of $\{u_n\}$ in E. Indeed, in view of the boundedness of $\{u_n\}$, passing to a subsequence if necessary, still denoted by $\{u_n\}$, we may assume that

$$u_n \rightharpoonup u_0 \quad \text{in} \quad E,$$

in view of Lemma 2.7, we have

$$u_n \to u_0 \quad \text{in} \quad L^{s(x)}(\mathbb{R}^N), \quad p(x)\le s(x)\ll p^*. \tag{3.13}$$

Moreover, by (2.5), direct calculation produces

$$\begin{aligned}\langle A'(u_n)-A'u_0),u_n-u_0\rangle &= \langle I'_{\lambda_n}(u_n)-I'_1(u_0),u_n-u_0\rangle\\ &\quad+\int_{\mathbb{R}^N}\big(\lambda_n f(x,u_n)-f(x,u_0)\big)(u_n-u_0)\,dx.\end{aligned} \tag{3.14}$$

It is clear that

$$\begin{aligned}\langle I'_{\lambda_n}(u_n)-I'_1(u_0),u_n-u_0\rangle &= \langle I'_{\lambda_n}(u_n),u_n-u_0\rangle+\langle I'_1(u_0),u_n-u_0\rangle\\ &\to 0.\end{aligned} \tag{3.15}$$

By virtue of (H_2), Remark 2.8, Lemma 2.6 and (3.13), one can deduce that

$$\begin{aligned}&\int_{\mathbb{R}^N}\big(\lambda_n f(x,u_n)-f(x,u_0)\big)(u_n-u_0)\,dx\\ &\le q^+\int_{\mathbb{R}^N} b(x)(\lambda_n|u_n|^{q(x)-1}+|u_0|^{q(x)-1})|u_n-u_0|\,dx\\ &= q^+\left\{\lambda_n\int_{\mathbb{R}^N} b(x)|u_n|^{q(x)-1}|u_n-u_0|\,dx+\int_{\mathbb{R}^N} b(x)|u_0|^{q(x)-1}|u_n-u_0|\,dx\right\}\\ &\le q^+\left\{6\big|b(x)\big|_{\frac{s(x)}{s(x)-q(x)}}\left||u_n|^{q(x)-1}\right|_{\frac{s(x)}{q(x)-1}}|u_n-u_0|_{s(x)}\right.\\ &\qquad\left.+3\big|b(x)\big|_{\frac{s(x)}{s(x)-q(x)}}\left||u_0|^{q(x)-1}\right|_{\frac{s(x)}{q(x)-1}}|u_n-u_0|_{s(x)}\right\}\\ &\to 0, \quad \text{as} \quad n\to\infty.\end{aligned} \tag{3.16}$$

Together (3.15) with (3.16), one deduces from (3.14) that

$$\langle A'(u_n)-A'(u_0),u_n-u_0\rangle \to 0 \quad \text{as} \quad n\to\infty.$$

Since A is of (S_+) type (see [7,11]), we obtain $u_n \to u$ in E.

Now from the last assertion of Theorem 2.10, we know that $I=I_1$ has infinitely many nontrivial critical points. Therefore, problem (1.1) possesses infinitely many nontrivial solutions. The proof of Theorem 1.1 is completed. □

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