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Nonlinear Parabolic Problems with Neumann-type Boundary Conditions and L^1-data

Abderrahmane El Hachimi[(1)] and Ahmed Jamea[(2)]

UFR Mathématiques Appliquées et Industrielles
Faculté des Sciences
B. P. 20, El Jadida, Maroc
[(1)]aelhachi@yahoo.fr; [(2)]a.jamea@yahoo.fr

Abstract

In this paper, we study existence, uniqueness and stability questions for the nonlinear parabolic equation:

$$\frac{\partial u}{\partial t} - \triangle_p u + \alpha(u) = f \qquad in\ ]0,\ T[\times\Omega,$$

with Neumann-type boundary conditions and initial data in L^1. Our approach is based essentially on the time discretization technique by Euler forward scheme.

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1 Introduction

In this work, we treat the nonlinear parabolic problem

$$\begin{aligned} \frac{\partial u}{\partial t} - \triangle_p u + \alpha(u) = f \text{ in } Q_T :=]0,\ T[\times\Omega, \\ |Du|^{p-2}\frac{\partial u}{\partial \eta} + \gamma(u) = g \text{ on } \Sigma_T :=]0,\ T[\times\partial\Omega, \\ u(0,.) = u_0 \text{ in } \Omega, \end{aligned} \tag{1}$$

where $\Delta_p u = \text{div}\left(|Du|^{p-2}Du\right)$, $1 < p < \infty$, Ω is a connected open bounded set in $\mathbb{R}^d$, d ≥ 3, with a connected Lipschitz boundary $\partial\Omega$, T is a fixed positive real number and α, γ are taken as continuous non decreasing real functions everywhere defined on $\mathbb{R}$ with $\alpha(0) = \gamma(0) = 0$. We will have in mind especially the case when initial data in L^1.

The usual weak formulations of parabolic problems with initial data in L^1 do not ensure existence and uniqueness of solutions. There then arose formulations which were more suitable than that of weak solutions. Through that work it is hoped that we can arrive at a definition of solution so that we can prove existence and uniqueness. For that, three notions of solutions have been adopted: Solutions named SOLA (Solution Obtained as the Limit of Approximations) defined by A. Dallaglio [6]. Renormalized solutions defined by R. Diperna and P. L. Lions [7]. Entropy solutions defined by Ph. Bénilan, L. Boccardo, T. Gallouet, R. Gariepy, M. Pierre, J.L. Vazquez in [4]. We will have interested here at entropy formulation. Many authors are interested has this type of formulations, see for example [1, 2, 3, 4, 19, 20, 25, 26].

The problem (1) is treated by F. Andereu, J. M. Mazón, S. Segura De león, J. Teledo [1] in the homogeneous case, i.e. $f = 0$, $g = 0$ and $\alpha = 0$, with γ is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ and $0 \in \gamma(0)$. Hulshof [12] considers the case where α is a uniformly Lipschitz continuous function, $\alpha(r) = 1$ for $r \in \mathbb{R}^+$, $\alpha \in C^1(\mathbb{R}^-)$, $\alpha' > 0$ on $\mathbb{R}^-$ and $\lim_{r\to-\infty} \alpha(r) = 0$ and some particular functions g. In [13], N. Igbida studies the case where α is the Heaviside maximal monotone graph. For $p = 2$, we obtain the heat equation, this equation is studied by many authors, see for example [14, 23] and the references therein. The elliptic case of problem (1) has been treated by many authors, see for example [3, 25, 26, 17], and the references therein.

We apply here a time discretization of given continuous problem by Euler forward scheme and we study existence, uniqueness and stability questions. We recall that the Euler forward scheme has been used by several authors while studying time discretization of nonlinear parabolic problems and we refer to the works [8, 9, 10, 15] for some details. This scheme is usually used to prove existence of solutions as well as to compute the numerical approximations.

The problem (1), or some special cases of it, arises in many different physical contexts, for example: Heat equation, non Newtonian fluids, diffusion phenomena, etc.

This paper is organized as follows: after some preliminary results in section 2, we discretize the problem (1) in section 3 by the Euler forward scheme and replace it by

$$\begin{aligned} U^n - \tau \triangle_p U^n + \tau\alpha(U^n) &= \tau f_n + U^{n-1} \ \ in\ \Omega, \\ |DU^n|^{p-2}\frac{\partial U^n}{\partial \eta} + \gamma(U^n) &= g_n \qquad on\ \partial\Omega, \\ U^0 &= u_0 \qquad\quad in\ \Omega, \end{aligned}$$

and show the existence and uniqueness of entropy solutions to the discretized problems. Section 4 is devoted to the analysis of stability of the discretized problem and in section 5 we study the asymptotic behavior of the solutions of the discrete dynamical system associated with the discretized problems. We shall finish this paper by showing the existence and uniqueness of entropy solution to the parabolic problem (1).

2 Preliminaries and Notations

In this section we give some notations, definitions and useful results we shall need in this work.
For a measurable set Ω of $\mathbb{R}^d$, $|\Omega|$ denotes its measure, the norm in $L^p(\Omega)$ is denoted by $\|.\|_p$ and $\|.\|_{1,p}$ denotes the norm in the Sobolev space $W^{1,p}(\Omega)$, C_i and C will denote various positive constants. For a Banach space X and $a < b$, $L^p(a, b; X)$ denotes the space of the measurable functions $u : [a,\ b] \to X$ such that

$$\left(\int_a^b \|u(t)\|_X^p \right)^{\frac{1}{p}} := \|u\|_{L^p(a,b;X)} < \infty.$$

For a given constant $k > 0$ we define the cut function $T_k : \mathbb{R} \to \mathbb{R}$ as

$$T_k(s) := \begin{cases} s & \text{if } |s| \le k, \\ k\ sign\ (s) & \text{if } |s| > k, \end{cases}$$

where

$$sign\ (s) := \begin{cases} 1 & \text{if } s > 0, \\ 0 & \text{if } s = 0, \\ -1 & \text{if } s < 0. \end{cases}$$

For a function $u = u(x)$, $x \in \Omega$, we define the truncated function $T_k u$ pointwise, i.e., for every $x \in \Omega$ the value of $(T_k u)$ at x is just $T_k(u(x))$.
Let the function $J_k : \mathbb{R} \to \mathbb{R}^+$ such that

$$J_k(x) = \int_0^x T_k(s) ds$$

(J_k it is the primitive function of T_k). We have

$$< \frac{\partial v}{\partial t}, T_k(v) >= \frac{d}{dt} \int_\Omega J_k(v) \quad in \quad L^1(]0,\ T[), \tag{2}$$

what implies that

$$\int_0^t < \frac{\partial v}{\partial s}, T_k(v) >= \int_\Omega J_k(v(t)) - \int_\Omega J_k(v(0)). \tag{3}$$

For $u \in W^{1,p}(\Omega)$, we denote by τu or u the trace of u on $\partial\Omega$ in the usual sense.
In ([4]) the authors introduce the following spaces

- $\mathcal{T}^{1,1}_{loc}(\Omega) = \left\{ u : \Omega \to \mathbb{R}\ \ measurable \ \ : \ \ T_k(u) \in W^{1,1}_{loc}(\Omega), \ \ for\ all \ \ k > 0 \right\},$

- $\mathcal{T}^{1,p}_{loc}(\Omega) = \left\{ u \in \mathcal{T}^{1,1}_{loc}(\Omega) \quad : DT_k(u) \in L^p_{loc}(\Omega), \quad for\ all \quad k > 0 \right\},$

- $\mathcal{T}^{1,p}(\Omega) = \left\{ u \in \mathcal{T}^{1,p}_{loc}(\Omega) \quad : \quad DT_k(u) \in L^p(\Omega), \quad for\ all \quad k > 0 \right\}.$

For bounded $\Omega' s$, we have

$$\mathcal{T}^{1,p}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \ \ measurable \quad T_k(u) \in W^{1,p}(\Omega), \quad for\ all \quad k > 0 \right\}.$$

Following [4], It is possible to give a sense to the derivative Du of a function $u \in \mathcal{T}^{1,p}_{loc}(\Omega)$, generalizing the usual concept of weak derivative in $W^{1,1}_{loc}(\Omega)$, thanks to the following result

Lemma 1 ([4]) *For every $u \in \mathcal{T}^{1,p}_{loc}(\Omega)$ there exist a unique measurable function $v : \Omega \to \mathbb{R}^d$ such that*

$$DT_k(u) = v\mathbf{1}_{\{|v|<k\}} \ \ a.e,$$

where $\mathbf{1}_B$ is the characteristic function of the measurable set $B \subset \mathbb{R}^d$.
Furthermore, $u \in W^{1,1}_{loc}(\Omega)$ if and only if $v \in L^1_{loc}(\Omega)$, and then $v \equiv Du$ in the usual weak sense.

We apply also the sets $\mathcal{T}^{1,p}_{tr}(\Omega)$ introduced in [2] as being the subset of functions in $\mathcal{T}^{1,p}(\Omega)$ for which a generalized notion of trace may be defined. More precisely $u \in \mathcal{T}^{1,p}_{tr}(\Omega)$ if $u \in \mathcal{T}^{1,p}(\Omega)$ and there exist a sequence $(u_n)_{n\in\mathbb{N}}$ in $W^{1,p}(\Omega)$ and a measurable function v on $\partial\Omega$ such that

a) $u_n \to u$ a.e. in Ω,

b) $DT_k(u_n) \to DT_k(u)$ in $L^1(\Omega)$ for every $k > 0$,

c) $u_n \to v$ a.e. on $\partial\Omega$.

The function v is the trace of u in the generalized sense introduced in [2]. For $u \in \mathcal{T}^{1,p}_{tr}(\Omega)$, the trace of u on $\partial\Omega$ is denoted by $tr(u)$ or u, the operator $tr(.)$ satisfied the following properties

i) if $u \in \mathcal{T}^{1,p}_{tr}(\Omega)$, then $\tau T_k(u) = T_k(tr(u)), \forall k > 0$,

ii) if $\varphi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, then $\forall u \in \mathcal{T}^{1,p}_{tr}(\Omega)$, we have $u - \varphi \in \mathcal{T}^{1,p}_{tr}(\Omega)$ and $tr(u - \varphi) = tr(u) - \tau\varphi$.

In the case where $u \in W^{1,p}(\Omega)$, $tr(u)$ coincides with τu.
Obviously, we have

$$W^{1,p}(\Omega) \subset \mathcal{T}^{1,p}_{tr}(\Omega) \subset \mathcal{T}^{1,p}(\Omega).$$

In [25], with Nonlinear Semigroup Theory, A. Siai demonstrated the following theorem

Theorem 2.1 ([25]) *If β, γ are non decreasing continuous functions on $\mathbb{R}$ such that $\beta(0) = \gamma(0) = 0$ and $f \in L^1(\Omega)$, $g \in L^1(\partial\Omega)$, then there exists an entropy solution $u \in T^{1,p}_{\mathrm{tr}}(\Omega)$ to the problem*

$$\begin{aligned} -\operatorname{div}[\mathbf{a}(., Du)] + \beta(u) &= f \quad \text{in } \Omega \\ \frac{\partial u}{\partial \nu_{\mathbf{a}}} + \gamma(\tau u) &= g \quad \text{on } \partial\Omega \end{aligned} \tag{4}$$

i.e. $\forall \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$

$$\int_\Omega a(., Du) DT_k(u-\varphi) + \int_\Omega \alpha(u) T_k(u-\varphi) + \int_{\partial\Omega} \gamma(u) T_k(u-\varphi) \leq \int_\Omega f T_k(u-\varphi)$$

$$+ \int_{\partial\Omega} g T_k(u-\varphi),$$

with $(\beta(u), \gamma(\tau u)) \in L^1(\Omega) \times L^1(\partial\Omega)$ and $\|(\beta(u), \gamma(\tau u))\|_1 \leq \|(f, g)\|_1$ and u is unique, up to an additive constant. Furthermore, if β or γ is one-to-one, then the entropy solution is unique. Where a is an operator of Leray-Lions type defined as follows

1) *$a : \Omega \times \mathbb{R}^d \to \mathbb{R}^d, (x, \xi) \mapsto a(x, \xi)$ is a Carathéodory function in the sense that a is continuous in ξ for almost every $x \in \Omega$, and measurable in x for every $\xi \in \mathbb{R}^d$.*

2) *There exists p, $1 < p < d$, and a constant $A_1 > 0$, so that,*

$$\langle a(x, \xi), \xi \rangle \geq A_1 |\xi|^p, \ \text{for a.e. } x \in \Omega \text{ and every } \xi \in \mathbb{R}^d.$$

3) *$\langle a(x, \xi_1) - a(x, \xi_2), \xi_1 - \xi_2 \rangle > 0$, if $\xi_1 \neq \xi_2$, for a.e. $x \in \Omega$.*

4) *There exist some $h_0 \in L^{p'}(\Omega)$, $p' = \frac{p}{p-1}$ and a constant $A_2 > 0$, such that*

$$|a(x, \xi)| \leq A_2 \left(h_0(x) + |\xi|^{p-1} \right) \ \text{for a.e. } x \in \Omega \text{ and every } \xi \in \mathbb{R}^d.$$

3 The semi-discrete problem

By the Euler forward scheme, we consider the following system

$$(Pn) \begin{cases} U^n - \tau \triangle_p U^n + \tau \alpha(U^n) = \tau f_n + U^{n-1} \text{ in } \Omega, \\ |DU^n|^{p-2} \dfrac{\partial U^n}{\partial \eta} + \gamma(U^n) = g_n \text{ on } \partial\Omega, \\ U^0 = u_0 \text{ in } \Omega, \end{cases}$$

where $N\tau = T$, $1 \leq n \leq N$ and $f_n(.) = \frac{1}{\tau} \int_{(n-1)\tau}^{n\tau} f(s, .) ds$, in Ω, $g_n(.) = \frac{1}{\tau} \int_{(n-1)\tau}^{n\tau} g(s, .) ds$ on $\partial\Omega$.
We assume the following hypotheses:

(H_1) α and γ are non decreasing continuous functions on $\mathbb{R}$ such that $\alpha(0) = \gamma(0) = 0$,

(H_2) $u_0 \in L^1(\Omega)$, $\quad f \in L^1(Q_T)$ and $g \in L^1(\Sigma_T)$.

Recently, in [4], a new concept of solution has been introduced for the elliptic equation

$$\begin{aligned} -\operatorname{div}[\mathbf{a}(x, Du)] &= f(x) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{5}$$

namely entropy solution. Following this idea we define the concept of entropy solution for the problems (Pn).

Definition 2 *An entropy solution to the discretized problems (Pn), is a sequence* $(U^n)_{0\leq n\leq N}$ *such that* $U^0 = u_0$ *and* U^n *is defined by induction as an entropy solution of the problem*

$$\begin{aligned} u - \tau\triangle_p u + \tau\alpha(u) &= \tau f_n + U^{n-1} \qquad in \;\; \Omega, \\ |Du|^{p-2}\frac{\partial u}{\partial \eta} + \gamma(u) &= g_n \qquad on \;\; \partial\Omega, \end{aligned}$$

i.e. $U^n \in \mathcal{T}^{1,p}_{tr}(\Omega)$ *and* $\forall \varphi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, $\forall k > 0$, *we have*

$$\int_\Omega U^n T_k(U^n - \varphi) \int_\Omega |DU^n|^{p-2} DU^n DT_k(U^n - \varphi) + \int_\Omega \tau\alpha(U^n) T_k(U^n - \varphi) +$$

$$\tau \int_{\partial\Omega} \gamma(U^n) T_k(U^n - \varphi) \leq \int_\Omega (\tau f_n + U^{n-1}) T_k(U^n - \varphi) + \tau \int_{\partial\Omega} g_n T_k(U^n - \varphi). \tag{6}$$

Lemma 3 *Let hypotheses* $(H_1) - (H_2)$ *be satisfied, if* $(U^n)_{0\leq n\leq N}$, $N \in \mathbb{N}$ *is an entropy solution of problems (Pn), then* $\forall n = 1, ..., N$, *we have* $U^n \in L^1(\Omega)$.

Proof. In inequality (6) we take $\varphi = 0$ as test function, we obtain

$$\tau \int_\Omega |DU^1|^{p-2} DU^1 DT_k(U^1) + \int_\Omega (\tau\alpha(U^1) + U^1) T_k(U^1) + \tau \int_{\partial\Omega} \gamma(U^1) T_k(U^1)$$

$$\leq \int_\Omega (\tau f_1 + u_0) T_k(U^1) + \tau \int_{\partial\Omega} g_1 T_k(U^1). \tag{7}$$

By assumption (H_1) and the properties of T_k, we get

$$\int_\Omega \tau\alpha(U^1) T_k(U^1) + \tau \int_{\partial\Omega} \gamma(U^1) T_k(U^1) \geq 0. \tag{8}$$

Now, since

$$\sum_{n=1}^{n=N} \tau \left(\|f_n\|_1 + \|g_n\|_{L^1(\partial\Omega)} \right) \leq \|f\|_{L^1(Q_T)} + \|g\|_{L^1(\Sigma_T)}, \tag{9}$$

and

$$\tau\int_\Omega |DU^1|^{p-2}DU^1 DT_k(U^1) = \tau\int_\Omega |DT_k(U^1)|^p \geq 0.$$

Thus, from inequality (7) we obtain,

$$0 \leq \int_\Omega U^1\frac{T_k(U^1)}{k} \leq \left(\|f\|_{L^1(Q_T)} + \|g\|_{L^1(\Sigma_T)} + \|u_0\|_1\right). \tag{10}$$

On the other hand, we have for each $x \in \Omega$

$$\lim_{k\to 0} U^1(x)\frac{T_k(U^1(x))}{k} = |U^1(x)|.$$

Then by Fatou's lemma, we deduce that $U^1 \in L^1(\Omega)$ and

$$\|U^1\|_1 \leq \|f\|_{L^1(Q_T)} + \|g\|_{L^1(\Sigma_T)} + \|u_0\|_1.$$

By induction, we deduce in the same manner that $U^n \in L^1(\Omega)$, $\forall n = 1, ..., N$. ■

Theorem 3.1 *Let hypotheses $(H_1) - (H_2)$ be satisfied and $1 < p < d$, then for all $N \in \mathbb{N}$ the problems (Pn) has a unique entropy solution $(U^n)_{0\leq n\leq N}$, such that for all $n = 1, ..., N$, $U^n \in \mathcal{T}^{1,p}_{tr}(\Omega) \cap L^1(\Omega)$.*

Proof. Existence. Let the problem

$$\begin{aligned} -\tau\triangle_p u + \overline{\alpha}(u) &= F \ in\ \Omega, \\ |Du|^{p-2}\frac{\partial u}{\partial \eta} + \gamma(u) &= G \ \ on\ \partial\Omega, \end{aligned} \tag{11}$$

where $u = U^1$, $F = \tau f_1 + u_0$ and $G = \tau g_1$. According to inequality (9) and hypothesis (H_2), we have $F \in L^1(\Omega)$, $G \in L^1(\partial\Omega)$ and, by hypothesis (H_1), the function defined by $\overline{\alpha}(s) = \tau\alpha(s) + s$ is non decreasing, continuous and satisfies $\overline{\alpha}(0) = 0$. Therefore, according to theorem 2.1, the problem (11) has an entropy solution U^1 in $\mathcal{T}^{1,p}_{tr}(\Omega)$.

By induction, using Lemma 3, we deduct in the same manner that for $n = 1, ..., N$, the problem

$$\begin{aligned} u - \tau\triangle_p u + \tau\alpha(u) &= \tau f_n + U^{n-1} \qquad in\ \ \Omega, \\ |Du|^{p-2}\frac{\partial u}{\partial \eta} + \gamma(u) &= g_n \qquad on\ \ \partial\Omega, \end{aligned}$$

has an entropy solution U^n in $\mathcal{T}^{1,p}_{tr}(\Omega) \cap L^1(\Omega)$.

Uniqueness. We firstly need the following lemma.

Lemma 4 *If $(U^n)_{0\leq n\leq N}$, $N \in \mathbb{N}$ is an entropy solution of (Pn), then for all $k > 0$, for all $n = 1, ..., N$ and for all $h > 0$, we have*

$$\tau\int_{\{h<|U^n|<k+h\}} |DU^n|^p \leq k\left(\int_{\{|U^n|>h\}} \tau|f_n| + \int_{\{|U^n|>h\}} |U^{n-1}| + \int_{\partial\Omega\cap\{|U^n|>h\}} \tau|g_n|\right).$$

Proof. Taking $\varphi = T_h(U^n)$ as test function in inequality (6), we have

$$\int_\Omega U^n T_k(U^n - T_h(U^n)) + \tau \int_\Omega |DU^n|^{p-2} DU^n DT_k(U^n - T_h(U^n))$$

$$+\tau \int_\Omega \alpha(U^n) T_k(U^n - T_h(U^n)) + \tau \int_{\partial\Omega} \gamma(U^n) T_k(U^n - T_h(U^n))$$

$$\leq \int_\Omega (\tau f_n + U^{n-1}) T_k(U^n - T_h(U^n)) + \tau \int_{\partial\Omega} g_n T_k(U^n - T_h(U^n)). \qquad (12)$$

By using the definition of T_k, we have

$$\int_\Omega U^n T_k(U^n - T_h(U^n)) \quad = \quad \int_{\overline{\Omega}_h} U^n T_k(U^n - hsign(U^n))$$

$$= \int_{\overline{\Omega}_h \cap \Omega_{(h,k)}} U^n (U^n - hsign(U^n)) + \int_{\overline{\Omega}_h \cap \overline{\Omega}_{(h,k)}} U^n sign(U^n - hsign(U^n)),$$

where

$$\overline{\Omega}_h = \{|U^n| > h\}, \Omega_{(h,k)} = \{|U^n - hsign(U^n)| \leq k\},$$

and

$$\overline{\Omega}_{(h,k)} = \{|U^n - hsign(U^n)| > k\}.$$

However

$$sign(U^n - hsign(U^n)) \mathbf{1}_{\overline{\Omega}_h} = sign(U^n) \mathbf{1}_{\overline{\Omega}_h};$$

Thus, we get

$$\int_\Omega U^n T_k(U^n - T_h(U^n)) \geq 0.$$

In the same manner, using the hypothesis (H_1) we obtain

$$\tau \int_\Omega \alpha(U^n) T_k(U^n - T_h(U^n)) + \tau \int_{\partial\Omega} \gamma(U^n) T_k(U^n - T_h(U^n)) \geq 0.$$

Now, we have

$$T_k\left(s - T_h(s)\right) = \begin{cases} s - hsign(s) & \text{if } h \leq |s| < k + h, \\ k & \text{if } |s| \geq k + h, \\ 0 & \text{if } |s| \leq h, \end{cases}$$

then, it follows that

$$\tau \int_{\{h<|U^n|<k+h\}} |DU^n|^p \leq k \left(\int_{\{|U^n|>h\}} \tau |f_n| + \int_{\{|U^n|>h\}} |U^{n-1}| + \int_{\partial\Omega \cap \{|U^n|>h\}} \tau |g_n| \right).$$

$\blacksquare$

Now, let $(U^n)_{0\leq n\leq N}$ and $(V^n)_{0\leq n\leq N}$, $N\in\mathbb{N}$ be two entropy solutions of problems (Pn) and let $\varphi\in W^{1,p}(\Omega)\cap L^\infty(\Omega)$ (for simplicity, we write $u = U^1,\ v=V^1$), then we have

$$\int_\Omega uT_k(u-\varphi)+\tau\int_\Omega |Du|^{p-2}DuDT_k(u-\varphi)+\tau\int_\Omega \alpha(u)T_k(u-\varphi)$$

$$+\tau\int_{\partial\Omega}\gamma(u)T_k(u-\varphi)\leq\int_\Omega(\tau f_n+U^{n-1})T_k(u-\varphi)+\tau\int_{\partial\Omega}g_nT_k(u-\varphi), \quad (13)$$

and

$$\int_\Omega vT_k(v-\varphi)+\tau\int_\Omega |Dv|^{p-2}DvDT_k(v-\varphi)+\tau\int_\Omega \alpha(v)T_k(v-\varphi)$$

$$+\tau\int_{\partial\Omega}\gamma(v)T_k(v-\varphi)\leq\int_\Omega(\tau f_n+U^{n-1})T_k(v-\varphi)+\tau\int_{\partial\Omega}g_nT_k(v-\varphi). \quad (14)$$

For the solution u, we take $\varphi=T_h(v)$ and for the solution v, we take $\varphi=T_h(u)$ as test functions and taking the limit as $h\to\infty$, we get by applying Dominated Convergence Theorem that

$$\int_\Omega (u-v)T_k(u-v)+\tau\lim_{h\to\infty}I_{k,h}+\tau\lim_{h\to\infty}J_{k,h}\leq 0, \quad (15)$$

where

$$I_{k,h}:=\int_\Omega |Du|^{p-2}DuDT_k(u-T_h(v))+\int_\Omega |Dv|^{p-2}DvDT_k(v-T_h(u)),$$

and

$$\begin{aligned} J_{k,h} \quad :=& \quad \int_\Omega \alpha(u)T_k(u-T_h(v))+\int_\Omega\alpha(v)T_k(v-T_h(u))+\int_{\partial\Omega}\gamma(u)T_k(u-T_h(v)) \\ &+ \quad \int_{\partial\Omega}\gamma(v)T_k(v-T_h(u)), \end{aligned}$$

by applying hypothesis (H_1), we get

$$\lim_{h\to\infty}J_{k,h}=\int_\Omega(\alpha(u)-\alpha(v))T_k(u-v)+\int_{\partial\Omega}(\gamma(u)-\gamma(v))T_k(u-v)\geq 0. \quad (16)$$

Now, we show that $\lim_{h\to\infty}I_{k,h}\geq 0$.

To prove this, we pose

$$\Omega_1(h)=\{|u|<h,\ |v|<h\},\qquad \Omega_2(h)=\{|u|<h,\ |v|\geq h\},$$

$$\Omega_3(h)=\{|u|\geq h,\ |v|<h\}\ \ \textit{and}\ \ \Omega_4(h)=\{|u|\geq h,\ |v|\geq h\},$$

and we spilt

$$I_{k,h} = I^1_{k,h} + I^2_{k,h} + I^3_{k,h} + I^4_{k,h},$$

where

$$\begin{aligned}
I^1_{k,h} &= \int_{\Omega_1(h)} \left(|Du|^{p-2}DuDT_k(u-v) + |Dv|^{p-2}DvDT_k(v-u)\right) \\
&= \int_{\Omega_1(h)} \left(|Du|^{p-2}Du - |Dv|^{p-2}Dv\right) DT_k(u-v) \\
&= \int_{\Omega_1(h)\cap\{|u-v|<k\}} \left(|Du|^{p-2}Du - |Dv|^{p-2}Dv\right)(Du - Dv) \geq 0,
\end{aligned}$$

and

$$I^2_{k,h} = \int_{\Omega_2(h)} |Du|^{p-2}DuDT_k(u - hsign(v)) + \int_{\Omega_2(h)} |Dv|^{p-2}DvDT_k(v-u).$$

We have

$$\int_{\Omega_2(h)} |Du|^{p-2}DuDT_k(u - hsign(v)) = \int_{\Omega_2(h)\cap\{|u-hsign(v)|<k\}} |Du|^p \geq 0,$$

and on the other hand, from the Hölder's inequality, we have

$$\left|\int_{\Omega_2(h)} |Dv|^{p-2}DvDT_k(v-u)\right|$$

$$\leq \left(\int_{\Omega(k,h)} |Dv|^p\right)^{\frac{1}{p'}} \left[\left(\int_{\Omega(k,h)} |Dv|^p\right)^{\frac{1}{p}} + \left(\int_{\Omega(k,h)} |Du|^p\right)^{\frac{1}{p}}\right]$$

$$\leq \left(\int_{\Omega_1(k,h)} |Dv|^p\right)^{\frac{1}{p'}} \left[\left(\int_{\Omega_1(k,h)} |Dv|^p\right)^{\frac{1}{p}} + \left(\int_{\Omega_2(k,h)} |Du|^p\right)^{\frac{1}{p}}\right],$$

where $\Omega(k,h) = \Omega_2(h)\cap\{|u-v|<k\}$, $\Omega_1(k,h) = \{h \leq |v| \leq h+k\}$, $\Omega_2(k,h) = \{h-k \leq |u| \leq h\}$ and $\frac{1}{p} + \frac{1}{p'} = 1$.
By lemma 4, we have

$$\tau\int_{\{h-k<|u|<h\}} |Du|^p \leq k\left(\int_{\{|u|>h-k\}} \tau|f_n| + \int_{\{|u|>h-k\}} |U^{n-1}| + \int_{\partial\Omega\cap\{|u|>h-k\}} \tau|g_n|\right).$$

Now, $\tau f_n \in L^1(\Omega)$, $\tau g_n \in L^1(\partial\Omega)$, $U^{n-1} \in L^1(\Omega)$ and $\lim_{h\to\infty} |\{|u| \geq h-k\}| = 0$, then

$$\lim_{h\to\infty}\int_{\{h-k<|u|<h\}} |Du|^p = 0.$$

In the same manner, we show that:

$$\lim_{h\to\infty}\int_{\{h<|v|<h+k\}}|Dv|^p = 0.$$

Hence

$$\lim_{h\to\infty} I^2_{k,h} \geq 0.$$

Similarly, we have

$$\lim_{h\to\infty} I^3_{k,h} = \lim_{h\to\infty}\int_{\Omega_3(h)}|Du|^{p-2}DuDT_k(u-v)+\int_{\Omega_3(h)}|Du|^{p-2}DuDT_k(v-hsign(u)) \geq 0.$$

Finally

$$\begin{aligned} I^4_{k,h} &= \int_{\Omega_4(h)}|Du|^{p-2}DuDT_k(u-hsign(v))+\int_{\Omega_4(h)}|Du|^{p-2}DuDT_k(v-hsign(u)) \\ &= \int_{\Omega_4(h)\cap\{|u-hsign(v)|<k\}}|Du|^p+\int_{\Omega_4(h)\cap\{|v-hsign(u)|<k\}}|Dv|^p \geq 0. \end{aligned}$$

It thus follows that

$$\lim_{h\to\infty} I_{k,h} \geq 0. \tag{17}$$

Therefore, by inequalities (15), (16) and (17), we get

$$\int_\Omega (u-v)T_k(u-v) \leq 0,$$

i.e.

$$\int_\Omega (u-v)\frac{1}{k}T_k(u-v) \leq 0.$$

Taking the limit as $k \to 0$, by Dominated Convergence Theorem, we get

$$\|u-v\|_1 \leq 0.$$

By induction, we prove that

$$\forall n = 1, ..., N, \quad \|U^n - V^n\|_1 = 0.$$

■

4 Stability

Now we give some a priori estimates for the discrete entropy solution $(U^n)_{1\leq n\leq N}$ which we use later to derive convergence results for the Euler forward scheme.

Theorem 4.1 *Let hypotheses $(H_1)-(H_2)$ be satisfied and $1<p<d$. Then, there exists a positive constant $C(u_0,f,g)$ depending on the data but not on N such that for all $n=1,...,N$, we have*

$$\begin{array}{ll} 1) & \|U^n\|_1 \le C(u_0,f,g), \\ 2) & \tau\sum_{i=1}^{n}\|\alpha(U^i)\|_1 + \tau\sum_{i=1}^{n}\|\gamma(U^i)\|_{L^1(\partial\Omega)} \le C(u_0,f,g), \\ 3) & \sum_{i=1}^{n}\|U^i-U^{i-1}\|_1 \le C(u_0,f,g), \\ 4) & \sum_{i=1}^{n}\tau\|T_k(U^i)\|_{1,p}^p \le k.C(u_0,f,g). \end{array}$$

Proof. 1) and 2): Let $\varphi=0$ as test function in inequality (6) and dividing by k, we obtain

$$\tau\frac{1}{k}\|DT_k(U^i)\|_p^p + \int_\Omega U^i\frac{1}{k}T_k(U^i) + \tau\int_\Omega \alpha(U^i)\frac{1}{k}T_k(U^i) + \tau\int_{\partial\Omega}\gamma(U^i)\frac{1}{k}T_k(U^i)$$

$$\le \tau\left(\|f_i\|_1 + \|g_i\|_{L^1(\partial\Omega)}\right) + \|U^{i-1}\|_1, \tag{18}$$

i.e.

$$\int_\Omega \left(U^i+\tau\alpha(U^i)\right)\frac{T_k(U^i)}{k} + \tau\int_{\partial\Omega}\gamma(U^i)\frac{T_k(U^i)}{k} \le \tau\left(\|f_i\|_1 + \|g_i\|_{L^1(\partial\Omega)}\right) + \|U^{i-1}\|_1, \tag{19}$$

Let $k\to 0$, by the properties of T_k and the Dominated Convergence Theorem we get,

$$\tau\|\alpha(U^i)\|_1 + \|U^i\|_1 + \tau\|\gamma(U^i)\|_{L^1(\partial\Omega)} \le \tau\left(\|f_i\|_1 + \|g_i\|_{L^1(\partial\Omega)}\right) + \|U^{i-1}\|_1. \tag{20}$$

Summing (20) from $i=1$ to n we obtain

$$\begin{aligned} \|U^n\|_1 + \tau\sum_{i=1}^{n}\left(\|\alpha(U^i)\|_1 + \|\gamma(U^i)\|_{L^1(\partial\Omega)}\right) &\le \sum_{i=1}^{n}\tau\left(\|f_i\|_1 + \|g_i\|_{L^1(\partial\Omega)}\right) + \|u_0\|_1 \\ &\le \|f\|_{L^1(Q_T)} + \|g\|_{L^1(\Sigma_T)} + \|u_0\|_1. \end{aligned}$$

Then inequalities 1) and 2) are satisfied.
3) Taking $\varphi = T_h\left(U^i - sign(U^i-U^{i-1})\right)$ as test function in inequality (6) and using the fact that:

$$\int_\Omega |DU^i|^{p-2}DU^iDT_k\left(U^i - T_h\left(U^i - sign(U^i-U^{i-1})\right)\right) = \int_{\Omega_k\cap\overline{\Omega}_h}|DU^i|^p \ge 0,$$

where

$$\Omega_3(k,h) = \left\{|U^i - T_h(U^i - sign(U^i-U^{i-1})| \le k\right\}$$

and

$$\overline{\Omega}_h = \left\{ |U^i - sign(U^i - U^{i-1})| > h \right\},$$

we obtain

$$\int_\Omega \left(U^i - U^{i-1}\right) T_k \left(U^i - T_h \left(U^i - sign(U^i - U^{i-1})\right)\right)$$

$$\leq k \left(\tau\|f_i\|_1 + \tau\|g_i\|_{L^1(\partial\Omega)} + \tau\|\alpha(U^i)\|_1 + \tau\|\gamma(U^i)\|_{L^1(\partial\Omega)}\right).$$

Taking the limit as $h \to \infty$ and using the Dominated Convergence Theorem, we get for $k = 1$

$$\|U^i - U^{i-1}\|_1 \leq \tau\|f_i\|_1 + \tau\|g_i\|_{L^1(\partial\Omega)} + \tau\|\alpha(U^i)\|_1 + \tau\|\gamma(U^i)\|_{L^1(\partial\Omega)}. \tag{21}$$

Summing (21) from $i = 1$ to n and applying the stability result 2) and inequality (9), we obtain

$$\sum_{i=1}^{n} \|U^i - U^{i-1}\|_1 \leq 2\|f\|_{L^1(Q_T)} + 2\|g\|_{L^1(\Sigma_T)} + \|u_0\|_1.$$

4) Taking $\varphi = 0$ as test function in inequality (6), we deduce by (8) that

$$\tau\|DT_k(U^i)\|_p^p \leq k \left(\tau\|f_i\|_1 + \tau\|g_i\|_{L^1(\partial\Omega)} + \|U^i - U^{i-1}\|_1\right). \tag{22}$$

Summing (22) from $i = 1$ to n and applying the stability result 3), we therefore get

$$\sum_{i=1}^{n} \tau\|DT_k(U^i)\|_p^p \leq k.C(u_0, f, g),.$$

Hence, by using Sobolev's inequality we deduct the stability result 4). ■

5 The semi-discrete dynamical system

This section aims to study the discrete dynamical system. We show existence of absorbing sets in $L^1(\Omega)$ and of the global attractor. (We refer to [27] for the definition of absorbing sets and global attractor).
By the results of theorem (3.1), problems (Pn) generates a continuous semigroup S_τ defined by

$$S_\tau U^{n-1} = U^n.$$

Proposition 5 *Let hypotheses $(H_1) - (H_2)$ be satisfied and $1 < p < d$. Then for τ small enough, there exists absorbing sets in $L^1(\Omega)$. More precisely, there exists a positive integer n_τ such that*

$$\|U^n\|_1 \leq C, \quad \forall n \geq n_\tau. \tag{23}$$

where C does not depend on τ.

Proof. By inequality (20), we have

$$y^n \leq y^{n-1} + \tau h_n,$$

where $y^n = \|U^n\|_1$ and $h_n = \|f_n\|_1 + \|g_n\|_{L^1(\partial\Omega)}$.
On the other hand, according to the stability results of theorem 4.1, there exists $n_\tau > 0$ such that

$$\tau \sum_{n=n_0}^{n=n_0+N} y_n \leq C_6 \quad \forall n_0 \geq n_\tau, \tag{24}$$

where C_6 does not depend on n_0.
By inequality (9), we have

$$\tau \sum_{n=n_0}^{n=n_0+N} \|h_n\|_1 \leq C_7 \ \ \forall n_0 \geq n_\tau.$$

Now, applying the discrete Gronwall's lemma [8, lemma 7.5], we therefore get

$$\|U^n\|_1 \leq C_8 \quad \forall n \geq n_\tau,$$

where C_8 is a constant not depending on τ.
Which implies the existence of absorbing sets in $L^1(\Omega)$. ■

Applying [27, theorem 1.1], we get the following result.

Corollary 6 *Let hypotheses $(H_1)-(H_2)$ be satisfied and $1 < p < d$. Then for τ small enough, the semigroup associated with problems (Pn) possesses a compact attractor $\mathcal{A}_\tau$ which is bounded in $L^1(\Omega)$.*

6 Convergence and existence result

Definition 7 *A function measurable $u : Q_T \to \mathbb{R}$ is an entropy solution of parabolic problem (1) in Q_T if $u \in C(0,T;L^1(\Omega))$, $T_k(u) \in L^p(0,T;W^{1,p}(\Omega))$ for all $k > 0$, and*

$$\begin{aligned}
&\int_0^t \int_\Omega |Du|^{p-2} Du DT_k(u-\varphi) + \int_0^t \int_\Omega \alpha(u) T_k(u-\varphi) + \int_0^t \int_{\partial\Omega} \gamma(u) T_k(u-\varphi) \\
&\quad \leq \ -\int_0^t \left\langle \frac{\partial \varphi}{\partial s},\ T_k(u-\varphi) \right\rangle + \int_\Omega J_k(u(0)-\varphi(0)) - \int_\Omega J_k(u(t)-\varphi(t)) \\
&\qquad + \ \int_0^t \int_\Omega f T_k(u-\varphi) + \int_0^t \int_{\partial\Omega} g T_k(u-\varphi),
\end{aligned} \tag{25}$$

for all $\varphi \in L^\infty(Q_T) \cap L^p(0,T;W^{1,p}(\Omega)) \cap W^{1,1}(0,T;L^1(\Omega))$ and $t \in [0,\ T]$.

Now, we state our main result of this work.

Theorem 6.1 *Let hypotheses $(H_1)-(H_2)$ be satisfied and $1<p<d$. Then the nonlinear parabolic problem* (1) *admits a unique entropy solution.*

Proof. Existence. Let us introduce a piecewise linear extension, called Rothe function, by

$$\begin{cases} u^N(0) := u_0, \\ u^N(t) := U^{n-1} + (U^n - U^{n-1})\frac{(t-t^{n-1})}{\tau}, \quad \forall\, t \in\,]t^{n-1},\ t^n], \quad n=1,...,N \ \ in\ \Omega, \end{cases} \tag{26}$$

and a piecewise constant function

$$\begin{cases} \overline{u}^N(0) := u_0, \\ \overline{u}^N(t) := U^n \quad \forall\, t \in\,]t^{n-1},\ t^n], \quad n=1,...,N \ \ in\ \Omega, \end{cases} \tag{27}$$

where $t^n := n\tau$.
As already shown, for any $N \in \mathbb{N}$, the solution $(U^n)_{1\le n\le N}$ of problems (Pn) is unique. Thus, u^N and $\overline{u}^N$ are uniquely defined and by construction, we have for any $t \in]t^{n-1},\ t^n]$ and $n=1,...,N$, that

$$\begin{aligned} &1) \quad \frac{\partial u^N(t)}{\partial t} = \frac{(U^n - U^{n-1})}{\tau}, \\ &2) \quad \overline{u}^N(t) - u^N(t) = (U^n - U^{n-1})\frac{t^n - t}{\tau}. \end{aligned}$$

By using the stability results of theorem 4.1, we deduce the following a priori estimates concerning the Rothe function u^N and the function $\overline{u}^N$.

Lemma 8 *Let hypotheses $(H_1)-(H_2)$ be satisfied and $1<p<d$. Then there exists a constant $C(T,u_0,f,g)$ not depending on N such that for all $N\in\mathbb{N}$, we have*

$$\|\overline{u}^N - u^N\|_{L^1(Q_T)} \le \frac{1}{N} C(T,u_0,f,g), \tag{28}$$

$$\|u^N\|_{L^1(Q_T)} \le C(T,u_0,f,g), \tag{29}$$

$$\|\overline{u}^N\|_{L^1(Q_T)} \le C(T,u_0,f,g), \tag{30}$$

$$\|\frac{\partial u^N}{\partial t}\|_{L^1(Q_T)} \le C(T,u_0,f,g), \tag{31}$$

$$\left\|T_k(\overline{u}^N)\right\|_{L^p(0,T,W^{1,p}(\Omega))} \le k.C(T,u_0,f,g). \tag{32}$$

Proof. We have

$$\begin{aligned} \|\overline{u}^N - u^N\|_{L^1(Q_T)} &= \sum_{n=1}^{N} \int_{t^{n-1}}^{t^n} \|U^n - U^{n-1}\|_1 \frac{(t^n - t)}{\tau} dt \\ &= \frac{\tau}{2} \sum_{n=1}^{N} \|U^n - U^{n-1}\|_1. \end{aligned}$$

Using inequality 4) of theorem 4.1, we deduce that

$$\|\overline{u}^N - u^N\|_{L^1(Q_T)} \leq \frac{1}{2N} TC(u_0, f, g).$$

In the same manner, we prove the estimates (29), (30), (31) and (32). ∎

Using estimates (29) and (31), we deduct that

$$\textit{the sequence } (u^N)_{N\in\mathbb{N}} \textit{ is relatively compact in } L^1(Q_T).$$

This implies the existence of a subsequence of $(u^N)_{N\in\mathbb{N}}$ converging to an element u in $L^1(Q_T)$.
And by estimate (28), we deduce hence that

$$\textit{the sequence } (\overline{u}^N)_{N\in\mathbb{N}} \textit{ converges to } u \textit{ in } L^1(Q_T).$$

On the other hand, by (32) we have that

$$\left(DT_k(\overline{u}^N)\right)_{N\in\mathbb{N}} \textit{ is uniformly bounded in } L^p(Q_T).$$

Hence there exists a subsequence, still denoted by $\left(DT_k(\overline{u}^N)\right)_{N\in\mathbb{N}}$ such that

$$\left(DT_k(\overline{u}^N)\right)_{N\in\mathbb{N}} \textit{ converges to an element } V \textit{ in } L^p(Q_T).$$

However

$$T_k(\overline{u}^N) \textit{ converges to } T_k(u) \textit{ in } L^p(Q_T).$$

Hence, it follows that

$$DT_k(\overline{u}^N) \textit{ converges to } DT_k(u) \textit{ weakly in } L^p(Q_T),$$

and by (32) we conclude that

$$T_k(u) \in L^p(0,T;W^{1,p}(\Omega)) \textit{ for all } k > 0.$$

We follow the same technique used in [1] to show that

$$\overline{u}^N \textit{ converges to } u \textit{ on } \Sigma_T.$$

Lemma 9 *The sequence $(u^N)_{N\in\mathbb{N}}$ converges to u in $C\left(0,\ T; L^1(\Omega)\right)$.*

Proof. Let $\varphi \in L^\infty(Q_T) \cap L^p(0,T;W^{1,p}(\Omega)) \cap W^{1,1}(0,T;L^1(\Omega))$, we rewrite (6) in the form

$$\int_0^t \left\langle \frac{\partial u^N}{\partial s}, T_k(\overline{u}^N - \varphi) \right\rangle + \int_0^t \int_\Omega |D\overline{u}^N|^{p-2} D\overline{u}^N DT_k(\overline{u}^N - \varphi)$$

$$+ \int_0^t \int_\Omega \alpha(\overline{u}^N) T_k(\overline{u}^N - \varphi) + \int_0^t \int_{\partial\Omega} \gamma(\overline{u}^N) T_k(\overline{u}^N - \varphi)$$

$$\leq \int_0^t \int_\Omega f_N T_k(\overline{u}^N - \varphi) + \int_0^t \int_{\partial\Omega} g_N T_k(\overline{u}^N - \varphi), \tag{33}$$

where $f_N(t,x) = f_n(x)$, $g_N(t,x) = g_n(x)$ $\forall t \in]t^{n-1},\ t^n]$, $n = 1, ..., N$.
Let $(t^n = n\tau_N)_{n=1}^N$ and $(t^m = m\tau_M)_{m=1}^M$ be two partitions of interval $[0,\ T]$ and let $\left(u^N(t), \overline{u}^N(t)\right)$, $\left(u^M(t), \overline{u}^M(t)\right)$ be the semi-discrete solutions defined by (26), (27) and corresponding to the partitions, respectively. The same method used in the proof of the uniqueness in the theorem 3.1, enables us to obtain for $k = 1$

$$\int_0^t \left\langle \frac{\partial(u^N - u^M)}{\partial s},\ T_1(\overline{u}^N - \overline{u}^M) \right\rangle \leq \int_0^t \int_\Omega |f_N - f_M| + \int_0^t \int_{\partial\Omega} |g_N - g_M|,$$

that is

$$\begin{aligned} \int_\Omega J_1\left(u^N(t) - u^M(t)\right) \quad &\leq \quad \left| \int_0^t \left\langle \frac{\partial(u^N - u^M)}{\partial s},\ T_1(\overline{u}^N - \overline{u}^M) - T_1(u^N - u^M) \right\rangle \right| \\ &+ \quad \|f_N - f_M\|_{L^1(Q_T)} + \|g_N - g_M\|_{L^1(\Sigma)}. \end{aligned}$$

However,

$$\begin{aligned} &\left| \int_0^t \left\langle \frac{\partial(u^N - u^M)}{\partial s},\ T_1(\overline{u}^N - \overline{u}^M) - T_1(u^N - u^M) \right\rangle \right| \\ &\leq \quad \left\| \frac{\partial(u^N - u^M)}{\partial s} \right\|_{L^1(Q_T)} \|T_1(\overline{u}^N - \overline{u}^M) - T_1(u^N - u^M)\|_{L^\infty(Q_T)} \\ &\leq \quad 2C(T, f, g, u_0)\|T_1(\overline{u}^N - \overline{u}^M) - T_1(u^N - u^M)\|_{L^\infty(Q_T)}. \end{aligned}$$

Now, as

$$\lim_{N,M\to\infty} \|T_1(\overline{u}^N - \overline{u}^M) - T_1(u^N - u^M)\|_{L^\infty(Q_T)} = 0,$$

we get

$$\lim_{N,M\to\infty} \left| \int_0^t \left\langle \frac{\partial(u^N - u^M)}{\partial s},\ T_1(\overline{u}^N - \overline{u}^M) - T_1(u^N - u^M) \right\rangle \right| = 0. \tag{34}$$

On the other hand, we have

$$\lim_{N,M\to\infty} \left(\|f_N - f_M\|_{L^1(Q_T)} + \|g_N - g_M\|_{L^1(\Sigma)} \right) = 0,$$

then, we obtain

$$\lim_{N,M\to\infty} \int_\Omega J_1\left(u^N(t) - u^M(t)\right) = 0. \tag{35}$$

Now, using the definition of J_k we have

$$\int_{\{|u^N - u^M| \leq 1\}} |u^N(t) - u^M(t)|^2 + \frac{1}{2} \int_{\{|u^N - u^M| \geq 1\}} |u^N(t) - u^M(t)| \leq \int_\Omega J_1\left(u^N(t) - u^M(t)\right).$$

Therefore, we obtain

$$\int_\Omega |u^N(t) - u^M(t)| = \int_{\{|u^N - u^M| \le 1\}} |u^N(t) - u^M(t)| + \int_{\{|u^N - u^M| \ge 1\}} |u^N(t) - u^M(t)|$$

$$\le C(\Omega) \left(\int_{\{|u^N - u^M| \le 1\}} |u^N(t) - u^M(t)|^2 \right)^{\frac{1}{2}} + \int_{\{|u^N - u^M| \ge 1\}} |u^N(t) - u^M(t)|$$

$$\le C(\Omega) \left(\int_\Omega J_1 \left(u^N(t) - u^M(t)\right) \right)^{\frac{1}{2}} + 2 \int_\Omega J_1 \left(u^N(t) - u^M(t)\right).$$

Then by (35), we conclude that $(u^N)_{N\in\mathbb{N}}$ is a Cauchy sequence in $C(0,\ T; \mathrm{L}^1(\Omega))$; Which implies that

$$(u^N)_{N\in\mathbb{N}} \textit{ converges to } u \textit{ in } C(0,\ T; \mathrm{L}^1(\Omega)). \tag{36}$$

■

It remains to prove that the limit function u is an entropy solution of the problem (1). Since $u^N(0) = U^0 = u_0$ for all $N \in \mathbb{N}$, then $u(0,.) = u_0$.
By (33) we get

$$\int_0^t \left\langle \frac{\partial u^N}{\partial s}, T_k(\overline{u}^N - \varphi) - T_k(u^N - \varphi) \right\rangle + \int_0^t \int_\Omega |D\overline{u}^N|^{p-2} D\overline{u}^N DT_k(\overline{u}^N - \varphi) +$$

$$\int_0^t \int_\Omega \alpha(\overline{u}^N) T_k(\overline{u}^N - \varphi) + \int_0^t \int_{\partial\Omega} \gamma(\overline{u}^N) T_k(\overline{u}^N - \varphi) \le - \int_0^t \left\langle \frac{\partial \varphi}{\partial s}, T_k(u^N - \varphi) \right\rangle$$

$$+ \int_\Omega J_k(u^N(0) - \varphi(0)) - \int_\Omega J_k(u^N(t) - \varphi(t)) + \int_0^t \int_\Omega f_N T_k(\overline{u}^N - \varphi)$$

$$+ \int_0^t \int_{\partial\Omega} g_N T_k(\overline{u}^N - \varphi). \tag{37}$$

By same manner, as used for the proof of the equality (34), we deduce that

$$\lim_{N\to\infty} \left(\int_0^t \left\langle \frac{\partial u^N}{\partial s}, T_k(\overline{u}^N - \varphi) - T_k(u^N - \varphi) \right\rangle \right) = 0. \tag{38}$$

We follow the same technique used in [19], we show that

$$\lim_{N\to\infty} \int_0^t \int_\Omega |D\overline{u}^N|^{p-2} D\overline{u}^N DT_k(\overline{u}^N - \varphi) = \int_0^t \int_\Omega |Du|^{p-2} DuDT_k(u - \varphi). \tag{39}$$

And by Lemma 9, we deduce that $u^N(t) \to u(t)$ in $L^1(\Omega)$ for all $t \in [0,\ T]$, which implies that

$$\int_\Omega J_k(u^N(t) - \varphi(t)) \to \int_\Omega J_k(u(t) - \varphi(t)) \qquad \forall t \in [0,\ T]. \tag{40}$$

Finally, taking the limits as $N \to \infty$, and using the above results, the continuities of α, γ and the facts that $f_N \to f$ in $L^1(Q_T)$, $g_N \to g$ in $L^1(\Sigma_T)$ and $T_k(\overline{u}^N - \varphi) \to T_k(u - \varphi)$ in $L^\infty(Q_T)$, we deduce that u is an entropy solution of the nonlinear parabolic problem (1).

Uniqueness. Let v another entropy solution of the nonlinear parabolic problem (1). Taking $\varphi = T_h(u^N)$ as test function in (25) and letting $h \to \infty$, we get

$$\begin{aligned}
&\int_\Omega J_k(v(t) - u^N(t)) + \int_0^t \left\langle \frac{\partial u^N}{\partial s},\ T_k(v - u^N) \right\rangle + \lim_{h\to\infty} \mathcal{II}_1^N(k,h) \\
&\qquad + \int_0^t \int_\Omega \alpha(v) T_k(v - u^N) + \int_0^t \int_{\partial\Omega} \gamma(v) T_k(v - u^N) \\
&\qquad \le \int_0^t \int_\Omega f T_k(v - u^N) + \int_0^t \int_{\partial\Omega} g T_k(v - u^N);
\end{aligned} \tag{41}$$

where

$$\mathcal{II}_1^N(k,h) = \int_0^t \int_\Omega |Dv|^{p-2} Dv D T_k(v - T_h(u^N)).$$

On the other hand, taking $\varphi = T_h(v)$ as a test function in the inequality (33) and taking the limit as $h \to \infty$, we get

$$\begin{aligned}
&\int_0^t \left\langle \frac{\partial u^N}{\partial s},\ T_k(\overline{u}^N - v) \right\rangle + \lim_{h\to\infty} \mathcal{II}_2^N(k,h) + \int_0^t \int_\Omega \alpha(\overline{u}^N) T_k(\overline{u}^N - v) + \\
&\int_0^t \int_{\partial\Omega} \gamma(\overline{u}^N) T_k(\overline{u}^N - v) \le \int_0^t \int_\Omega f_N T_k(\overline{u}^N - v) + \int_0^t \int_{\partial\Omega} g_N T_k(\overline{u}^N - v),
\end{aligned} \tag{42}$$

where

$$\mathcal{II}_2^N(k,h) = \int_0^t \int_\Omega |D\overline{u}^N|^{p-2} D\overline{u}^N D T_k(\overline{u}^N - T_h(v)).$$

Adding (41) and (42), we get

$$\begin{aligned}
&\int_\Omega J_k(v(t) - u^N(t)) + \int_0^t \left\langle \frac{\partial u^N}{\partial s},\ T_k(v - u^N) + T_k(\overline{u}^N - v) \right\rangle \\
&+ \lim_{h\to\infty} \mathcal{II}^N(k,h) + \int_0^t \int_\Omega \left[\alpha(v) T_k(v - u^N) + \alpha(\overline{u}^N) T_k(\overline{u}^N - v)\right]
\end{aligned}$$

$$+\int_0^t \int_{\partial\Omega} \left[\gamma(v)T_k(v-u^N)+\gamma(\overline{u}^N)T_k(\overline{u}^N-v)\right]$$

$$\leq \int_0^t \int_\Omega \left[fT_k(v-u^N)+f_N T_k(\overline{u}^N-v)\right] + \int_0^t \int_{\partial\Omega} \left[gT_k(v-u^N)+g_N T_k(\overline{u}^N-v)\right],$$

where

$$\mathcal{II}^N(k,h) = \mathcal{II}_1^N(k,h) + \mathcal{II}_2^N(k,h).$$

Taking the limit as $N \to \infty$, using the above convergence results and the hypothesis (H_1), we get

$$\int_\Omega J_k(v(t)-u(t)) + \lim_{N\to\infty} \lim_{h\to\infty} \mathcal{II}^N(k,h) \leq 0. \tag{43}$$

Applying the technique used in the proof of uniqueness in theorem 3.1, we deduce that

$$\lim_{N\to\infty} \lim_{h\to\infty} \mathcal{II}^N(k,h) \geq 0. \tag{44}$$

Therefore the inequality (43) becomes

$$\int_\Omega J_k(v(t)-u(t)) \leq 0.$$

i.e.

$$\int_\Omega \frac{J_k(v(t)-u(t))}{k} \leq 0.$$

However

$$\lim_{k\to 0} \frac{J_k(x)}{k} = |x|.$$

Then, by Fatou's lamma, we get

$$\|v(t)-u(t)\|_1 \leq 0, \quad \forall t \in [0,\ T].$$

■

Remark 10 *The above results can be generalized, for example if the p-Laplacian operator $\Delta_p u$ is replaced by the operator $a(.,Du)$ defined in the theorem 2.1.*

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