Electronic Journal of Qualitative Theory of Differential Equations
2014, No. **37**, 1–14; http://www.math.u-szeged.hu/ejqtde/

New results on almost periodic solutions for a Nicholson's blowflies model with a linear harvesting term

Aiping Zhang ✉

School of Science, Hunan University of Technology, Zhuzhou, Hunan 412000, P. R. China

Received 4 February 2014, appeared 31 July 2014
Communicated by Leonid Berezansky

Abstract. In this paper, we study the global dynamic behavior of a non-autonomous delayed Nicholson's blowflies model with a linear harvesting term. Under proper conditions, we employ a novel argument to establish a criterion on the global exponential stability of positive almost periodic solutions for the model. Moreover, we also provide a numerical example to support the theoretical results.

Keywords: Nicholson's blowflies model, linear harvesting term, positive almost periodic solution, global exponential stability.

2010 Mathematics Subject Classification: 34C25, 34K13.

1 Introduction

Since the exploitation of biological resources and the harvest of population species are commonly practiced in fishery, forestry, and wildlife management, the study of population dynamics with harvesting is an important subject in mathematical bioeconomics, which is related to the optimal management of renewable resources (see [3,7,13]). Recently, Berezansky et al. [2] presented the following Nicholson's blowflies model

$$x'(t) = -\delta x(t) + px(t-\tau)e^{-ax(t-\tau)} - Hx(t-\sigma),\ \delta, p, \tau, a, H, \sigma \in (0,\ +\infty), \tag{1.1}$$

where $Hx(t-\sigma)$ is a linear harvesting term, $x(t)$ is the size of the population at time t, p is the maximum per capita daily egg production, $\frac{1}{a}$ is the size at which the population reproduces at its maximum rate, δ is the per capita daily adult death rate, and τ is the generation time. Furthermore, the authors in [1, 12, 15, 16] extended (1.1) to non-autonomous equations with periodic time-varying coefficient and delays, and they also established some criteria to guarantee the existence of positive periodic solutions for these generalized models by applying the method of coincidence degree and the fixed-point theorem in cones. L. Berezansky et al. [2] formulated an open problem: what can be said about the dynamic behavior of (1.1).

✉ Email: aipingzhang2012@aliyun.com

It is well known that the focus in theoretical models of population and community dynamics must not be only on how populations depend on their own population densities or the population densities of other organisms, but also on how populations change in response to the physical environment. To consider almost periodic environmental factors, it is reasonable to study the non-autonomous Nicholson's blowflies model with almost periodic coefficients and delays. Most recently, some sufficient conditions were obtained in [10, 11, 14] to ensure the local existence and exponential stability of positive almost periodic solution for the non-autonomous Nicholson's blowflies model with a linear harvesting term. However, as pointed out by Liu [8], it is difficult to study the global dynamic behavior of the Nicholson's blowflies model with a linear harvesting term. So far, there is no literature considering the global exponential stability of positive almost periodic solutions for (1.1) and its generalized equations. Thus, it is worthwhile to continue to investigate the global dynamic behavior of positive almost periodic solutions for non-autonomous Nicholson's blowflies model with the linear harvesting term.

Motivated by the above discussions, the main purpose of this paper is to establish some criteria for the global dynamic behavior of positive almost periodic solutions for a general Nicholson's blowflies model with the linear harvesting term given by

$$\begin{aligned} x'(t) = & -a(t)x(t) + \sum_{j=2}^{m} \beta_j(t)x(t-\tau_j(t))e^{-\gamma_j(t)x(t-\tau_j(t))} \\ & + \beta_1(t)x(t-\tau_1(t))e^{-\gamma_1(t)x(t)} - H(t)x(t-\sigma(t)), \end{aligned} \tag{1.2}$$

where $a, H, \sigma, \gamma_j \colon R \to (0, +\infty)$ and $\beta_j, \tau_j \colon R \to [0, +\infty)$ are almost periodic functions for $j = 1, 2, \ldots, m$. Obviously, (1.1) is a special case of (1.2) with constant coefficients and delays.

For convenience, we introduce some notations. In the following part of this paper, given a bounded continuous function g defined on R, let g^+ and g^- be defined as

$$g^+ = \sup_{t \in R} |g(t)|, \quad g^- = \inf_{t \in R} |g(t)|.$$

It will be assumed that

$$\gamma_j^- \geq 1 \ (j = 1, 2, \ldots, m), \quad r := \max\{\max_{1 \leq j \leq m} \tau_j^+, \quad \sigma^+\}. \tag{1.3}$$

Let $C = C([-r, 0], R)$ be the continuous functions space equipped with the usual supremum norm $\|\cdot\|$, and let $C_+ = C([-r, 0], (0, +\infty))$. If $x(t)$ is continuous and defined on $[-r + t_0, \sigma)$ with $t_0, \sigma \in R$, then we define $x_t \in C$, where $x_t(\theta) = x(t + \theta)$ for all $\theta \in [-r, 0]$.

We also consider admissible initial conditions

$$x_{t_0} = \varphi, \quad \varphi \in C_+. \tag{1.4}$$

We denote by $x_t(t_0, \ \varphi)(x(t; t_0, \varphi))$ an admissible solution of the admissible initial value problem (1.2) and (1.4). Also, let $[t_0, \eta(\varphi))$ be the maximal right-interval of the existence of $x_t(t_0, \varphi)$.

Since the function $\frac{1-x}{e^x}$ is decreasing on the interval $[0, 1]$ with the range $[0, 1]$, it follows easily that there exists a unique $\kappa \in (0, 1)$ such that

$$\frac{1-\kappa}{e^\kappa} = \frac{1}{e^2}. \tag{1.5}$$

Obviously,

$$\sup_{x\geq\kappa}\left|\frac{1-x}{e^x}\right| = \frac{1}{e^2}. \tag{1.6}$$

Moreover, since xe^{-x} increases on $[0,1]$ and decreases on $[1,+\infty)$, let $\widetilde{\kappa}$ be the unique number in $(1,+\infty)$ such that

$$\kappa e^{-\kappa} = \widetilde{\kappa}e^{-\widetilde{\kappa}}. \tag{1.7}$$

2 Preliminary results

In this section, we shall first recall some basic definitions, lemmas which are used in what follows.

Definition 2.1 ([4,5])**.** A continuous function $u\colon R \to R$ is said to be **almost periodic on** R if, for any $\epsilon > 0$, the set $T(u,\epsilon) = \{\delta : |u(t+\delta) - u(t)| < \epsilon \text{ for all } t \in R\}$ is relatively dense, i.e., for any $\epsilon > 0$, it is possible to find a real number $l = l(\epsilon) > 0$ with the property that, for any interval with length $l(\epsilon)$, there exists a number $\delta = \delta(\epsilon)$ in this interval such that $|u(t+\delta) - u(t)| < \epsilon$ for all $t \in R$.

From the theory of almost periodic functions in [4,5], it follows that for any $\epsilon > 0$, it is possible to find a real number $l = l(\epsilon) > 0$, for any interval with length $l(\epsilon)$, there exists a number $\delta = \delta(\epsilon)$ in this interval such that

$$\begin{cases} |a(t+\delta) - a(t)| < \epsilon,\ |H(t+\delta) - H(t)| < \epsilon,\ |\beta_j(t+\delta) - \beta_j(t)| < \epsilon, \\ |\tau_j(t+\delta) - \tau_j(t)| < \epsilon,\ \ |\gamma_j(t+\delta) - \gamma_j(t)| < \epsilon,\ \ |\sigma(t+\delta) - \sigma(t)| < \epsilon, \end{cases} \tag{2.1}$$

for all $t \in R$ and $j = 1,2,\ldots,m$.

Lemma 2.2 ([8, Theorem 2.1])**.** *Assume that*

$$\inf_{t\in R}\left\{\beta_1(t)e^{-\widetilde{\kappa}} - H(t)\right\} > 0, \quad \textit{and} \ \ \tau_1(t) \equiv \sigma(t) \ \ \textit{for all} \ \ t \in R. \tag{2.2}$$

Then, the solution $x_t(t_0,\varphi) \in C_+$ for all $t \in [t_0,\eta(\varphi))$, the set of $\{x_t(t_0,\varphi) : t \in [t_0,\ \eta(\varphi))\}$ is bounded, and $\eta(\varphi) = +\infty$.

Lemma 2.3 ([8, Theorem 3.1])**.** *Suppose that all conditions in Lemma 2.2 are satisfied. Let*

$$\liminf_{t\to+\infty}\left\{\sum_{j=2}^{m}\frac{\beta_j(t)}{a(t)} + \left[\frac{\beta_1(t)}{a(t)} - \frac{H(t)}{a(t)}\right]\right\} > 1. \tag{2.3}$$

Then, there exist two positive constants K_1 and K_2 such that

$$K_1 \leq \liminf_{t\to+\infty} x(t;t_0,\varphi) \leq \limsup_{t\to+\infty} x(t;t_0,\varphi) \leq K_2.$$

Lemma 2.4 ([6, Lemma 2.3])**.** *Let (2.2) hold. Suppose that there exists a positive constant M such that*

$$\max_{1\leq j\leq m}\gamma_j^+ \leq \frac{\widetilde{\kappa}}{M}, \tag{2.4}$$

and

$$\sup_{t\in R}\left\{-a(t) + \frac{1}{eM}\sum_{j=1}^{m}\frac{\beta_j(t)}{\gamma_j(t)}\right\} < 0, \quad \inf_{t\in R}\left\{-a(t) + e^{-\kappa}\sum_{j=2}^{m}\frac{\beta_j(t)}{\gamma_j(t)}\right\} > 0. \tag{2.5}$$

Then, the set of $\{x_t(t_0,\varphi) : t \in [t_0,\ \eta(\varphi))\}$ *is bounded, and* $\eta(\varphi) = +\infty$. *Moreover, there exists* $t_\varphi > t_0$ *such that*

$$\kappa < x(t; t_0, \varphi) < M \quad \text{for all} \ \ t \geq t_\varphi. \tag{2.6}$$

Lemma 2.5. *Suppose* (2.2), (2.4) *and* (2.5) *hold, and*

$$\sup_{t \in R} \Big\{ -a(t) + \sum_{j=2}^{m} \beta_j(t) \frac{1}{e^2} + \beta_1(t) e^{-\kappa}(M+1) + H(t) \Big\} < 0. \tag{2.7}$$

Moreover, assume that $x(t) = x(t; t_0, \varphi)$ *is a solution of equation* (1.2) *with initial condition* (1.4) *and* φ' *is bounded continuous on* $[-r, 0]$. *Then for any* $\epsilon > 0$, *there exists* $l = l(\epsilon) > 0$, *such that every interval* $[\alpha, \alpha + l]$ *contains at least one number* δ *for which there exists* $N > 0$ *satisfying*

$$|x(t+\delta) - x(t)| \leq \epsilon, \quad \text{for all} \ \ t > N. \tag{2.8}$$

Proof. Define a continuous function $\Gamma(\mu)$ by setting

$$\Gamma(\mu) = \sup_{t \in R} \Big\{ -[a(t) - \mu] + \sum_{j=2}^{m} \beta_j(t) \frac{1}{e^2} e^{\mu r} + \beta_1(t) e^{-\kappa}(M + e^{\mu r}) + H(t) e^{\mu r} \Big\}, \quad \mu \in [0,1].$$

Then, we have

$$\Gamma(0) = \sup_{t \in R} \Big\{ -a(t) + \sum_{j=2}^{m} \beta_j(t) \frac{1}{e^2} + \beta_1(t) e^{-\kappa}(M+1) + H(t) \Big\} < 0,$$

which implies that there exist two constants $\eta > 0$ and $\lambda \in (0,1]$ such that

$$\Gamma(\lambda) = \sup_{t \in R} \Big\{ -[a(t) - \lambda] + \sum_{j=2}^{m} \beta_j(t) \frac{1}{e^2} e^{\lambda r} + \beta_1(t) e^{-\kappa}(M + e^{\lambda r}) + H(t) e^{\lambda r} \Big\} < -\eta < 0. \tag{2.9}$$

For $t \in (-\infty, t_0 - r]$, we add the definition of $x(t)$ with $x(t) \equiv x(t_0 - r)$. Set

$$\begin{aligned}
\epsilon(\delta, t) = & -[a(t+\delta) - a(t)]x(t+\delta) \\
& + \sum_{j=2}^{m} [\beta_j(t+\delta) - \beta_j(t)] x(t+\delta-\tau_j(t+\delta)) e^{-\gamma_j(t+\delta)x(t+\delta-\tau_j(t+\delta))} \\
& + \sum_{j=2}^{m} \beta_j(t) \Big[x(t+\delta-\tau_j(t+\delta)) e^{-\gamma_j(t+\delta)x(t+\delta-\tau_j(t+\delta))} \\
& \qquad\qquad - x(t-\tau_j(t)+\delta) e^{-\gamma_j(t+\delta)x(t-\tau_j(t)+\delta)} \Big] \\
& + \sum_{j=2}^{m} \beta_j(t) \Big[x(t-\tau_j(t)+\delta) e^{-\gamma_j(t+\delta)x(t-\tau_j(t)+\delta)} \\
& \qquad\qquad - x(t-\tau_j(t)+\delta) e^{-\gamma_j(t)x(t-\tau_j(t)+\delta)} \Big] \\
& + [\beta_1(t+\delta) - \beta_1(t)] x(t+\delta-\tau_1(t+\delta)) e^{-\gamma_1(t+\delta)x(t+\delta)} \\
& + \beta_1(t) \Big[x(t+\delta-\tau_1(t+\delta)) e^{-\gamma_1(t+\delta)x(t+\delta)} - x(t-\tau_1(t)+\delta) e^{-\gamma_1(t)x(t+\delta)} \Big] \\
& - [H(t+\delta) - H(t)] x(t+\delta-\sigma(t+\delta)) \\
& - H(t)[x(t+\delta-\sigma(t+\delta)) - x(t-\sigma(t)+\delta)], \quad t \in R.
\end{aligned} \tag{2.10}$$

By Lemma 2.2, the solution $x(t)$ is bounded and

$$\kappa < x(t) < M \quad \text{for all } t \geq t_{\varphi}, \tag{2.11}$$

which implies that the right side of (1.2) is also bounded, and $x'(t)$ is a bounded function on $[t_0 - r, +\infty)$. Thus, in view of the fact that $x(t) \equiv x(t_0 - r)$ for $t \in (-\infty, t_0 - r]$, we obtain that $x(t)$ is uniformly continuous on R. From (2.1), for any $\epsilon > 0$, there exists $l = l(\epsilon) > 0$, such that every interval $[\alpha, \alpha + l]$, $\alpha \in R$, contains a δ for which

$$|\epsilon(\delta, t)| \leq \frac{1}{2}\eta\epsilon \quad \text{for all } t \in R. \tag{2.12}$$

Let $N_0 \geq \max\{t_0,\ t_0 - \delta,\ t_{\varphi} + r,\ t_{\varphi} + r - \delta\}$. For $t \in R$, denote $u(t) = x(t+\delta) - x(t)$. Then, for all $t \geq N_0$, we get

$$\begin{aligned} \frac{du(t)}{dt} = & - a(t)[x(t+\delta) - x(t)] \\ & + \sum_{j=2}^{m} \beta_j(t) \left[x(t - \tau_j(t) + \delta) e^{-\gamma_j(t) x(t - \tau_j(t) + \delta)} - x(t - \tau_j(t)) e^{-\gamma_j(t) x(t - \tau_j(t))} \right] \\ & + \beta_1(t) \left[x(t - \tau_1(t) + \delta) e^{-\gamma_1(t) x(t+\delta)} - x(t - \tau_1(t) + \delta) e^{-\gamma_1(t) x(t)} \right] \\ & + \beta_1(t) \left[x(t - \tau_1(t) + \delta) e^{-\gamma_1(t) x(t)} - x(t - \tau_1(t)) e^{-\gamma_1(t) x(t)} \right] \\ & - H(t)[x(t - \sigma(t) + \delta) - x(t - \sigma(t))] + \epsilon(\delta, t). \end{aligned} \tag{2.13}$$

From (1.6), (1.7), (2.10), (2.13) and the inequalities

$$|e^{-s} - e^{-t}| = e^{-(s + \theta(t-s))}|s - t| \leq e^{-\kappa}|s - t|, \quad \text{where } s, t \in [\kappa, \widetilde{\kappa}], \quad 0 < \theta < 1, \tag{2.14}$$

and

$$|se^{-s} - te^{-t}| = \left| \frac{1 - (s + \theta(t - s))}{e^{s + \theta(t-s)}} \right| |s - t| \leq \frac{1}{e^2}|s - t|, \text{ where } s, t \in [\kappa, +\infty),\ 0 < \theta < 1, \tag{2.15}$$

we obtain

$$\begin{aligned} & D^-(e^{\lambda s}|u(s)|)|_{s=t} \\ & \leq \lambda e^{\lambda t}|u(t)| + e^{\lambda t} \Bigg\{ - a(t)|x(t+\delta) - x(t)| \\ & \quad + \Bigg| \sum_{j=2}^{m} \beta_j(t) \left[x(t - \tau_j(t) + \delta) e^{-\gamma_j(t) x(t - \tau_j(t) + \delta)} - x(t - \tau_j(t)) e^{-\gamma_j(t) x(t - \tau_j(t))} \right] \\ & \quad + \beta_1(t) \left[x(t - \tau_1(t) + \delta) e^{-\gamma_1(t) x(t+\delta)} - x(t - \tau_1(t) + \delta) e^{-\gamma_1(t) x(t)} \right] \\ & \quad + \beta_1(t) \left[x(t - \tau_1(t) + \delta) e^{-\gamma_1(t) x(t)} - x(t - \tau_1(t)) e^{-\gamma_1(t) x(t)} \right] \\ & \quad - H(t) \left[x(t - \sigma(t) + \delta) - x(t - \sigma(t)) \right] + \epsilon(\delta, t) \Bigg| \Bigg\} \\ & = \lambda e^{\lambda t}|u(t)| + e^{\lambda t} \Bigg\{ - a(t)|u(t)| \\ & \quad + \Bigg| \sum_{j=2}^{m} \frac{\beta_j(t)}{\gamma_j(t)} \left[\gamma_j(t) x(t - \tau_j(t) + \delta) e^{-\gamma_j(t) x(t - \tau_j(t) + \delta)} - \gamma_j(t) x(t - \tau_j(t)) e^{-\gamma_j(t) x(t - \tau_j(t))} \right] \end{aligned}$$

$$
\begin{aligned}
&+\beta_1(t)\Big[x(t-\tau_1(t)+\delta)e^{-\gamma_1(t)x(t+\delta)} - x(t-\tau_1(t)+\delta)e^{-\gamma_1(t)x(t)}\Big] \\
&+\beta_1(t)\Big[x(t-\tau_1(t)+\delta)e^{-\gamma_1(t)x(t)} - x(t-\tau_1(t))e^{-\gamma_1(t)x(t)}\Big] \\
&-H(t)\big[x(t-\sigma(t)+\delta) - x(t-\sigma(t))\big] + \epsilon(\delta,t)\Big|\Big\} \\
\le\ & \lambda e^{\lambda t}|u(t)| + e^{\lambda t}\Big\{ -a(t)|u(t)| + \sum_{j=2}^{m}\beta_j(t)\frac{1}{e^2}|u(t-\tau_j(t))| + \beta_1(t)Me^{-\kappa}|u(t)| \\
&+\beta_1(t)e^{-\kappa}|u(t-\tau_1(t))| + H(t)e^{\lambda\sigma(t)}e^{\lambda(t-\sigma(t))}|u(t-\sigma(t))| + |\epsilon(\delta,t)|\Big\} \\
=\ & -[a(t)-\lambda]e^{\lambda t}|u(t)| + \sum_{j=2}^{m}\beta_j(t)\frac{1}{e^2}e^{\lambda\tau_j(t)}e^{\lambda(t-\tau_j(t))}|u(t-\tau_j(t))| + \beta_1(t)Me^{-\kappa}e^{\lambda t}|u(t)| \\
&+\beta_1(t)e^{-\kappa}e^{\lambda\tau_1(t)}e^{\lambda(t-\tau_1(t))}|u(t-\tau_1(t))| \\
&+H(t)e^{\lambda\sigma(t)}e^{\lambda(t-\sigma(t))}|u(t-\sigma(t))| + e^{\lambda t}|\epsilon(\delta,t)|, \quad \text{for all } t \ge N_0.
\end{aligned} \tag{2.16}
$$

Let

$$U(t) = \sup_{-\infty<s\le t}\{e^{\lambda s}|u(s)|\}. \tag{2.17}$$

It is obvious that $e^{\lambda t}|u(t)| \le U(t)$, and $U(t)$ is non-decreasing.

Now, we distinguish two cases to finish the proof.

Case one.

$$U(t) > e^{\lambda t}|u(t)| \quad \text{for all } t \ge N_0. \tag{2.18}$$

We claim that

$$U(t) \equiv U(N_0) \quad \text{is a constant for all } t \ge N_0. \tag{2.19}$$

Assume, by way of contradiction, that (2.19) does not hold. Then, there exists $t_1 > N_0$ such that $U(t_1) > U(N_0)$. Since

$$e^{\lambda t}|u(t)| \le U(N_0) \quad \text{for all } t \le N_0.$$

There must exist $\beta \in (N_0,\ t_1)$ such that

$$e^{\lambda\beta}|u(\beta)| = U(t_1) \ge U(\beta),$$

which contradicts (2.18). This contradiction implies that (2.19) holds. It follows that there exists $t_2 > N_0$ such that

$$|u(t)| \le e^{-\lambda t}U(t) = e^{-\lambda t}U(N_0) < \epsilon \quad \text{for all } t \ge t_2. \tag{2.20}$$

Case two. There is a $t_0^* \ge N_0$ that $U(t_0^*) = e^{\lambda t_0^*}|u(t_0^*)|$. Then, in view of (2.13) and (2.20), we get

$$\begin{aligned}
0 &\leq D^{-}(e^{\lambda s}|u(s)|)|_{s=t_0^*} \\
&\leq -[a(t_0^*)-\lambda]e^{\lambda t_0^*}|u(t_0^*)| \\
&\quad + \sum_{j=2}^{m}\beta_j(t_0^*)\frac{1}{e^2}e^{\lambda\tau_j(t_0^*)}e^{\lambda(t_0^*-\tau_j(t_0^*))}|u(t_0^*-\tau_j(t_0^*))| + \beta_1(t_0^*)Me^{-\kappa}e^{\lambda t}|u(t_0^*)| \\
&\quad + \beta_1(t_0^*)e^{-\kappa}e^{\lambda\tau_1(t_0^*)}e^{\lambda(t_0^*-\tau_1(t_0^*))}|u(t_0^*-\tau_1(t_0^*))| \\
&\quad + H(t_0^*)e^{\lambda\sigma(t_0^*)}e^{\lambda(t_0^*-\sigma(t_0^*))}|u(t_0^*-\sigma(t_0^*))| \\
&\quad + e^{\lambda t_0^*}|\epsilon(\delta,t_0^*)| \\
&\leq \Big\{-[a(t_0^*)-\lambda] + \sum_{j=2}^{m}\beta_j(t_0^*)\frac{1}{e^2}e^{\lambda r} + \beta_1(t_0^*)Me^{-\kappa} \\
&\quad + \beta_1(t_0^*)e^{-\kappa}e^{\lambda r} + H(t_0^*)e^{\lambda r}\Big\}U(t_0^*) + \frac{1}{2}\eta\epsilon e^{\lambda t_0^*} \\
&< -\eta U(t_0^*) + \eta\epsilon e^{\lambda t_0^*},
\end{aligned} \tag{2.21}$$

which yields

$$e^{\lambda t_0^*}|u(t_0^*)| = U(t_0^*) < \epsilon e^{\lambda t_0^*}, \quad \text{and} \quad |u(t_0^*)| < \epsilon. \tag{2.22}$$

For any $t > t_0^*$, with the same approach as that in deriving of (2.22), we can show

$$e^{\lambda t}|u(t)| < \epsilon e^{\lambda t}, \quad \text{and} \quad |u(t)| < \epsilon, \tag{2.23}$$

if $U(t) = e^{\lambda t}|u(t)|$.

On the other hand, if $U(t) > e^{\lambda t}|u(t)|$ and $t > t_0^*$. We can choose $t_0^* \leq t_3 < t$ such that

$$U(t_3) = e^{\lambda t_3}|u(t_3)| \quad \text{and} \quad U(s) > e^{\lambda s}|u(s)| \quad \text{for all } s \in (t_3,\ t],$$

which, together with (2.23), yields

$$|u(t_3)| < \epsilon.$$

With a similar argument as that in the proof of Case one, we can show that

$$U(s) \equiv U(t_3) \quad \text{is a constant for all } s \in (t_3,\ t], \tag{2.24}$$

which implies that

$$|u(t)| < e^{-\lambda t}U(t) = e^{-\lambda t}U(t_3) = |u(t_3)|e^{-\lambda(t-t_3)} < \epsilon.$$

In summary, there must exist $N > \max\{t_0^*,\ N_0,\ t_2\}$ such that $|u(t)| \leq \epsilon$ holds for all $t > N$. The proof of Lemma 2.5 is now complete.

□

3 Main results

In this section, we establish sufficient conditions on the existence, uniqueness, and global exponential stability of positive almost periodic solutions of (1.2).

Theorem 3.1. *Under the assumptions of Lemma 2.5, equation (1.2) has at least one positive almost periodic solution $x^*(t)$. Moreover, $x^*(t)$ is globally exponentially stable, i.e., there exist constants K_{φ,x^*} and t_{φ,x^*} such that*

$$|x(t;t_0,\varphi) - x^*(t)| < K_{\varphi,x^*}e^{-\lambda t} \text{ for all } t > t_{\varphi,x^*}, \text{ where } \lambda \text{ is defined in (2.9).}$$

Proof. Let $v(t) = v(t;t_0,\varphi^v)$ be a solution of equation (1.2) with initial conditions satisfying the assumptions in Lemma 2.5. We also add the definition of $v(t)$ with $v(t) \equiv v(t_0 - r)$ for all $t \in (-\infty, t_0 - r]$. Set

$$\begin{aligned}
&\epsilon(k,t)\\
&\quad = -[a(t+t_k) - a(t)]v(t+t_k)\\
&\qquad + \sum_{j=2}^{m}[\beta_j(t+t_k) - \beta_j(t)]v(t+t_k-\tau_j(t+t_k))e^{-\gamma_j(t+t_k)v(t+t_k-\tau_j(t+t_k))}\\
&\qquad + \sum_{j=2}^{m}\beta_j(t)\Big[v(t+t_k-\tau_j(t+t_k))e^{-\gamma_j(t+t_k)v(t+t_k-\tau_j(t+t_k))}\\
&\qquad - v(t-\tau_j(t)+t_k)e^{-\gamma_j(t+t_k)v(t-\tau_j(t)+t_k)}\Big]\\
&\qquad + \sum_{j=2}^{m}\beta_j(t)\Big[v(t-\tau_j(t)+t_k)e^{-\gamma_j(t+t_k)v(t-\tau_j(t)+t_k)} - v(t-\tau_j(t)+t_k)e^{-\gamma_j(t)v(t-\tau_j(t)+t_k)}\Big]\\
&\qquad + [\beta_1(t+t_k) - \beta_1(t)]v(t+t_k-\tau_1(t+t_k))e^{-\gamma_1(t+t_k)x(t+t_k)}\\
&\qquad + \beta_1(t)\Big[v(t+t_k-\tau_1(t+t_k))e^{-\gamma_1(t+t_k)v(t+t_k)} - v(t-\tau_1(t)+t_k)e^{-\gamma_1(t)v(t+t_k)}\Big]\\
&\qquad - [H(t+t_k) - H(t)]v(t+t_k-\sigma(t+t_k))\\
&\qquad - H(t)\big[v(t+t_k-\sigma(t+t_k)) - v(t-\sigma(t)+t_k)\big], \quad t \in R,
\end{aligned} \tag{3.1}$$

where $\{t_k\}$ is any sequence of real numbers. By Lemma 2.3, the solution $v(t)$ is bounded and

$$\kappa < v(t) < M \quad \text{for all} \quad t \geq t_{\varphi^v}, \tag{3.2}$$

which implies that the right side of (1.2) is also bounded, and $v'(t)$ is a bounded function on $[t_0 - r, +\infty)$. Thus, in view of the fact that $v(t) \equiv v(t_0 - r)$ for $t \in (-\infty, t_0 - r]$, we obtain that $v(t)$ is uniformly continuous on R. Then, from the almost periodicity of a, H, σ, τ_j, γ_j and β_j, we can select a sequence $\{t_k\} \to +\infty$ such that

$$\left.\begin{array}{l}
|a(t+t_k) - a(t)| \leq \frac{1}{k}, \ \ |H(t+t_k) - H(t)| \leq \frac{1}{k},\\
|\tau_j(t+t_k) - \tau_j(t)| \leq \frac{1}{k}, \ \ |\sigma(t+t_k) - \sigma(t)| \leq \frac{1}{k}\\
|\beta_j(t+t_k) - \beta_j(t)| \leq \frac{1}{k}, \ \ |\gamma_j(t+t_k) - \gamma_j(t)| \leq \frac{1}{k}, \ \ |\epsilon(k,t)| \leq \frac{1}{k}
\end{array}\right\} \text{ for all } j, t. \tag{3.3}$$

Since $\{v(t+t_k)\}_{k=1}^{+\infty}$ is uniformly bounded and equiuniformly continuous, by the Ascoli–Arzelà lemma and diagonal selection principle, we can choose a subsequence $\{t_{k_j}\}$ of $\{t_k\}$, such that $v(t+t_{k_j})$ (for convenience, we still denote by $v(t+t_k)$) uniformly converges to a continuous function $x^*(t)$ on any compact set of R, and

$$\kappa \leq x^*(t) \leq M \quad \text{for all} \quad t \in R. \tag{3.4}$$

Now, we prove that $x^*(t)$ is a solution of (1.2). In fact, for any $t \geq t_0$ and $\Delta t \in R$, from (3.3), we have

$$\begin{aligned}
x^*(t+\Delta t) &- x^*(t) \\
&= \lim_{k\to+\infty}[v(t+\Delta t+t_k) - v(t+t_k)] \\
&= \lim_{k\to+\infty}\int_t^{t+\Delta t}\Big\{-a(\mu+t_k)v(\mu+t_k) \\
&\qquad +\sum_{j=2}^{m}\beta_j(\mu+t_k)v(\mu+t_k-\tau_j(\mu+t_k))e^{-\gamma_j(\mu+t_k)v(\mu+t_k-\tau_j(\mu+t_k))} \\
&\qquad +\beta_1(\mu+t_k)v(\mu+t_k-\tau_1(\mu+t_k))e^{-\gamma_1(\mu+t_k)v(\mu+t_k)} \\
&\qquad -H(\mu+t_k)v(\mu+t_k-\sigma(\mu+t_k))\Big\}d\mu \\
&= \lim_{k\to+\infty}\int_t^{t+\Delta t}\Big\{-a(\mu)v(\mu+t_k) \\
&\qquad +\sum_{j=2}^{m}\beta_j(\mu)v(\mu+t_k-\tau_j(\mu))e^{-\gamma_j(\mu)v(\mu+t_k-\tau_j(\mu))} \\
&\qquad +\beta_1(\mu)v(\mu+t_k-\tau_1(\mu))e^{-\gamma_1(\mu)v(\mu+t_k)} \\
&\qquad -H(\mu)v(\mu-\sigma(\mu)+t_k)+\epsilon(k,\mu)\Big\}d\mu \\
&= \int_t^{t+\Delta t}\Big\{-a(\mu)x^*(\mu)+\sum_{j=2}^{m}\beta_j(\mu)x^*(\mu-\tau_j(\mu))e^{-\gamma_j(\mu)x^*(\mu-\tau_j(\mu))} \\
&\qquad +\beta_1(\mu)x^*(\mu-\tau_1(\mu))e^{-\gamma_1(\mu)x^*(\mu)}-H(\mu)x^*(\mu-\sigma(\mu))\Big\}d\mu \\
&\quad +\lim_{k\to+\infty}\int_t^{t+\Delta t}\epsilon(k,\mu)d\mu \\
&= \int_t^{t+\Delta t}\Big\{-a(\mu)x^*(\mu)+\sum_{j=2}^{m}\beta_j(\mu)x^*(\mu-\tau_j(\mu))e^{-\gamma_j(\mu)x^*(\mu-\tau_j(\mu))} \\
&\qquad +\beta_1(\mu)x^*(\mu-\tau_1(\mu))e^{-\gamma_1(\mu)x^*(\mu)}-H(\mu)x^*(\mu-\sigma(\mu))\Big\}d\mu,
\end{aligned} \tag{3.5}$$

where $t+\Delta t \geq t_0$. Consequently, (3.5) implies that

$$\begin{aligned}
\frac{d}{dt}\{x^*(t)\} = & -a(t)x^*(t)+\sum_{j=2}^{m}\beta_j(t)x^*(t-\tau_j(t))e^{-\gamma_j(t)x^*(t-\tau_j(t))} \\
& +\beta_1(t)x^*(t-\tau_1(t))e^{-\gamma_1(t)x^*(t)}-H(t)x^*(t-\sigma(t)).
\end{aligned} \tag{3.6}$$

Therefore, $x^*(t)$ is a solution of (1.2).

Secondly, we prove that $x^*(t)$ is an almost periodic solution of (1.2). From Lemma 2.5, for any $\varepsilon > 0$, there exists $l = l(\varepsilon) > 0$, such that every interval $[\alpha, \alpha+l]$ contains at least one number δ for which there exists $N > 0$ satisfing

$$|v(t+\delta)-v(t)| \leq \varepsilon \text{ for all } t > N. \tag{3.7}$$

Then, for any fixed $s \in R$, we can find a sufficiently large positive integer $N_1 > N$ such that for any $k > N_1$,

$$s+t_k > N, \ \ |v(s+t_k+\delta)-v(s+t_k)| \leq \varepsilon. \tag{3.8}$$

Let $k \to +\infty$, we obtain

$$|x^*(s+\delta) - x^*(s)| \le \varepsilon,$$

which implies that $x^*(t)$ is an almost periodic solution of equation (1.2).

Finally, we prove that $x^*(t)$ is globally exponentially stable.

Let $x(t) = x(t; t_0, \varphi)$ and $y(t) = x(t) - x^*(t)$, where $t \in [t_0 - r, +\infty)$. Then

$$\begin{aligned} y'(t) = & -a(t)y(t) \\ & + \sum_{j=2}^{m} \beta_j(t) \left[x(t-\tau_j(t)) e^{-\gamma_j(t)x(t-\tau_j(t))} - x^*(t-\tau_j(t)) e^{-\gamma_j(t)x^*(t-\tau_j(t))} \right] \\ & + \beta_1(t) \left[x(t-\tau_1(t)) e^{-\gamma_1(t)x(t)} - x^*(t-\tau_1(t)) e^{-\gamma_1(t)x^*(t)} \right] - H(t)y(t-\sigma(t)). \end{aligned} \tag{3.9}$$

It follows from Lemma 2.4 that there exists $t_{\varphi,\varphi^*} > t_0$ such that

$$\kappa \le x(t),\ x^*(t) \le M \ \text{ for all } \ t \in [t_{\varphi,\varphi^*} - r,\ +\infty). \tag{3.10}$$

We consider the Lyapunov functional

$$V(t) = |y(t)| e^{\lambda t}. \tag{3.11}$$

Calculating the upper left derivative of $V(t)$ along the solution $y(t)$ of (3.9), we have

$$\begin{aligned} D^-(V(t)) \le & -a(t)|y(t)|e^{\lambda t} \\ & + \sum_{j=2}^{m} \beta_j(t) |x(t-\tau_j(t)) e^{-\gamma_j(t)x(t-\tau_j(t))} - x^*(t-\tau_j(t)) e^{-\gamma_j(t)x^*(t-\tau_j(t))}| e^{\lambda t} \\ & + \beta_1(t) |x(t-\tau_1(t)) e^{-\gamma_1(t)x(t)} - x^*(t-\tau_1(t)) e^{-\gamma_1(t)x^*(t)}| e^{\lambda t} \\ & + H(t)|y(t-\sigma(t))|e^{\lambda t} + \lambda |y(t)| e^{\lambda t}, \ \text{for all} \ t > t_{\varphi,\varphi^*}. \end{aligned} \tag{3.12}$$

We claim that

$$V(t) = |y(t)|e^{\lambda t} < e^{\lambda t_{\varphi,\varphi^*}} \Big(\max_{t \in [t_0 - r,\ t_{\varphi,\varphi^*}]} |x(t) - x^*(t)| + 1 \Big) =: K_{\varphi,\varphi^*} \ \text{ for all } \ t > t_{\varphi,\varphi^*}. \tag{3.13}$$

Contrarily, there must exist $t_* > t_{\varphi,\varphi^*}$ such that

$$V(t_*) = K_{\varphi,\varphi^*} \quad \text{and} \quad V(t) < K_{\varphi,\varphi^*} \quad \text{for all } \ t \in [t_0 - r,\ t_*). \tag{3.14}$$

Since

$$\kappa \le \gamma_j(t_*) x(t_* - \tau_j(t_*)),\ \gamma_j(t_*) x^*(t_* - \tau_j(t_*)) \le \gamma_j^+ M \le \widetilde{\kappa}, \qquad j = 1, 2, \ldots, m.$$

Together with (1.5), (1.6), (1.7), (2.18), (2.19), (3.12) and (3.14), we obtain

$$\begin{aligned} 0 \le & \, D^-(V(t_*)) \\ \le & -a(t_*)|y(t_*)|e^{\lambda t_*} \\ & + \sum_{j=2}^{m} \beta_j(t_*) \left| x(t_* - \tau_j(t_*)) e^{-\gamma_j(t_*)x(t_* - \tau_j(t_*))} - x^*(t_* - \tau_j(t_*)) e^{-\gamma_j(t_*)x^*(t_* - \tau_j(t_*))} \right| e^{\lambda t_*} \\ & + \beta_1(t_*) \left| x(t_* - \tau_1(t_*)) e^{-\gamma_1(t_*)x(t_*)} - x^*(t_* - \tau_1(t_*)) e^{-\gamma_1(t_*)x^*(t_*)} \right| e^{\lambda t_*} \\ & + H(t_*)|y(t_* - \sigma(t_*))|e^{\lambda t_*} + \lambda |y(t_*)| e^{\lambda t_*} \end{aligned}$$

$$
\begin{aligned}
\leq & -a(t_*)|y(t_*)|e^{\lambda t_*} + \sum_{j=2}^{m} \frac{\beta_j(t_*)}{\gamma_j(t_*)} \Big| \gamma_j(t_*)x(t_* - \tau_j(t_*))e^{-\gamma_j(t_*)x(t_*-\tau_j(t_*))} \\
& \qquad - \gamma_j(t_*)x^*(t_* - \tau_j(t_*))e^{-\gamma_j(t_*)x^*(t_*-\tau_j(t_*))} \Big| e^{\lambda t_*} \\
& + \beta_1(t_*) \Big| x(t_* - \tau_1(t_*))e^{-\gamma_1(t_*)x(t_*)} - x(t_* - \tau_1(t_*))e^{-\gamma_1(t_*)x^*(t_*)} \Big| e^{\lambda t_*} \\
& + \beta_1(t_*) \Big| x(t_* - \tau_1(t_*))e^{-\gamma_1(t_*)x^*(t_*)} - x^*(t_* - \tau_1(t_*))e^{-\gamma_1(t_*)x^*(t_*)} \Big| e^{\lambda t_*} \\
& + H(t_*)|y(t_* - \sigma(t_*))|e^{\lambda t_*} + \lambda|y(t_*)|e^{\lambda t_*} \\
\leq & -[a(t_*) - \lambda]|y(t_*)|e^{\lambda t_*} + \sum_{j=2}^{m} \beta_j(t_*) \frac{1}{e^2} |y(t_* - \tau_j(t_*))|e^{\lambda t_*} \\
& + \beta_1(t_*)M \Big| e^{-\gamma_1(t_*)x(t_*)} - e^{-\gamma_1(t_*)x^*(t_*)} \Big| e^{\lambda t_*} \\
& + \beta_1(t_*)|x(t_* - \tau_1(t_*)) - x^*(t_* - \tau_1(t_*))|e^{-\gamma_1(t_*)x^*(t_*)}e^{\lambda t_*} \\
& + H(t_*)|y(t_* - \sigma(t_*))|e^{\lambda t_*} \\
\leq & -[a(t_*) - \lambda]|y(t_*)|e^{\lambda t_*} + \sum_{j=2}^{m} \beta_j(t_*) \frac{1}{e^2} |y(t_* - \tau_j(t_*))|e^{\lambda(t_*-\tau_j(t_*))}e^{\lambda\tau_j(t_*)} \\
& + \beta_1(t_*)Me^{-\kappa}|y(t_*)|e^{\lambda t_*} \\
& + \beta_1(t_*)e^{-\kappa}|y(t_* - \tau_1(t_*))|e^{\lambda(t_*-\tau_1(t_*))}e^{\lambda\tau_1(t_*)} \\
& + H(t_*)|y(t_* - \sigma(t_*))|e^{\lambda(t_*-\sigma(t_*))}e^{\lambda\sigma(t_*)} \\
\leq & \left\{ -[a(t_*) - \lambda] + \sum_{j=2}^{m} \beta_j(t_*) \frac{1}{e^2} e^{\lambda r} + \beta_1(t_*)e^{-\kappa}(M + e^{\lambda r}) + H(t_*)e^{\lambda r} \right\} K_{\varphi,\varphi^*}.
\end{aligned}
$$

Thus,

$$
0 \leq -[a(t_*) - \lambda] + \sum_{j=2}^{m} \beta_j(t_*) \frac{1}{e^2} e^{\lambda r} + \beta_1(t_*)e^{-\kappa}(M + e^{\lambda r}) + H(t_*)e^{\lambda r},
$$

which contradicts with (2.13). Hence, (3.13) holds. It follows that

$$
|y(t)| < K_{\varphi,\varphi^*}e^{-\lambda t} \quad \text{for all} \quad t > t_{\varphi,\varphi^*}.
$$

This completes the proof of Theorem 3.1.

□

4 An example

In this section, we present an example to check the validity of our results obtained in the previous sections.

Example 4.1. Consider the following Nicholson's blowflies model with a linear harvesting term:

$$
\begin{aligned}
x'(t) = & -\frac{30 + 15|\cos t|}{100} x(t) + \frac{100 - \sin\sqrt{2}t}{100 + \sin\sqrt{2}t} x(t - 2e^{\sin^4 t})e^{-x(t-2e^{\sin^4 t})} \\
& + \frac{1}{100}(3 + \cos^4\sqrt{3}t)x(t - 2e^{\cos^4 t})e^{-x(t)} - \frac{1}{100}(2 + \cos^4\sqrt{3}t)x(t - 2e^{\cos^4 t}).
\end{aligned}
\tag{4.1}
$$

Obviously,

$$a^+ = 0.45,\ a^- = 0.3, \quad \beta_2^- = \frac{99}{101}, \quad \beta_2^+ = \frac{101}{99},$$

$$\gamma_i^- = \gamma_i^+ = 1, \quad i = 1,2,$$

$$\beta_1(t) = \frac{1}{100}(3+\cos^4\sqrt{3}t), \quad H(t) = \frac{1}{100}(2+\cos^4 t)$$

$$\tau_2(t) = 2e^{\sin^4 t}, \quad \tau_1(t) = \sigma(t) = 2e^{\cos^4 t}, \quad r = 2e.$$

Note $\kappa \approx 0.7215355$ and $\tilde{\kappa} \approx 1.342276$. Let $M = 1.33$. Then

$$\begin{aligned}
&a^- M = 0.3 \times 1.33 \approx 0.399,\\
&\frac{\beta_2^+}{\gamma_2^-}\frac{1}{e} = \frac{101}{99}\frac{1}{e} \approx 0.3753113, \quad \frac{\beta_1^+}{\gamma_1^-}\frac{1}{e} = \frac{101}{99}\frac{1}{e} \approx 0.016,\\
&\frac{\beta_2^-}{\gamma_2^+}e^{-\kappa} = \frac{99}{101}e^{-\kappa} \approx \frac{99}{101}e^{-0.7215355} \approx 0.4763816,\\
&\beta_2^+\frac{1}{e^2} = \frac{101}{99}\frac{1}{e^2} \approx 0.1380693, \quad \beta_1^+ e^{-\kappa}(M+1) + H^+ \approx 0.09,
\end{aligned}$$

which imply that (4.1) satisfies the assumptions of Theorem 3.1. Therefore, equation (4.1) has a unique positive almost periodic solution $x^*(t)$, which is globally exponentially stable with the exponential convergent rate $\lambda \approx 0.005$. The numerical simulation in Figure 4.1 strongly supports the conclusion.

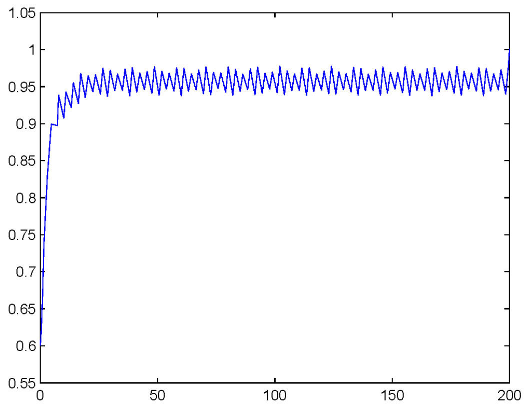

Figure 4.1: Numerical solution $x(t)$ of equation (4.1) for initial value $\varphi(s) \equiv 0.6,\ s \in [-2e, 0]$.

Remark 4.2. We remark that the results in [6, 8–11, 14] and the references therein cannot be applied to prove the global exponential stability of positive almost periodic solutions for (4.1). This implies that the results of this paper are new and they complement previously known results. In particular, in this present paper, we employ a novel proof to establish some criteria to guarantee the global dynamic behavior of positive almost periodic solutions for non-autonomous Nicholson's blowflies model with the linear harvesting term.

Acknowledgements

This work was supported by the Scientific Research Fund of Hunan Provincial Natural Science Foundation of China (Grant No. 12JJ3007). The author would like to express the sincere appreciation to the editor and reviewer for their helpful comments in improving the presentation and quality of the paper.

References

[1] P. Amster, A. Déboli, Existence of positive periodic solutions of a generalized Nicholson's blowflies model with a nonlinear harvesting term, *Appl. Math. Lett.* **25**(2012), No. 9, 1203–1207. MR2930746

[2] L. Berezansky, E. Braverman, L. Idels, Nicholson's blowflies differential equations revisited: main results and open problems, *Appl. Math. Modelling* **34**(2010), 1405–1417. MR2592579; url

[3] C. W. Clark, *Mathematical bioeconomics. The optimal management of renewable resources*, Wiley, New York, 1990. MR1044994

[4] A. M. Fink, *Almost periodic differential equations*, Lecture Notes in Mathematics, Vol. 377, Springer, Berlin, 1974. MR1566844

[5] C. Y. He, *Almost periodic differential equations,* Higher Education Publishing House, Beijing, 1992, (in Chinese).

[6] A. Jiang, Global exponential stability of periodic solutions for a Nicholson's blowflies model with a linear harvesting term, *Appl. Math. Inf. Sci.* **8**(2014), No. 5, 1–7. MR3178775

[7] A. W. Leung, Optimal harvesting-coefficient control of steady-state prey–predator diffusive Volterra–Lotka system, *Appl. Math. Optim* **31**(1995), 219–241. MR1309307; url

[8] B. Liu, Global dynamic behaviors for a delayed Nicholson's blowflies model with a linear harvesting term, *Electron. J. Qual. Theory Differ. Equ.* **2013**, No. 45, 1–13. MR3090578

[9] B. Liu, New results on the positive almost periodic solutions for a model of hematopoiesis, *Nonlinear Anal. Real World Appl.* **17**(2014), 252–264. MR3158473

[10] X. Liu, J. Meng, The positive almost periodic solution for Nicholson-type delay systems with linear harvesting terms, *Appl. Math. Modell.* **36**(2012), 3289–3298. MR2910618

[11] F. Long, Positive almost periodic solution for a class of Nicholson's blowflies model with a linear harvesting term, *Nonlinear Anal. Real World Appl.* **13**(2012), No. 2, 686–693. MR2846873

[12] F. Long, M. Yang, Positive periodic solutions of delayed Nicholson's blowflies model with a linear harvesting term, *Electron. J. Qual. Theory Differ. Equ.* **2011**, No. 41, 1–11. MR2805761

[13] L. J. Troutman, *Variational calculus and optimal control*, Springer, New York, 1996. MR1363262; url

[14] L. Wang, Almost periodic solution for Nicholson's blowflies model with patch structure and linear harvesting terms, *Appl. Math. Modell.* **37**(2013), 2153–2165. MR3002307

[15] W. Zhao, C. Zhu, H. Zhu, On positive periodic solution for the delay Nicholson's blowflies model with a harvesting term, *Appl. Math. Modell.* **36**(2012), No. 7, 3335–3340. MR2910623

[16] Q. Zhou, The positive periodic solution for Nicholson-type delay system with linear harvesting terms, *Appl. Math. Modell.* **37**(2013), 5581–5590. MR3028415