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Fractional differential inclusions with anti-periodic boundary conditions in Banach spaces

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Abstract. The main purpose of this paper is to provide the theory of differential inclusions by new existence results of solutions for boundary value problems of differential inclusions with fractional order and with anti-periodic boundary conditions in Banach spaces. We prove existence theorems of solutions under both convexity and nonconvexity conditions on the multivalued side. Meanwhile, the compactness of the set solutions is also established.

Keywords: fractional differential inclusions, Caputo fractional derivative, anti-periodic boundary conditions, fixed point, measure of noncompactness.

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1 Introduction

During the past two decades, fractional differential equations and fractional differential inclusions have gained considerable importance due to their applications in various fields, such as physics, mechanics and engineering. For some of these applications, one can see [16, 22, 27, 33] and the references therein. El Sayed et al. [15] initiated the study of fractional multivalued differential inclusions. For some recent development on initial value problems for differential equations and inclusions of fractional order we refer the reader to the references [1, 32, 34–38].

Some applied problems in physics require fractional differential equations and inclusions with boundary conditions. Recently, many authors have studied differential inclusions with various boundary conditions. Some of these works have been done in finite dimensional spaces and of positive integer order, for example, Ibrahim et al. [25] and Gomma [17] considered a functional multivalued three-point boundary value problem of second-order. Gomma [18] studied four-point boundary value problems for non-convex differential inclusions.

Several results have been obtained for fractional differential equations and inclusions with various boundary value conditions in finite dimensional spaces. We refer, for example, to Agarwal et al. [1] who established conditions for the existence of solutions for various classes

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of initial and boundary value problems for fractional differential equations and inclusions involving the Caputo derivative, Ouahab [31] studied a fractional differential inclusion with Dirichlet boundary conditions under both convexity and nonconvexity conditions on the multivalued right-hand side and Ntouyas et al. [30] discussed the existence of solutions for a boundary value problem of fractional differential inclusions with three-point integral boundary conditions involving convex and non-convex multivalued maps.

For some recent works on boundary value problems for fractional differential inclusions in infinite dimensional spaces, we refer to Benchohra et al. [8] who established the existence of solutions of nonlinear fractional differential inclusions with two point boundary conditions.

Anti-periodic boundary conditions appear in a variety of situation, see [2,3,12,13] and the references therein.

Let $2 < \alpha < 3$ and E be a Banach space. In this paper we consider the following two fractional boundary value problems :

$$\begin{cases} {}^cD^\alpha\, x(t) = f(t, x(t)), \quad \text{a.e. on } J = [0, b], \\ x(0) = -x(b), \quad x^{(1)}(0) = -x^{(1)}(b), \quad x^{(2)}(0) = -x^{(2)}(b), \end{cases} \tag{1.1}$$

and

$$\begin{cases} {}^cD_g^{\alpha-2}\, x^{(2)}(t) \in F(t, x(t)), \quad \text{a.e. on } J = [0, b], \\ x(0) = -x(b), \quad x^{(1)}(0) = -x^{(1)}(b), \quad x^{(2)}(0) = -x^{(2)}(b), \end{cases} \tag{1.2}$$

where ${}^cD^\alpha$ is the Caputo derivative of order α with the lower limit zero, $f\colon J \times E \to E$ is a function, ${}^cD_g^\alpha$ is the Caputo derivative of order α in the generalized (weak) sense with the lower limit zero, which is specified later, and $F\colon\ J \times E \to 2^E$ is a multifunction.

Firstly, we shall prove that if $h \in L^1(J, E)$ and $x \in AC^3(J, E)$ is a solution of the fractional boundary value problem

$$\begin{cases} {}^cD^{\,\alpha} x(t) = h(t), \quad \text{a.e. on } J = [0, b], \\ x(0) = -x(b), \quad x^{(1)}(0) = -x^{(1)}(b), \quad x^{(2)}(0) = -x^{(2)}(b), \end{cases} \tag{1.3}$$

then

$$\begin{aligned} x(t) = {} & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)\, ds - \frac{1}{2\Gamma(\alpha)} \int_0^b (b-s)^{\alpha-1} h(s)\, ds \\ & + \frac{(b-2t)}{4\Gamma(\alpha-1)} \int_0^b (b-s)^{\alpha-2} h(s)\, ds \\ & + \frac{t(b-t)}{4\Gamma(\alpha-2)} \int_0^b (b-s)^{\alpha-3} h(s)\, ds, \quad t \in J. \end{aligned} \tag{1.4}$$

Moreover, if $h \in AC(J, E)$ and $x\colon J \to E$ such that (1.4) holds, then $x \in AC^3(J, E)$ and satisfies (1.3).

In [28], Lan et al. pointed out that we can neither prove that if $h \in C([0, 1], \mathbb{R})$, then x given in (1.4) is a solution of (1.3) nor show that if $h \in C(J, \mathbb{R})$ satisfies (1.3), then x and h satisfy (1.4) although these results have been widely used in some papers such as [2, Lemma 2.1], [3, Lemma 1.2] and [12, Lemma 2.7]. Due to the requirement $h \in AC(J, E)$, the continuity assumption on f is not sufficient. To overcome this difficulty we shall impose a Lipschitz type condition on f. Motivated by the work of Lan et al. [28] we introduce a correct formula of solutions of (1.1).

Also, due to the requirement $h \in AC(J, E)$, a problem arises when we study the problem (1.1) in the case when f is a multifunction. This problem arises because we do not know the conditions that guarantee the closedness of the set of absolutely continuous selections for a multifunction. To avoid this problem we consider the Caputo derivative in the generalized sense when we study the problem (1.2). We shall show that if $x \in AC^3(J, E)$ is a solutions of (1.2), then the following fractional boundary value problem

$$\begin{cases} {}^cD^{\alpha}x(t) \in F(t, x(t)), & \text{a.e. on } J = [0, b], \\ x(0) = -x(b), \quad x^{(1)}(0) = -x^{(1)}(b), & x^{(2)}(0) = -x^{(2)}(b) \end{cases}$$

holds. We note that Ahmed [2] considered (1.1) in infinite dimensional Banach spaces while Cerna [12] considered (1.2) in finite dimensional spaces. It is important to note that, based on the remark of Lan et al. [28], and mentioned in the preceding paragraph, the proofs in the paper of both Ahmed [2] and Cerna [12] require additional assumptions.

The present paper is organized as follows: in Section 2 we collect some background material and basic results from multivalued analysis and fractional calculus to be used later. In Section 3 we prove an existence result for (1.1) and in Section 4 we establish various existence results for (1.2) and we prove the compactness of the set of solutions. At the end of the paper we give examples in order to illustrate the feasibility of our assumptions.

The proofs rely on the methods and results for boundary value fractional differential inclusions, the properties of noncompact measure and fixed point techniques.

2 Preliminaries and notation

Let $C(J, E)$ be the space of E-valued continuous functions on J with the uniform norm $\|x\| = \sup\{\|x(t)\|, x \in J\}$, $L^1(J, E)$ be the quotient space of all E-valued Bochner integrable functions on J with the norm $\|f\|_{L^1(J,E)} = \int_0^b \|f(t)\| \, dt$, $P_b(E) = \{B \subseteq E : B$ is nonempty and bounded$\}$, $P_{cl}(E) = \{B \subseteq E : B$ is nonempty and closed$\}$, $P_k(E) = \{B \subseteq E : B$ is nonempty and compact$\}$, $P_{ck}(E) = \{B \subseteq E : B$ is nonempty, convex and compact$\}$, $P_{cl,cv}(E) = \{B \subseteq E : B$ is nonempty, closed and convex$\}$, $\mathrm{Conv}(B)$ (respectively, $\overline{\mathrm{Conv}}(B)$) be the convex hull (respectively, convex closed hull in E) of a subset B.

Let $G\colon J \to 2^E$ be a multifunction. By S^1_G we will denote the set of integrable selections of G; i.e. $S^1_G = \{f \in L^1(J, E) : f(t) \in G(t)$ a.e.$\}$. This set may be empty. For $P_{cl}(E)$-valued measurable multifunction, it is nonempty if and only if $t \to \inf\{\|x\| : x \in G(t)\} \in L^1(J, \mathbb{R}^+)$. In particular, this is the case if $t \to \sup\{\|x\| : x \in G(t)\} \in L^1(J, \mathbb{R}^+)$ (such a multifunction is said to be integrably bounded). Note that $S^1_G \subseteq L^1(J, E)$ is closed and it is convex if and only if for almost all $t \in J$, $G(t)$ is convex set in E.

Definition 2.1. Let X and Y be two topological spaces. A multifunction $G\colon X \to P(Y)$ is said to be upper semicontinuous if $G^{-1}(V) = \{x \in X : G(x) \subseteq V\}$ is an open subset of X for every open $V \subseteq Y$. G is called closed if its graph $\Gamma_G = \{(x, y) \in X \times Y : y \in G(x)\}$ is closed subset of the topological space $X \times Y$. G is said to be completely continuous if $G(B)$ is relatively compact for every bounded subset B of X.

If the multifunction G is completely continuous with nonempty compact values, then G is *u.s.c.* if and only if G is closed.

Lemma 2.2 ([26, Theorem 1.3.5]). *Let X_0, X be (not necessarily separable) Banach spaces, and let $F\colon J \times X_0 \to P_k(X)$ be such that*

(i) *for every $x \in X_0$ the multifunction $F(\cdot, x)$ has a strongly measurable selection;*

(ii) *for a.e. $t \in J$ the multifunction $F(t, \cdot)$ is upper semicontinuous.*

Then for every strongly measurable function $z\colon J \to X_0$ there exists a strongly measurable function $f\colon J \to X$ such that $f(t) \in F(t, z(t))$, a.e.

Remark 2.3 ([26, Theorem 1.3.1])**.** For single-valued or compact-valued multifunctions acting on a separable Banach space the notions measurability and strongly measurable coincide. So, if X_0, X are separable Banach spaces we can replace strongly measurable with measurable in the above lemma.

Definition 2.4. A sequence $\{f_n : n \in \mathbb{N}\} \subset L^1(J, E)$ is said to be semi-compact if:

(i) it is integrably bounded , i.e. there is $q \in L^1(J, \mathbb{R}^+)$ such that

$$\|f_n(t)\| \leq q(t) \quad \text{a.e. } t \in J;$$

(ii) the set $\{f_n(t) : n \in \mathbb{N}\}$ is relatively compact in E a.e. $t \in J$.

Lemma 2.5 ([26])**.** *Every semi-compact sequence in $L^1(J, E)$ is weakly compact in $L^1(J, E)$.*

Definition 2.6. Let (X, d) be a metric space. A multifunction $G\colon [a, b] \to 2^X$ is said to be absolutely continuous if for any $\epsilon > 0$ there is $\delta < 0$ such that if $\{a_i, b_i\}_{i=1}^{i=n}$ (with arbitrary $n \in \mathbb{N}$), $a_1 < b_1 \leq a_2 < b_2 \leq \cdots \leq a_n < b_n$ and $\sum_{i=1}^{n}(b_i - a_i) < \delta$, then $\sum_{i=1}^{n} h(F(b_i), F(a_i)) < \epsilon$, where h is the Hausdorff distance.

Lemma 2.7 ([7, Theorem 3])**.** *Suppose that (X, d) is a metric space, $G\colon [a, b] \to 2^X$ is absolutely continuous with compact values, $t_0 \in [a, b]$ and $x_0 \in G(t_0)$. Then G admits an absolutely continuous selection g satisfying $g(t_0) = x_0$.*

For more about multifunctions we refer to [4, 11, 23, 24, 26].

Let $(\mathcal{A}, \geq)$ be a partially ordered set. A function $\gamma\colon P_b(E) \to \mathcal{A}$ is called a measure of noncompactness (MNC) in E if

$$\gamma(\overline{\operatorname{Conv}} B) = \gamma(B),$$

for every $B \in P_b(E)$.

The Hausdorff measure of noncompactness is defined on $P_b(E)$ as

$$\chi(B) = \inf\{\epsilon > 0 : B \text{ can be covered by finitely many balls of radius} \leq \epsilon\},$$

Lemma 2.8 (Generalized Cantor's intersection [4])**.** *If $(B_n)_{n \geq 1}$ is a decreasing sequence of nonempty closed subsets of E and $\lim_{n\to\infty} \chi(B_n) = 0$, then $\cap_{n=1}^{\infty} B_n$ is nonempty and compact.*

Lemma 2.9 ([10, p. 125]))**.** *Let B be a bounded set in E. Then for every $\varepsilon > 0$ there is a sequence $(x_n)_{n\geq 1}$ in B such that*

$$\chi(B) \leq 2\chi\{x_n : n \geq 1\} + \varepsilon.$$

Lemma 2.10 ([29])**.** *Let $\chi_{C(J,E)}$ be the Hausdorff measure of noncompactness on $C(J, E)$. If $W \subseteq C(J, E)$ is bounded, then for every $t \in J$,*

$$\chi(W(t)) \leq \chi_{C(J,E)}(W),$$

where $W(t) = \{x(t) : x \in W\}$. Furthermore, if W is equicontinuous on J, then the map $t \to \chi\{x(t) : x \in W\}$ is continuous on J and $\chi_{C(J,E)}(W) = \sup_{t\in J} \chi\{x(t) : x \in W\}$.

Lemma 2.11 ([20]). *Let $C \subseteq L^1(J,E)$ be countable with $\|u(t)\| \leq h(t)$ for a.e. $t \in J$, and every $u \in C$, where $h \in L^1(J,\mathbb{R}^+)$. Then the function $\varphi(t) = \chi\{u(t) : u \in C\}$ belongs to $L^1(J,\mathbb{R}^+)$ and satisfies*

$$\chi\left\{\int_0^b u(s)\,ds : u \in C\right\} \leq 2\int_0^b \chi\{u(s) : u \in C\}\,ds.$$

Lemma 2.12 ([5, Lemma 4]). *Let $\{f_n : n \in \mathbb{N}\} \subset L^P(J,E)$, $P \geq 1$ be an integrably bounded sequence such that*

$$\chi\{f_n(t) : n \geq 1\} \leq \gamma(t), \quad a.e.\ t \in J,$$

where $\gamma \in L^1(J,\mathbb{R}^+)$. Then for each $\epsilon > 0$ there exists a compact $K_\epsilon \subseteq E$, a measurable set $J_\epsilon \subset J$, with measure less than ϵ, and a sequence of functions $\{g_n^\epsilon\} \subset L^P(J,E)$ such that $\{g_n^\epsilon(t) : n \geq 1\} \subseteq K_\epsilon$, for all $t \in J$ and

$$\|f_n(t) - g_n^\epsilon(t)\| < 2\gamma(t) + \epsilon, \quad \text{for every } n \geq 1 \text{ and every } t \in J - J_\epsilon.$$

We need the following lemma which is related to [4, Theorem 1.1.4].

Lemma 2.13. *Let (K_n) be a sequence of subsets of E. Suppose there is a compact convex subset $K \subseteq E$ such that for any neighborhood U of K there is a natural number N so that for any $m \geq N : K_m \subseteq U$. Then $\cap_{j\geq 1} \overline{\mathrm{Conv}} \cup \{K_j : n \geq j\} \subseteq K$.*

Definition 2.14 ([6]). The Riemann–Liouville fractional integral of order $q > 0$ with lower limit zero for a function $f \in L^P(J,E)$, $P \in [1,\infty)$ is defined as follows:

$$I^q f(t) = (g_q * f)(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)\,ds, \quad t \in J,$$

where the integration is in the sense of Bochner, Γ is the Euler gamma function defined by $\Gamma(q) = \int_0^\infty t^{q-1}e^{-t}\,dt$, $g_q(t) = \frac{t^{q-1}}{\Gamma(q)}$, for $t > 0$, $g_q(t) = 0$, for $t \leq 0$ and $*$ denotes the convolution of functions. For $q = 0$, we set $I^0 f(t) = f(t)$. It is known that $I^q I^\beta f(t) = I^{q+\beta} f(t)$, $\beta, q \geq 0$.

Note that by applying the Young inequality, it follows that

$$\|I^{\,q} f\|_{L^P(J,E)} = \|g_q * f\|_{L^P(J,E)} \leq \|g_q\|_{L^1(J,\mathbb{R})}\|f\|_{L^P(J,E)} = g_{q+1}(b)\|f\|_{L^P(J,E)}.$$

Then I^q maps $L^P(J,E)$ to $L^P(J,E)$.

Definition 2.15 ([6]). Let $q > 0$ and m the smallest integer greater than or equal to q. The Riemann–Liouville fractional derivative of order q for a function $f \in L^1(J,E)$, $g_{m-q} * f \in W^{m,1}(J,E)$ is defined by

$$\begin{aligned} D^q f(t) &= \frac{d^m}{dt^m} I^{m-q} f(t) = \frac{d^m}{dt^m}(g_{m-q} * f)(t) \\ &= \frac{1}{\Gamma(m-q)} \frac{d^m}{dt^m} \int_0^t (t-s)^{m-q-1} f\,(s)\,ds, \quad t \in J, \end{aligned}$$

where

$$W^{m,1}(J,E) = \left\{ f : \exists \varphi \in L^1(J,E) : f(t) = \sum_{k=0}^{m-1} c_k \frac{t^k}{k!} + \int_0^t \frac{(t-s)^{m-1}}{(m-1)!}\varphi(s)\,ds, \quad t \in J \right\}.$$

Note that $\varphi(t) = f^{(m)}(t)$ and $c_k = f^{(k)}(0)$, $k = 0,1,\ldots,m-1$. Let $W_0^{m,1}(J,E) = \{f \in W^{m,1}(J,E) : f^{(k)}(0) = 0, k = 0,1,\ldots,m-1\}$.

Lemma 2.16 ([6, Lemma 1.8])**.** *Let $q > 0$, m be the smallest integer greater than or equal to q and $1 < P < \infty$. Let $\mathbb{L}_q$ be an operator with domain $L^P(J,E)$, defined by $\mathbb{L}_q(f) = I^q f = g_q * f$ and $\mathfrak{L}_q$ be an operator with domain $R_0^{m,P}(J,E) = \{f \in L^P(J,E) : g_{m-q} * f \in W_0^{m,P}(J,E)\}$ and defined by $\mathfrak{L}_q(f) = D^q f$. Then $\mathfrak{L}_q = \mathbb{L}_q^{-1}$.*

To know more about fractional calculus see [7,27,33]. The proof of the following lemma is the same way as in the scalar case (see [27,33]).

Lemma 2.17. *Let $AC(J,E)$ be the space of absolutely continuous functions defined on J to E and $q \in (0,1)$.*

(i) *If $f \in AC(J,E)$ and E is separable, then $I^q(D^q f(t)) = f(t)$, a.e. $t \in J$.*

(ii) *I^q maps $AC(J,E)$ to $AC(J,E)$.*

Proof. **(i)** Let $f \in AC(J,E)$. From Lemma 2.16, it suffices to show that $g_{1-q} * f \in W_0^{m,1}(J,E)$. Since E is separable, f has a Bochner integrable derivative $f^{(1)}$ almost everywhere and

$$f(s) = f(0) + \int_0^s f^{(1)}(x)\,dx.$$

Then

$$\begin{aligned}
(g_{1-q} * f)(t) &= \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q} f(s)\,ds \\
&= \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q} \left[f(0) + \int_0^s f^{(1)}(x)\,dx \right] ds \\
&= \frac{f(0)}{\Gamma(1-q)} \int_0^t (t-s)^{-q}\,ds + \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q} \left(\int_0^s f^{(1)}(x)\,dx \right) ds \\
&= \frac{f(0)}{\Gamma(2-q)} t^{1-q} + \frac{1}{\Gamma(1-q)} \int_0^t \left(\int_0^s (t-s)^{-q} f^{(1)}(x)\,dx \right) ds.
\end{aligned}$$

The first term is an absolutely continuous function because $t^{1-q} = (1-q)\int_0^t x^{-q}\,dx$. The second term is also a primitive of a Bochner integrable function and hence it is absolutely continuous. Moreover, $(g_{1-q} * f)(0) = 0$. So, $g_{1-q} * f \in W_0^{m,1}(J,E)$.

(ii) Let $f \in AC(J,E)$. By arguing as in (i), we get $I^{1-q} f \in AC(J,E)$. Again, since $0 < 1-q < 1$, then $I^q f = I^{1-(1-q)} f \in AC(J,E)$.

□

We denote by $C^m(J,E)$ the Banach space of m times continuously differentiable functions with the norm $\|f\|_m = \sup_{t\in J} \sum_{k=0}^m \|f^{(k)}(t)\|$ and

$$\begin{aligned}
AC^m(J,E) &= \{f \in C^{m-1}(J,E) : f^{(m-1)} \in AC(J,E)\} \\
&= \{f \in C^{m-1}(J,E) : f^{(m)} \in L^1(J,E)\}.
\end{aligned}$$

Definition 2.18 ([6])**.** Let $q > 0$ and m be the smallest integer greater than or equal to q. The *Caputo derivative of order q* for given function $f \in AC^m(J,E)$ is defined by

$$^cD^q f(t) = I^{m-q}(f^{(m)}(t)) = \frac{1}{\Gamma(m-q)} \int_0^t (t-s)^{m-q-1} f^{(m)}(s)\,ds, \quad t \in J.$$

We need the following lemma.

Lemma 2.19. *Let E be a separable Banach space and $f \in AC^3(J, E)$. Then*

$$I^\alpha({}^cD^\alpha f(t)) = f(t) + b_1 + tb_2 + t^2 b_3, \quad a.e.\ t \in J,$$

where b_1, b_2 and b_3 are elements in E.

Proof. In view of Definition 2.18 we have

$$I^\alpha({}^cD^\alpha f(t)) = I^\alpha(I^{3-\alpha}(f^{(3)}(t)) = I^3 f^{(3)}(t) = f(t) - \sum_{k=0}^{k=2} \frac{f^{(k)}(0)}{k!} t^k.$$

□

The following lemma is essential and its proof is similar to the proofs of Theorems 2.4 and 2.7 in [28].

Lemma 2.20. *Let E be a separable Banach space.*

(1) *If $h \in L^1(J, E)$ and $x \in AC^3(J, E)$ satisfy (1.3), then (1.4) holds.*

(2) *Let $h \in AC(J, E)$ and $x \colon J \to E$ such that (1.4) holds. Then $x \in AC^3(J, E)$ and satisfies (1.3).*

Proof. **(1)** Since $x \in AC^3(J, E)$, then by Lemma 2.19, there are b_1, b_2, b_3 in E such that

$$I^\alpha({}^cD^\alpha x(t)) = x(t) + b_1 + t\, b_2 \ + t^2 b_3, \quad \text{a.e. } t \in J\ .$$

Because x is a solution of (1.3)

$$I^\alpha h(t) = x(t) + b_1 + t\, b_2 \ + t^2 b_3, \quad \text{a.e. } t \in J\ .$$

Therefore,

$$\begin{aligned} x(t) &= I^\alpha h(t) - b_1 - tb_2 \ - t^2 b_3 \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)\, ds - b_1 - t\, b_2 \ - t^2 b_3, \quad \text{a.e. } t \in \ J. \end{aligned} \tag{2.1}$$

Since $h \in L^1(J, E)$ and $\alpha - 1 > 1$, then the function z defined by

$$z(s) := (b-s)^{\alpha-1} h(s)$$

belongs to $L^1(J, E)$. Then, $I^\alpha h \in AC(J, E)$. Hence, the two functions in both sides of (2.1) are continuous and thus (2.1) holds for every $t \in J$.

Then for $t \in J$

$$x^{(1)}(t) = \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} h(s)\, ds - b_2 \ - 2tb_3, \tag{2.2}$$

and

$$x^{(2)}(t) = \frac{1}{\Gamma(\alpha-2)} \int_0^t (t-s)^{\alpha-3} h(s)\, ds - 2b_3. \tag{2.3}$$

By applying the boundary conditions $x(0) = -x(b)$, $x^{(1)}(0) = -x^{(1)}(b)$, $x^{(2)}(0) = -x^{(2)}(b)$ and (2.1)–(2.3) we find that

$$b_1 = \frac{1}{2\Gamma(\alpha)} \int_0^b (b-s)^{\alpha-1} h(s)\,ds - \frac{b}{4\Gamma(\alpha-1)} \int_0^b (b-s)^{\alpha-2} h(s)\,ds,$$

$$b_2 = \frac{1}{2\Gamma(\alpha-1)} \int_0^b (b-s)^{\alpha-2} h(s)\,ds - \frac{b}{4\Gamma(\alpha-2)} \int_0^b (b-s)^{\alpha-3} h(s)\,ds,$$

and

$$b_3 = \frac{1}{4\Gamma(\alpha-2)} \int_0^b (b-s)^{\alpha-3} h(s)\,ds.$$

(2) Assume that $h \in AC(J,E)$ and (1.4) holds. Since $\alpha - 1 > 1$, the equation (1.4) gives us

$$\begin{aligned} x^{(1)}(t) = {} & \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} h(s)\,ds - \frac{1}{2\Gamma(\alpha-1)} \int_0^b (b-s)^{\alpha-2} h(s)\,ds \\ & + \frac{(b-2t)}{4\Gamma(\alpha-2)} \int_0^b (b-s)^{\alpha-3} h(s)\,ds. \end{aligned} \tag{2.4}$$

Since for each $t \in J$

$$\int_0^t (t-s)^{\alpha-3} h(s)\,ds < \infty,$$

then (2.4) implies

$$\begin{aligned} x^{(2)}(t) &= \frac{1}{\Gamma(\alpha-2)} \int_0^t (t-s)^{\alpha-3} h(s)\,ds - \frac{1}{2\Gamma(\alpha-2)} \int_0^b (b-s)^{\alpha-3} h(s)\,ds \\ &= I^{\alpha-2} h(t) - \frac{1}{2\Gamma(\alpha-2)} \int_0^b (b-s)^{\alpha-3} h(s)\,ds. \end{aligned} \tag{2.5}$$

Because $h \in AC(J,E)$ and $\alpha - 2 \in (0,1)$, then Lemma 2.17(ii) implies $I^{\alpha-2}h(t) \in AC(J,E)$ and thus $x^{(2)} \in AC(J,E)$, which means that $x \in AC^3(J,E)$. Moreover, because $3-\alpha \in (0,1)$, then from the definition of the Riemann–Liouville fractional derivative of order $3-\alpha$ and (2.5) we get

$$\begin{aligned} D^{3-\alpha} h(t) &= \frac{1}{\Gamma(\alpha-2)} \frac{d}{dt} \int_0^t (t-s)^{\alpha-3} h(s)\,ds \\ &= \frac{d}{dt}(I^{\alpha-2} h(t)) = x^{(3)}(t), \quad t \in J. \end{aligned}$$

Again, since $h \in AC(J,E)$ and $3-\alpha \in (0,1)$, then by Lemma 2.17 (i) and the last equality we get for a.e. $t \in J$

$$^{c}D^{\alpha} x(t) = I^{3-\alpha} x^{(3)}(t) = I^{3-\alpha} D^{3-\alpha} h(t) = h(t).$$

□

We need also to the following auxiliary results:

Lemma 2.21 (Kakutani–Glicksberg–Fan theorem [19]). *Let W be a nonempty compact and convex subset of a locally convex topological vector space. If $R\colon W \to P_{P_{cl,cv}}(W)$ is an u.s.c. multifunction, then it has a fixed point.*

Lemma 2.22 ([26, Prop. 3.5.1]). *Let W be a closed subset of a Banach space X and $R\colon W \to P_k(X)$ be a closed multifunction which is γ-condensing on every bounded subset of W, where γ is a monotone measure of noncompactness defined on X. If the set of fixed points for R is a bounded subset of X then it is compact.*

The following fixed point theorem for contraction multivalued is proved by Govitz and Nadler [14].

Lemma 2.23. *Let (X, d) be a complete metric space. If $R\colon X \to P_{cl}(X)$ is contraction, then R has a fixed point.*

3 Existence of solutions for the problem (1.1)

In the rest of the paper, E will denote a separable Banach space. In this section, we give an existence result of solutions of (1.1).

Theorem 3.1. *Let E be a separable Banach space, $f\colon J \times E \to E$ be a function. We assume the following hypothesis:*

(H_1) *For each $\rho > 0$, there exists $L_\rho > 0$ such that for all $s, t \in J$ and $x, y \in E$ with $\|x\| \le \rho$, $\|y\| \le \rho$ we have*
$$\|f(s, x) - f(t, y)\| \le L_\rho \max\{|s - t|, \|x - y\|\}.$$

Then, the problem (1.1) has a solution provided that there is a positive real number r such that

$$(L_r \max\{b, r\} + \|f(0, 0)\|) \frac{(2\alpha^2 - \alpha + 6)b^\alpha}{2\Gamma(\alpha + 1)} < r. \tag{3.1}$$

Proof. According to condition (H_1), there exists $L_r > 0$ such that for all $s, t \in J$ and $x, y \in E$ with $\|x\| \le r$, $\|y\| \le r$ we have

$$\|f(s, x) - f(t, y)\| \le L_r \max\{|s - t|, \|x - y\|\}. \tag{3.2}$$

Let us introduce a function $T\colon C(J, E) \to C(J, E)$ defined by

$$\begin{aligned}(Tx)(t) = {}& \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} f(s, x(s))\, ds - \frac{1}{2\Gamma(\alpha)} \int_0^b (b - s)^{\alpha-1} f(s, x(s))\, ds \\ &+ \frac{(b - 2t)}{4\Gamma(\alpha - 1)} \int_0^b (b - s)^{\alpha-2} f(s, x(s))\, ds \\ &+ \frac{t(b - t)}{4\Gamma(\alpha - 2)} \int_0^b (b - s)^{\alpha-3} f(s, x(s))\, ds, \quad t \in J.\end{aligned} \tag{3.3}$$

Firstly we prove, by using Schauder's fixed point theorem, that T has a fixed point. The proof will be given in several steps.
Step 1. Let $B_0 = \{x \in C(J, E) : \|x\| \le r\}$. Obviously, B_0 is a bounded, closed and convex subset of $C(J, E)$. We claim that $T(B_0) \subseteq B_0$. Let $x \in B_0$. We note that, by (3.2), for $t \in J$

$$\begin{aligned}\|f(t, x(t))\| &\le \|f(t, x(t)) - f(0, 0)\| + \|f(0, 0)\| \\ &\le L_r\ \max\{b, r\} + \|f(0, 0)\|\end{aligned} \tag{3.4}$$

Let $y = T(x)$. Then (3.3) and (3.4) imply that for $t \in J$

$$
\begin{aligned}
\|y(t)\| &\le (L_r \max\{b,r\} + \|f(0,0)\|) \\
&\quad \times \left[\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\, ds + \frac{1}{2\Gamma(\alpha)} \int_0^b (b-s)^{\alpha-1}\, ds \right.\\
&\qquad \left. + \frac{|b-2t|}{4\Gamma(\alpha-1)} \int_0^b (b-s)^{\alpha-2}\, ds + \frac{t(b-t)}{4\Gamma(\alpha-2)} \int_0^b (b-s)^{\alpha-3}\, ds \right] \\
&\le (L_r \ \max\{b,r\} + \|f(0,0)\|) \\
&\quad \times \left[\frac{b^\alpha}{\Gamma(\alpha+1)} + \frac{b^\alpha}{2\Gamma(\alpha+1)} + \frac{b^\alpha}{4\Gamma(\alpha)} + \frac{b^\alpha}{4\Gamma(\alpha-1)} \right] \\
&\le (L_r \max\{b,r\} + \|f(0,0)\|) \left[\frac{3b^\alpha}{2\Gamma(\alpha+1)} + \frac{b^\alpha}{4\Gamma(\alpha)} + \frac{b^\alpha}{4\Gamma(\alpha-1)} \right] \\
&= (L_r \max\{b,r\} + \|f(0,0)\|) \frac{(\alpha^2+6)b^\alpha}{4\Gamma(\alpha+1)} \\
&\le (L_r \max\{b,r\} + \|f(0,0)\|) \frac{(2\alpha^2-\alpha+6)b^\alpha}{2\Gamma(\alpha+1)} \\
&\le r.
\end{aligned}
$$

Therefore, $T(B_0) \subseteq B_0$.

Step 2. Let $Z = T(B_0)$. We claim that Z is equicontinuous. Let $y \in Z$. Then there is $x \in B_r$ with $y = T(x)$. Therefore, for $t, t+\lambda \in J$ we have

$$
\begin{aligned}
\|y(t+\lambda) - y(t)\| &\le \frac{1}{\Gamma(\alpha)} \left\| \int_0^t \left[(t+\lambda-s)^{\alpha-1} - (t-s)^{\alpha-1} \right] f(s,x(s))\, ds \right\| \\
&\quad + \frac{1}{\Gamma(\alpha)} \left\| \int_t^{t+\lambda} (t+\lambda-s)^{\alpha-1} f(s,x(s))\, ds \right\| \\
&\quad + \frac{2\lambda}{4\Gamma(\alpha-1)} \int_0^b (b-s)^{\alpha-2} \|f(s,x(s))\|\, ds \\
&\quad + \frac{|\lambda b - 2t\lambda - \lambda^2|}{4\Gamma(\alpha-2)} \int_0^b (b-s)^{\alpha-3} \|f(s,x(s))\|\, ds \\
&\le \frac{L_r \ \max\{b,r\} + \|f(0,0)\|}{\Gamma(\alpha)} \int_0^t \left[(t+\lambda-s)^{\alpha-1} - (t-s)^{\alpha-1} \right] ds \\
&\quad + \frac{L_r \ \max\{b,r\} + \|f(0,0)\|}{\Gamma(\alpha)} \int_t^{t+\lambda} (t+\lambda-s)^{\alpha-1}\, ds \\
&\quad + \frac{2\lambda(L_r \ \max\{b,r\} + \|f(0,0)\|)}{4\Gamma(\alpha-1)} \int_0^b (b-s)^{\alpha-2}\, ds \\
&\quad + \frac{(L_r \ \max\{b,r\} + \|f(0,0)\|)(|\lambda b - 2t\lambda - \lambda^2|)}{4\Gamma(\alpha-2)} \int_0^b (b-s)^{\alpha-3}\, ds.
\end{aligned}
$$

This inequality implies $\|y(t+\lambda) - y(t)\| \to 0$ as $\lambda \to 0$, independently of x. Therefore $Z = T(B_0)$ is equicontinuous.

Now for every $n \ge 1$, set $B_n = \overline{\text{Conv}}\, T(B_{n-1})$. From Step 1, $B_1 = \overline{\text{Conv}}\, T(B_0) \subseteq B_0$. Also $B_2 = \overline{\text{Conv}}\, T(B_1) \subseteq \overline{\text{Conv}}\, T(B_0) \subseteq B_1$. By induction, the sequence (B_n), $n \ge 1$ is a decreasing sequence of nonempty, closed convex and bounded subsets of $C(J,E)$. Our goal is to show that the subset $B = \cap_{n=1}^{\infty} B_n$ is nonempty and compact in $C(J,E)$. By Lemma 2.8, it is enough to show that

$$
\lim_{n\to\infty} \chi_{C(J,E)}(B_n) = 0, \tag{3.5}
$$

where $\chi_{C(J,E)}$ is the Hausdorff measure of noncompactness on $C(J,E)$.
Step 3. Our aim in this step is to show that the relation (3.5) is satisfied. Let $n \geq 1$ be a fixed natural number and $\varepsilon > 0$. In view of Lemma 2.9, there exists a sequence (y_k), $k \geq 1$ in $T(B_{n-1})$ such that

$$\chi_{C(J,E)}(B_n) = \chi_{C(J,E)}T(B_{n-1}) \leq 2\chi_{C(J,E)}\{y_k : k \geq 1\} + \varepsilon.$$

From Step 2, B_{n-1} is equicontinuous. This together with Lemma 2.10 and by using the nonsingularity of χ, the above inequality becomes

$$\chi_{C(J,E)}(B_n) \leq 2\sup_{t\in J}\chi\{y_k(t) : k \geq 1\} + \varepsilon. \tag{3.6}$$

Because $y_k = T(B_{n-1})$, $k \geq 1$ there is $x_k \in B_{n-1}$ such that $y_k = T(x_k)$, $k \geq 1$. Let $t \in J$ be fixed. Note that from (3.2) for every natural number m, n we have

$$\|f(t, x_m(t)) - f(t, x_n(t))\| \leq L_r\|x_m(t) - x_n(t)\|.$$

Then

$$\begin{aligned}\chi\{f(t, x_k(t)) : k \geq 1\} &\leq L_r\chi\{x_k(t) : k \geq 1\}\\ &\leq L_r\chi_{C(J,E)}(B_{n-1}).\end{aligned} \tag{3.7}$$

From (3.7) and the properties of χ, for $t \in J$ we get

$$\begin{aligned}\chi\{y_k(t) : k \geq 1\} \leq{}& \frac{L_r}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\chi\{x_k(s) : k \geq 1\}\,ds\\ &+ \frac{L_r}{2\Gamma(\alpha)}\int_0^b (b-s)^{\alpha-1}\chi\{x_k(s)) : k \geq 1\}\,ds\\ &+ \frac{L_r b}{4\Gamma(\alpha-1)}\int_0^b (b-s)^{\alpha-2}\chi\{x_k(s)) : k \geq 1\}\,ds\\ &+ \frac{b^2}{4\Gamma(\alpha-2)}\chi\left\{\int_0^b (b-s)^{\alpha-3}f(s, x_k(s))\,ds : k \geq 1\right\}.\end{aligned}$$

Therefore,

$$\begin{aligned}\chi\{y_k(t) : k \geq 1\} \leq L_r\chi(B_{n-1})&\left[\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\,ds\right.\\ &+ \frac{1}{2\Gamma(\alpha)}\int_0^b (b-s)^{\alpha-1}\,ds\\ &\left.+ \frac{b}{4\Gamma(\alpha-1)}\int_0^b (b-s)^{\alpha-2}\,ds\right]\\ + \frac{b^2}{4\Gamma(\alpha-2)}&\chi\left\{\int_0^b (b-s)^{\alpha-3}f(s, x_k(s))\,ds : k \geq 1\right\}.\end{aligned} \tag{3.8}$$

Now in order to estimate the quantity $\chi\{\int_0^b (b-s)^{\alpha-3}f(s, x_k(s))\,ds : k \geq 1\}$, we note that (3.4) implies for any $k \geq 1$ and for $t \in J$

$$\|f(s, x_k(s))\| \leq L_r\max\{b, r\} + \|f(0,0)\|. \tag{3.9}$$

Then, by (3.7), (3.9) and Lemma 2.12, there exist a compact $K_\epsilon \subseteq E$, a measurable set $J_\epsilon \subset J$ with measure less than ϵ, and a sequence of functions $\{z_k^\epsilon\} \subset L^1(J,E)$ such that for all $s \in J$, $\{z_k^\epsilon(s) : k \geq 1\} \subseteq K_\epsilon$ and

$$\|f(s,x_k(s)) - z_k^\epsilon(s)\| < 2L_r\chi_{C(J,E)}(B_{n-1}) + \epsilon,$$

for every $k \geq 1$ and every $s \in J - J_\epsilon$. Therefore, for any $k \geq 1$

$$\begin{aligned}\left\|\int_{J-J_\epsilon}(b-s)^{\alpha-3}(f(s,x_k(s)) - z_k^\epsilon(s))\,ds\right\| \\ \leq (2L_r\chi_{C(J,E)}(B_{n-1}) + \epsilon)\frac{b^{\alpha-2}}{\alpha-2}.\end{aligned} \tag{3.10}$$

Also, for any $k \geq 1$

$$\begin{aligned}\left\|\int_{J_\epsilon}(b-s)^{\alpha-3}f(s,x_k(s)\,ds\right\| \\ \leq (L_r\max\{b,r\} + \|f(0,0)\|)\int_{J_\epsilon}(b-s)^{\alpha-3}\,ds.\end{aligned} \tag{3.11}$$

This together with (3.10) we have

$$\begin{aligned}\chi\left\{\int_0^b(b-s)^{\alpha-3}f(s,x_k(s))\,ds : k \geq 1\right\} &\leq \chi\left\{\int_{J-J_\epsilon}(b-s)^{\alpha-3}(f(s,x_k(s)) - z_k^\epsilon(s))\,ds : k \geq 1\right\} \\ &\quad + \chi\left\{\int_{J-J_\epsilon}(b-s)^{\alpha-3}z_k^\epsilon(s)\,ds : k \geq 1\right\} \\ &\quad + \chi\left\{\int_{J_\epsilon}(b-s)^{\alpha-3}f(s,x_k(s)\,ds : k \geq 1\right\} \\ &\leq (2L_r\chi_{C(J,E)}(B_{n-1}) + \epsilon)\frac{b^{\alpha-2}}{\alpha-2}. \\ &\quad + (L_r\ \max\{b,r\} + \|f(0,0)\|)\int_{J_\epsilon}(b-s)^{\alpha-3}\,ds.\end{aligned}$$

From this inequality and by taking into account that ε is arbitrary, we get

$$\begin{aligned}\chi_{C(J,E)}(B_n) &\leq 2L_r\chi_{C(J,E)}(B_{n-1})\left[\frac{3b^\alpha}{2\Gamma(\alpha+1)} + \frac{b^\alpha}{4\Gamma(\alpha)}\right] \\ &\quad + L_r\chi_{C(J,E)}(B_{n-1})\frac{b^\alpha}{\Gamma(\alpha-1)} \\ &= \zeta\chi_{C(J,E)}(B_{n-1}),\end{aligned} \tag{3.12}$$

where

$$\begin{aligned}\zeta &= 2L_r\left[\frac{3b^\alpha}{2\Gamma(\alpha+1)} + \frac{b^\alpha}{4\Gamma(\alpha)} + \frac{b^\alpha}{2\Gamma(\alpha-1)}\right] \\ &= L_r\frac{(2\alpha^2-\alpha+6)b^\alpha}{2\Gamma(\alpha+1)}.\end{aligned}$$

By means of a finite number of steps, we obtain from (3.12) for every $n \in \mathbb{N}$,

$$\chi_{C(J,E)}(B_n) \leq \zeta^{n-1}\chi_{C(J,E)}(B_{n-1}). \tag{3.13}$$

Observe that (3.1) implies

$$\begin{aligned} r\zeta &= rL_r\frac{(2\alpha^2-\alpha+6)b^\alpha}{2\Gamma(\alpha+1)} \\ &\le (L_r\max\{b,r\}+\|f(0,0)\|)\frac{(2\alpha^2-\alpha+6)b^\alpha}{2\Gamma(\alpha+1)} \\ &< r. \end{aligned}$$

Then $\zeta < 1$. By passing to the limit as $n \to +\infty$ in (3.13) we obtain (3.5) and so our aim in this step is verified. Therefore, the set $B = \cap_{n=1}^{\infty} B_n$ is a nonempty and compact subset of $C(J,E)$. Moreover, every B_n being bounded, closed and convex, B is also bounded closed and convex.

Step 4. Let us verify that $T(B) \subseteq B$.

Indeed, $T\ (B) \subseteq T(B_n) \subseteq \overline{\text{Conv}}\,T(B_n) = B_{n+1}$, for every $n \ge 1$. Therefore, $T(B) \subset \cap_{n=2}^{\infty} B_n$. On the other hand, $B_n \subset B_1$ for every $n \ge 1$. So, $T(B) \subset \cap_{n=2}^{\infty} B_n = \cap_{n=1}^{\infty} B_n = B$.

Step 5. The function $T|_B \colon B \to 2^B$ is continuous. Consider a sequence $\{x_n\}_{n\ge1}$ in B with $x_n \to x$ in B and let $y_n = T(x_n)$. We have to show that $\lim_{n\to\infty} y_n = T(x)$. For any $n \ge 1$ and $t \in J$

$$\begin{aligned} y_n(t) = {} & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,x_n(s))\,ds - \frac{1}{2\Gamma(\alpha)}\int_0^b (b-s)^{\alpha-1} f(s,x_n(s))\,ds \\ & + \frac{(b-2t)}{4\Gamma(\alpha-1)}\int_0^b (b-s)^{\alpha-2} f(s,x_n(s))\,ds \\ & + \frac{t(b-t)}{4\Gamma(\alpha-2)}\int_0^b (b-s)^{\alpha-3} f(s,x_n(s))\,ds. \end{aligned} \tag{3.14}$$

Note that for every $t \in J$

$$\|f(t,x_n(t))(t)\| \le L_r\max\{b,r\}+\|f(0,0)\|.$$

Furthermore,

$$\lim_{n\to\infty}\|f(t,x_n(t))-f(t,x(t))\| = 0.$$

Therefore, by passing to the limit as $n \to \infty$ in (3.14) we get $\lim_{n\to\infty} y_n = T(x)$.

As a consequence of Steps 1–5 and Schauder's fixed point theorem, there is $x \in B$ such that $x = T(x)$. That is, for any $t \in J$

$$\begin{aligned} x(t) = {} & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} h(s)\,ds - \frac{1}{2\Gamma(\alpha)}\int_0^b (b-s)^{\alpha-1} h(s)\,ds \\ & + \frac{(b-2t)}{4\Gamma(\alpha-1)}\int_0^b (b-s)^{\alpha-2} h(s)\,ds \\ & + \frac{t(b-t)}{4\Gamma(\alpha-2)}\int_0^b (b-s)^{\alpha-3} h(s)\,ds, \end{aligned}$$

where $h(t) = f(t,x(t))$. Obviously h is continuous, and hence $x^{(1)}(t)$ exists. Then there is a positive number η such that $\|x^{(1)}(t)\| \le \eta,\ t \in J$.

Thus, by (H_1) for $t,s \in J$

$$\begin{aligned} \|h(t)-h(s)\| &\le L_r\max\{|t-s|,\eta|t-s|\} \\ &\le |t-s|L_r\max\{1,\eta\}. \end{aligned}$$

This means that $h \in AC(J,E)$ and hence by Lemma 2.20(2) the function x is a solution for (1.1). □

In the following corollary we simplify the condition (3.1).

Corollary 3.2. *Assume that the assumption (H_1) is satisfied with $L_\rho \leq \sigma$, for any $\rho > 0$ then the problem (1.1) has a solution provided that*

$$\frac{\sigma(2\alpha^2-\alpha+6)b^\alpha}{2\Gamma(\alpha+1)} < 1. \tag{3.15}$$

Proof. From (3.15) we can take r such that

$$\max\left\{ b, \frac{\|f(0,0)\|\frac{(2\alpha^2-\alpha+6)b^\alpha}{2\Gamma(\alpha+1)}}{1-\frac{\sigma(2\alpha^2-\alpha+6)b^\alpha}{2\Gamma(\alpha+1)}} \right\} < r. \tag{3.16}$$

We need only to check that $T(B_0) \subseteq B_0$. As in Step 1, let $x \in B_0$ and $y = T(x)$. Then by (3.16) for any $t \in J$

$$\|y(t)\| \leq (\sigma r + \|f(0,0)\|)\frac{(2\alpha^2-\alpha+6)b^\alpha}{2\Gamma(\alpha+1)} < r.$$

□

Theorem 3.3. *Let $G\colon J \to P_k(E)$ be an absolutely continuous multifunction. Then the fractional boundary value problem*

$$\begin{cases} {}^cD_w^\alpha\, x(t) \in G(t), & \text{a.e. on } J=[0,b], \\ x(0) = -x(b), \quad x^{(1)}(0) = -x^{(1)}(b), & x^{(2)}(0) = -x^{(2)}(b), \end{cases} \tag{3.17}$$

has a solution.

Proof. In virtue of Lemma 2.7 there is $h \in AC(J,E)$ such that $h(t) \in G(t)$, a.e. We define $x\colon J \to E$, by

$$\begin{aligned} x(t) = {} & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)\,ds - \frac{1}{2\Gamma(\alpha)}\int_0^b (b-s)^{\alpha-1}h(s)\,ds \\ & + \frac{(b-2t)}{4\Gamma(\alpha-1)}\int_0^b (b-s)^{\alpha-2}h(s)\,ds \\ & + \frac{t(b-t)}{4\Gamma(\alpha-2)}\int_0^b (b-s)^{\alpha-3}h(s)\,ds. \end{aligned}$$

By Lemma 2.20 (2), $x \in AC^3(J,E)$ and satisfies (3.17). □

4 Existence of solutions for the problem (1.2)

To give existence results of solutions for the problem (1.2) we present the definition of the Caputo derivative in the generalized sense [27, Sec. 2.4].

Definition 4.1. Let $q > 0$ and m the smallest integer greater than or equal to q. The generalized or weak Caputo derivative of order q with lower limit zero for a function $f \in J \to E$ is defined by

$$^cD_g^q f(t) = D^q \left[f(t) - \sum_{k=0}^{m-1} \frac{f^{(k)}(0)t^k}{k!} \right].$$

So, the generalized or weak Caputo derivative $^cD_g^q f(t)$ is defined for function f for which the Riemann–Liouville fractional derivative exists. In particular, when $q \in (0,1)$, we have

$$^cD_g^q f(t) = D^q[f(t) - f(0)] = \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q}(f(t) - f(0))\, ds. \tag{4.1}$$

As in the scalar case (see Theorems 2.1 and 2.2 [27]) if $f \in C^m(J,E)$ then $^cD_g^q f(t) = {}^cD^q f(t)$. But it is enough that $f \in AC^m(J,E)$.

Lemma 4.2. *Let $q > 0$ and m be the smallest integer greater than or equal to q. If $f \in AC^m(J,E)$, then $^cD_g^q f(t)$ exists almost everywhere on J and if q is not natural number, then*

$$^cD_g^q f(t) = {}^c D^q f(t), \quad a.e.$$

Also, if $f \in C^m(J,E)$, then $^cD_g^q f(t)$ is continuous and if q is not natural number, then

$$^cD_g^q f(t) = {}^c D^q f(t), \quad \forall t \in J.$$

Next, we need the following auxiliary lemma.

Lemma 4.3. **(1)** *Let $z \in C(J,E)$ and $x \in C^2(J,E)$ such that $x(0) = -x(b)$, $x^{(1)}(0) = -x^{(1)}(b)$. If x is a solution to the fractional boundary value problem*

$$\begin{cases} ^cD_g^{\alpha-2}\, x^{(2)}(t) = z(t), & a.e.\ t \in J, \\ x^{(2)}(0) = -x^{(2)}(b), \end{cases} \tag{4.2}$$

then

$$\begin{aligned} x(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} z(s)\, ds - \frac{1}{2\Gamma(\alpha)} \int_0^b (b-s)^{\alpha-1} z(s)\, ds \\ &\quad + \frac{(b-2t)}{4\Gamma(\alpha-1)} \int_0^b (b-s)^{\alpha-2} z(s)\, ds \\ &\quad + \frac{t(b-t)}{4\Gamma(\alpha-2)} \int_0^b (b-s)^{\alpha-3} z(s)\, ds, \quad t \in J. \end{aligned} \tag{4.3}$$

(2) *Let $z \in C(J,E)$ and $x \colon J \to E$ such that (4.3) holds. Then $x \in C^2(J,E), x(0) = -x(b)$, $x^{(1)}(0) = -x^{(1)}(b)$ and satisfies (4.2).*

Proof. **(1)** Since x is a solution of (4.2), then

$$D^{\alpha-2}(x^{(2)}(t) - x^{(2)}(0)) = z(t), \quad \text{a.e. } t \in J.$$

This equation implies

$$I^{\alpha-2} D^{\alpha-2}(x^{(2)}(t) - x^{(2)}(0)) = I^{\alpha-2} z(t), \quad \text{a.e. } t \in J.$$

Therefore,

$$x^{(2)}(t) = I^{\alpha-2}z(t) + x^{(2)}(0).$$

From this equation together the condition $x^{(2)}(0) = -x^{(2)}(b)$ we get

$$x^{(2)}(t) = I^{\alpha-2}z(t) - \frac{1}{2}I^{\alpha-2}z(b).$$

By integrating both sides in this equation we get for a.e. $t \in J$

$$x^{(1)}(t) = I^{\alpha-1}z(t) - \frac{t}{2}I^{\alpha-2}z(b) + c_1.$$

By using the condition $x^{(1)}(0) = -x^{(1)}(b)$, we obtain for a.e. $t \in J$

$$x^{(1)}(t) = I^{\alpha-1}z(t) - \frac{t}{2}I^{\alpha-2}z(b) - \frac{1}{2}I^{\alpha-1}z(b) + \frac{b}{4}I^{\alpha-2}z(b).$$

Again, by integrating both sides in this equation we have for a.e. $t \in J$

$$x(t) = I^{\alpha}z(t) - \frac{t^2}{4}I^{\alpha-2}z(b) - \frac{t}{2}I^{\alpha-1}z(b) + \frac{tb}{4}I^{\alpha-2}z(b) + c_2.$$

Applying the condition $x(0) = -x(b)$ we get for a.e. $t \in J$

$$\begin{aligned} x(t) &= I^{\alpha}z(t) - \frac{t^2}{4}I^{\alpha-2}z(b) - \frac{t}{2}I^{\alpha-1}z(b) + \frac{tb}{4}I^{\alpha-2}z(b) \\ &\quad - \frac{1}{2}I^{\alpha}z(b) + \frac{b^2}{8}I^{\alpha-2}z(b) + \frac{b}{4}I^{\alpha-1}z(b) - \frac{b^2}{8}I^{\alpha-2}z(b) \\ &= I^{\alpha}z(t) - \frac{1}{2}I^{\alpha}z(b) + \frac{b-2t}{4}I^{\alpha-1}z(b) + \frac{t(b-t)}{4}I^{\alpha-2}z(b). \end{aligned}$$

The two functions on both sides of this equation are continuous, thus it holds for every $t \in J$.

(2) Clearly $x(0) = -x(b)$. Since $\alpha - 1 > 1$, the equation (4.3) gives us

$$\begin{aligned} x^{(1)}(t) &= \frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}z(s)\,ds - \frac{1}{2\Gamma(\alpha-1)}\int_0^b (b-s)^{\alpha-2}z(s)\,ds \\ &\quad + \frac{(b-2t)}{4\Gamma(\alpha-2)}\int_0^b (b-s)^{\alpha-3}z(s)\,ds. \end{aligned} \tag{4.4}$$

Note that $x^{(1)}(0) = -x^{(1)}(b)$. Moreover, since for each $t \in J$

$$\int_0^t (t-s)^{\alpha-3}z(s)\,ds < \infty,$$

then the equation (4.4) implies

$$\begin{aligned} x^{(2)}(t) &= \frac{1}{\Gamma(\alpha-2)}\int_0^t (t-s)^{\alpha-3}z(s)\,ds - \frac{1}{2\Gamma(\alpha-2)}\int_0^b (b-s)^{\alpha-3}z(s)\,ds \\ &= \frac{1}{\Gamma(\alpha-2)}\int_0^t (t-s)^{\alpha-3}z(s)\,ds \\ &\quad - \frac{1}{2\Gamma(\alpha-2)}\int_0^b (b-s)^{\alpha-3}z(s)\,ds \\ &= I^{\alpha-2}z(t) - \frac{1}{2\Gamma(\alpha-2)}\int_0^b (b-s)^{\alpha-3}z(s)\,ds. \end{aligned} \tag{4.5}$$

One can easily check that

$$x^{(2)}(0) = -x^{(2)}(b) = \frac{-1}{2\Gamma(\alpha-2)} \int_0^b (b-s)^{\alpha-3} z(s)\, ds.$$

Now, let $q = \alpha - 2$. Then $q \in (0,1)$. In virtue of the equation (4.5) we get

$$I^{\alpha-2} z(t) = x^{(2)}(t) - x^{(2)}(0), \quad t \in J.$$

This equation together with (4.1) implies

$$\begin{aligned} {}^cD_g^q x^{(2)}(t) &= D^q[x^{(2)}(t) - x^{(2)}(0)] \\ &= D^q I^q z(t) = z(t), \quad \text{a.e. } t \in J. \end{aligned}$$

Then, $x^{(2)}$satisfies the fractional boundary value problem

$$\begin{cases} {}^cD_g^q x^{(2)}(t) = z(t), \quad \text{a.e. } t \in J \\ x^{(2)}(0) = -x^{(2)}(b) = \dfrac{-1}{2\Gamma(\alpha-2)} \displaystyle\int_0^b (b-s)^{\alpha-3} z(s)\, ds. \end{cases}$$

Therefore, the function x given by (4.3) is a solution for (4.2).

□

Remark 4.4. If $x \in AC^3(J,E)$, then by Lemma 4.2, for almost every $t \in J$

$$\begin{aligned} {}^cD_g^{\alpha-2}\, x^{(2)}(t) &= {}^cD^{\alpha-2}\, x^{(2)}(t) \\ &= \frac{1}{\Gamma(3-\alpha)} \int_0^t (t-s)^{2-\alpha} x^{(3)}(s)\, ds \\ &= {}^cD^{\alpha} x(t). \end{aligned}$$

This means that if $z \in AC(J,E)$, then the function x given by (4.3) is a solution for the fractional boundary value problem

$$\begin{cases} {}^cD^{\alpha} x(t) = z(t), \quad \text{a.e. } t \in J, \\ x(0) = -x(b), \quad x^{(1)}(0) = -x^{(1)}(b), \quad x^{(2)}(0) = -x^{(2)}(b). \end{cases}$$

As a result of Lemma 4.3 we can give the concept of solutions for (1.2) in the following definition.

Definition 4.5. A function $x \in C^2(J,E)$ is said to be a solution of (1.2) if there is $f \in L^1(J,E)$ with $f(t) \in F(t, x(t))$, a.e. $t \in J$ such that

$$\begin{aligned} x(t) = &\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds - \frac{1}{2\Gamma(\alpha)} \int_0^b (b-s)^{\alpha-1} f(s)\, ds \\ &+ \frac{(b-2t)}{4\Gamma(\alpha-1)} \int_0^b (b-s)^{\alpha-2} f(s)\, ds + \frac{t(b-t)}{4\Gamma(\alpha-2)} \int_0^b (b-s)^{\alpha-3} f(s)\, ds. \end{aligned} \tag{4.6}$$

Now we are in position to give existence results of solutions of (1.2). We consider first the case when the values of the multifunction F are convex.

4.1 Convex case

Theorem 4.6. *Let $\alpha \in (2,3]$, $F\colon J \times E \to P_{ck}(E)$ be a multifunction. We assume the following hypotheses:*

(H_2) *For every $x \in E$, $t \longrightarrow F(t,x)$ is measurable, for every $t \in J$, $x \longrightarrow F(t,x)$ is upper semicontinuous.*

(H_3) *There exists a function $\varphi \in L^{\frac{3}{q}}(J,\mathbb{R}^+)$, $0 < q < 3\alpha - 6$, and a nondecreasing continuous function $\Omega\colon [0,\infty) \to (0,\infty)$ such that for any $x \in E$*

$$\|F(t,x)\| \le \varphi(t)\,\Omega(\|x\|), \quad a.e.\ t \in J. \tag{4.7}$$

(H_4) *There exists a function $\beta \in L^{\frac{3}{q}}(J,\mathbb{R}^+)$, $0 < q < 3\alpha - 6$, satisfying*

$$\delta = \frac{b^{\alpha-1}(\alpha+4)}{\Gamma(\alpha)}\|\beta\|_{L^1_{(J,\,\mathbb{R}^+)}} + \frac{b^2}{\Gamma(\alpha-2)}\rho\|\beta\|_{L^{\frac{3}{q}}(J,\,\mathbb{R}^+)} < 1, \tag{4.8}$$

and for every bounded subset $D \subseteq E$, $\chi(F(t,D)) \le \beta(t)\chi(D)$, for a.e. $t \in J$, where $\rho = b^{\alpha-2-\frac{q}{3}}/\eta^{1-\frac{q}{3}}$ and $\eta = 3(\alpha-3)/(3-q)+1$.

Then the problem (1.2) has a solution provided that there is $r > 0$ such that

$$\frac{(\alpha+5)\Omega(r)b^{\alpha-1}}{4\Gamma(\alpha)}\|\varphi\|_{L^1(J,\mathbb{R}^+)} + \frac{\Omega(r)}{4\Gamma(\alpha-2)}\|\varphi\|_{L^{\frac{3}{q}}(J,\mathbb{R}^+)}\frac{b^{\alpha-\frac{q}{3}}}{\eta^{1-\frac{q}{3}}} \le r. \tag{4.9}$$

Proof. At first, in view of (H_2), Lemma 2.2 and Remark 2.3, for every $x \in C(J,E)$, the multifunction $t \to F(t,x(t))$ has a measurable selection and by (H_3) this selection belongs to $S^1_{F(\cdot,x(\cdot))}$. So, we can introduce the multifunction $R\colon C(J,E) \to 2^{C(J,E)}$ which is defined as: let $x \in C(J,E)$. A function $y \in R(x)$ if and only if there is $f \in S^1_{F(\cdot,x(\cdot))}$ such that for each $t \in J$

$$\begin{aligned} y(t) = {} & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)\,ds - \frac{1}{2\Gamma(\alpha)}\int_0^b (b-s)^{\alpha-1} f(s)\,ds \\ & + \frac{(b-2t)}{4\Gamma(\alpha-1)}\int_0^b (b-s)^{\alpha-2} f(s)\,ds \\ & + \frac{t(b-t)}{4\Gamma(\alpha-2)}\int_0^b (b-s)^{\alpha-3} f(s)\,ds. \end{aligned}$$

It is easy to see that any fixed point for R is a mild solution for (1.2). So our goal is to prove, by using Lemma 2.21, that R has a fixed point. The proof will be given in several steps.
Step 1. Let $D_0 = \{x \in C(J,E) : \|x\| \le r\}$. We claim that $R(D_0) \subseteq D_0$. To prove that, let $x \in B_0$ and $y \in R(x)$. By recalling the definition of R, using (4.9) and applying Hölder's inequality,

there is $f \in S^1_{F(\cdot,x(\cdot))}$ such that for every $t \in J$

$$\begin{aligned}
\|y(t)\| &\le \frac{\Omega(r)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\varphi(s)ds + \frac{\Omega(r)}{2\Gamma(\alpha)} \int_0^b (b-s)^{\alpha-1}\varphi(s)\, ds \\
&\quad + \frac{|b-2t|\Omega(r)}{4\Gamma(\alpha-1)} \int_0^b (b-s)^{\alpha-2}\varphi(s)\, ds + \frac{t(b-t)\Omega(r)}{4\Gamma(\alpha-2)} \int_0^b (b-s)^{\alpha-3}\varphi(s)\, ds \\
&\le \frac{\Omega(r)b^{\alpha-1}}{\Gamma(\alpha)}\|\varphi\|_{L^1(J,\mathbb{R}^+)} + \frac{\Omega(r)b^{\alpha-1}}{2\Gamma(\alpha)}\|\varphi\|_{L^1(J,\mathbb{R}^+)} + \frac{\Omega(r)b^{\alpha-1}}{4\Gamma(\alpha-1)}\|\varphi\|_{L^1(J,\mathbb{R}^+)} \\
&\quad + \frac{b^2\Omega(r)}{4\Gamma(\alpha-2)}\|\varphi\|_{L^{\frac{3}{q}}(J,\mathbb{R}^+)} \left(\int_0^b (b-s)^{\frac{3(\alpha-3)}{3-q}}\, ds\right)^{\frac{3-q}{3}} \\
&\le \frac{\Omega(r)b^{\alpha-1}}{\Gamma(\alpha)}\|\varphi\|_{L^1(J,\mathbb{R}^+)}\left(\frac{3}{2} + \frac{\alpha-1}{4}\right) + \frac{b^2\Omega(r)}{4\Gamma(\alpha-2)}\|\varphi\|_{L^{\frac{3}{q}}(J,\mathbb{R}^+)} \frac{b^{\alpha-2-\frac{q}{3}}}{\eta^{1-\frac{q}{3}}} \\
&\le \frac{(\alpha+5)\Omega(r)b^{\alpha-1}}{4\Gamma(\alpha)}\|\varphi\|_{L^1(J,\mathbb{R}^+)} + \frac{\Omega(r)}{4\Gamma(\alpha-2)}\|\varphi\|_{L^{\frac{3}{q}}(J,\mathbb{R}^+)} \frac{b^{\alpha-\frac{q}{3}}}{\eta^{1-\frac{q}{3}}} \\
&\le r.
\end{aligned}$$

Therefore, $R(D_0) \subseteq D_0$.

Step 2. Let $M = R(D_0)$. We claim that M is equicontinuous. Let $y \in M$. Then there is $x \in D_0$ with $y \in R(x)$. By recalling the definition of R, there is $f \in S^1_{F(\cdot,x(\cdot))}$ such that

$$\begin{aligned}
y(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds - \frac{1}{2\Gamma(\alpha)} \int_0^b (b-s)^{\alpha-1} f(s)\, ds \\
&\quad + \frac{(b-2t)}{4\Gamma(\alpha-1)} \int_0^b (b-s)^{\alpha-2} f(s)\, ds + \frac{t(b-t)}{4\Gamma(\alpha-2)} \int_0^b (b-s)^{\alpha-3} f(s)\, ds.
\end{aligned}$$

By arguing as in Step 2 in the proof of Theorem 3.1, for any $t,\ t+\lambda \in J$ we have

$$\begin{aligned}
\|y(t+\lambda) - y(t)\| &\le \frac{1}{\Gamma(\alpha)} \left\| \int_0^t \left[(t+\lambda-s)^{\alpha-1} - (t-s)^{\alpha-1}\right] f(s)\, ds \right\| \\
&\quad + \frac{1}{\Gamma(\alpha)} \left\| \int_t^{t+\lambda} (t+\lambda-s)^{\alpha-1} f(s)\, ds \right\| \\
&\quad + \frac{2\lambda}{4\Gamma(\alpha-1)} \int_0^b (b-s)^{\alpha-2} \|f(s)\|\, ds \\
&\quad + \frac{|\lambda b - 2t\lambda - \lambda^2|}{4\Gamma(\alpha-2)} \int_0^b (b-s)^{\alpha-3} \|f(s)\|\, ds.
\end{aligned}$$

Not that by the Hölder's inequality we have

$$\left\| \int_t^{t+\lambda} (t+\lambda-s)^{\alpha-1} f(s)\, ds \right\| \le \Omega(r)\|\varphi\|_{L^{\frac{3}{q}}(J,\mathbb{R}^+)} \frac{\lambda^{1-\frac{q}{3}}}{\eta^{1-\frac{q}{3}}}.$$

The previous two inequalities imply that $\|y(t+\lambda) - y(t)\| \to 0$ as $\lambda \to 0$, independently of x. Therefore $M = R(D_0)$ is equicontinuous.

Now for every $n \ge 1$, set $D_n = \overline{\mathrm{Conv}}\, R(D_{n-1})$ and $D = \cap_{n=1}^{\infty} D_n$. Clearly the sequence (D_n), $n \ge 1$ is a decreasing sequence of nonempty, closed, convex and bounded subsets of $C(J,E)$. Our goal is to use Lemma 2.8 to show that D is nonempty and compact in $C(J,E)$. So we show that

$$\lim_{n\to\infty} \chi(B_n) = 0, \tag{4.10}$$

where $\chi_{C(J,E)}$ is the Hausdorff measure of noncompactness on $C(J,E)$.

Step 3. Our aim in this step is to show that the relation (4.10) is satisfied. Let $n \geq 1$ be a fixed natural number and $\varepsilon > 0$. In view of Lemma 2.9, there exists a sequence (y_k), $k \geq 1$ in $R(D_{n-1})$ such that

$$\chi_{C(J,E)}(D_n) = \chi_{C(J,E)}R(D_{n-1}) \leq 2\chi_{C(J,E)}\{y_k : k \geq 1\} + \varepsilon.$$

By applying Lemma 2.10 and by using the nonsingularity of χ, the above inequality becomes

$$\chi_{C(J,E)}(D_n) \leq \; 2\sup_{t\in J} \chi\{y_k(t) : k \geq 1\} + \varepsilon. \tag{4.11}$$

Now, since $y_k \in R(D_{n-1})$, $k \geq 1$ there is $x_k \in D_{n-1}$ such that $y_k \in R(x_k)$, $k \geq 1$. By recalling the definition of R, repeating the same procedure as in Step 3 in the proof of Theorem 3.1, applying Lemma 2.11 and using (H_4), for every $k \geq 1$ there is $f_k \in S^1_{F(\cdot,x_k(\cdot))}$ such that for every $t \in J$

$$\begin{aligned}
\chi\{y_k(t) : k \geq 1\} &\leq \frac{b^{\alpha-1}(\alpha+4)}{2\Gamma(\alpha)} \int_0^b \beta(s)\chi\{x_k(s) : k \geq 1\}\, ds \\
&\quad + \frac{b^2}{4\Gamma(\alpha-2)} \chi\left\{\int_0^b (b-s)^{\alpha-3} f_k(s)\, ds : k \geq 1\right\} \\
&\leq \frac{b^{\alpha-1}(\alpha+4)}{2\Gamma(\alpha)} \chi_{C(J,E)}(D_{n-1})\, \|\beta\|_{L^1_{(J,\,\mathbb{R}^+)}} \\
&\quad + \frac{b^2}{4\Gamma(\alpha-2)} \chi\left\{\int_0^b (b-s)^{\alpha-3} f_k(s)\, ds : k \geq 1\right\}.
\end{aligned} \tag{4.12}$$

Now, we use the same procedure as in Step 3 in the proof of Theorem 3.1, to estimate the quantity $\chi\{ \int_0^b (b-s)^{\alpha-3} f_k(s)\, ds : k \geq 1\}$. We note that from (H_2), for any $k \geq 1$ and for a.e. $t \in J$, $\|f_k(t)\| \leq \varphi(t)\ \Omega(r)$. Consequently, $f_k \in L^{\frac{3}{q}}(J,E), k \geq 1$. Furthermore, from (H_4) it holds that for a.e. $t \in J$

$$\begin{aligned}
\chi(\{f_k(t) : k \geq 1\} &\leq \chi\{F(s, x_k(t)) : k \geq 1\} \\
&\leq \beta(t)\ \chi\{x_k(t) : k \geq 1\} \\
&\leq \beta(t)\ \chi(D_{n-1}(t)) = \beta(t)\ \chi_{C(J,E)}(D_{n-1}) = \gamma(t).
\end{aligned} \tag{4.13}$$

Note that $\gamma \in L^{\frac{3}{q}}(J, \mathbb{R}^+)$. Then, by virtue of Lemma 2.12, there exists a compact $K_\epsilon \subseteq E$, a measurable set $J_\epsilon \subset J$, with measure less than ϵ, and a sequence of functions $\{g_k^\epsilon\} \subset L^{\frac{2}{q}}(J,E)$ such that for all $s \in J$, $\{g_k^\epsilon(s) : k \geq 1\} \subseteq K$ and

$$\|f_k(s) - g_k^\epsilon(s)\| < 2\gamma(s) + \epsilon, \quad \text{for every } k \geq 1 \text{ and every } s \in J - J_\epsilon.$$

Then, using Hölder's inequality we obtain

$$\begin{aligned}
&\left\| \int_{J-J_\epsilon} (b-s)^{\alpha-3}(f_k(s) - g_k^\epsilon(s))\, ds \right\| \\
&\quad \leq \|f_k - g_k^\epsilon\|_{L^{\frac{3}{q}}(J-J_\epsilon,\ E)} \left(\int_{J-J_\epsilon} (b-s)^{\frac{3(\alpha-3)}{3-q}}\, ds \right)^{\frac{3-q}{3}} \\
&\quad \leq \left(2\|\gamma\|_{L^{\frac{2}{q}}(J-J_\epsilon,\ \mathbb{R}^+)} + \epsilon b^{\frac{q}{3}} \right) \frac{b^{\alpha-2-\frac{q}{3}}}{\eta^{1-\frac{q}{3}}}.
\end{aligned} \tag{4.14}$$

Also, by Hölder's inequality we get for any $k \geq 1$

$$\begin{aligned}
&\left\| \int_{J_\epsilon} (b-s)^{\alpha-3} f_k(s)\, ds \right\| \\
&\quad \leq \Omega(r) \int_{J_\epsilon} (b-s)^{\alpha-3} \varphi(s)\, ds \\
&\quad \leq \Omega(r) \|\varphi_k\|_{L^{\frac{3}{q}}(J_\epsilon,\, \mathbb{R}^+)} \left(\int_{J_\epsilon} (b-s)^{\frac{3(\alpha-3)}{3-q}}\, ds \right)^{\frac{3-q}{3}} \\
&\quad \leq \Omega(r) \|\varphi_k\|_{L^{\frac{3}{q}}(J_\epsilon,\, \mathbb{R}^+)} \frac{b^{\alpha-2-\frac{q}{3}}}{\eta^{1-\frac{q}{3}}}.
\end{aligned} \tag{4.15}$$

By (4.14) and (4.15), we have

$$\begin{aligned}
&\chi \left\{ \int_0^b (b-s)^{\alpha-3} f_k(s)\, ds : k \geq 1 \right\} \\
&\quad \leq \chi \left\{ \int_{J-J_\epsilon} (b-s)^{\alpha-3} (f_k(s) - g_k^\epsilon(s))\, ds : k \geq 1 \right\} \\
&\qquad + \chi \left\{ \int_{J-J_\epsilon} (b-s)^{\alpha-3} g_k^\epsilon(s)\, ds : k \geq 1 \right\} \\
&\qquad + \chi \left\{ \int_{J_\epsilon} (b-s)^{\alpha-3} f_k(s)\, ds : k \geq 1 \right\} \\
&\quad \leq \left(2\|\gamma\|_{L^{\frac{3}{q}}(J-J_\epsilon,\mathbb{R}^+)} + \epsilon b^{\frac{q}{3}} \right) \frac{b^{\alpha-2-\frac{q}{3}}}{\eta^{1-\frac{q}{3}}} \\
&\qquad + \Omega(r) \|\varphi_k\|_{L^{\frac{3}{q}}(J_\epsilon,\mathbb{R}^+)} \frac{b^{\alpha-2-\frac{q}{3}}}{\eta^{1-\frac{q}{3}}}.
\end{aligned}$$

From this inequality with (4.13) and by taking into account that ε is arbitrary, we get

$$\begin{aligned}
\chi \left\{ \int_0^b (b-s)^{\alpha-3} f_k(s)\, ds : k \geq 1 \right\} &\leq 2\|\gamma\|_{L^{\frac{3}{q}}(\overline{J_p}-J_\epsilon,\, \mathbb{R}^+)} \frac{b^{\alpha-2-\frac{q}{3}}}{\eta^{1-\frac{q}{3}}} \\
&\leq 2\|\beta\|_{L^{\frac{3}{q}}(J,\, \mathbb{R}^+)}\, \chi_{C(J,E)}(D_{n-1}) \frac{b^{\alpha-2-\frac{q}{3}}}{\eta^{1-\frac{q}{3}}}.
\end{aligned}$$

Again from the fact that ε is arbitrary, this inequality with (4.12) and (4.13) gives us

$$\begin{aligned}
\chi_{C(J,E)}(D_n) &\leq \frac{b^{\alpha-1}(\alpha+4)}{\Gamma(\alpha)} \chi_{C(J,E)}(D_{n-1}) \|\beta\|_{L^1_{(J,\mathbb{R}^+)}} \\
&\quad + \frac{b^2}{\Gamma(\alpha-2)} \rho \chi_{C(J,E)}(D_{n-1}) \|\beta\|_{L^{\frac{3}{q}}(J,\mathbb{R}^+)} \\
&= \delta \chi_{C(J,E)}(D_{n-1}),
\end{aligned}$$

where

$$\delta = \frac{b^{\alpha-1}(\alpha+4)}{\Gamma(\alpha)} \|\beta\|_{L^1_{(J,,\, \mathbb{R}^+)}} + \frac{b^2}{\Gamma(\alpha-2)} \rho \|\beta\|_{L^{\frac{3}{q}}(J\ ,\, \mathbb{R}^+)}.$$

By means of a finite number of steps, we can write

$$0 \leq \chi_{C(J,E)}(D_n) \leq \delta^{n-1} \chi_{C(J,E)}(D_1), \quad \forall\, n \geq 1.$$

Since this inequality is true for every $n \in \mathbb{N}$, by (4.8) and by passing to the limit as $n \to +\infty$, we obtain (4.10) and so our aim in this step is verified. Hence, D is a nonempty and compact subset of $C(J,E)$. Moreover, it is convex. Note that $R(D) \subseteq D$.

Step 4. The graph of the multivalued function $R|_D \colon D \to 2^D$ is closed. Consider a sequence $\{x_n\}_{n\geq 1}$ in D with $x_n \to x$ in B and let $y_n \in R(x_n)$ with $y_n \to y$ in $C(J,E)$. We have to show that $y \in R(x)$. By recalling the definition of R, for any $n \geq 1$, there is $f_n \in S^1_{F(\cdot,x_n(\cdot))}$ such that

$$\begin{aligned} y_n(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_n(s)\,ds - \frac{1}{2\Gamma(\alpha)} \int_0^b (b-s)^{\alpha-1} f_n(s)\,ds \\ + \frac{(b-2t)}{4\Gamma(\alpha-1)} \int_0^b (b-s)^{\alpha-2} f_n(s)\,ds + \frac{t(b-t)}{4\Gamma(\alpha-2)} \int_0^b (b-s)^{\alpha-3} f_n(s)\,ds. \end{aligned}$$

Observe that for every $n \geq 1$ and for a.e. $t \in J$

$$\begin{aligned} \|f_n(t)\| &\leq \varphi(t)\Omega(\|x_n(t)\|) \\ &\leq \varphi(t)\ \Omega(\|x_n\|_{C(J,E)}) \\ &\leq \varphi(t)\Omega(r). \end{aligned}$$

This shows that the set $\{f_n : n \geq 1\}$ is integrably bounded. In addition, the set $\{f_n(t) : n \geq 1\}$ is relatively compact for a.e. $t \in J$ because assumption (H_3) together with the convergence of $\{x_n\}_{n\geq 1}$ implies that

$$\chi\{f_n(t) : n \geq 1\} \leq \chi\{F(t, \{x_n(t) : n \geq 1\})\} \leq \beta(t)\chi\{x_n(t) : n \geq 1\} = 0.$$

Hence, the sequence$\{f_n\}_{n\geq 1}$ is semicompact, therefore, by Lemma 2.5, it is weakly compact in $L^1(J,E)$. So, without loss of generality we can assume that f_n converges weakly to a function $f \in L^1(J,E)$. From Mazur's theorem, there is a sequence (z_n), $n \geq 1$ of convex combinations of f_n such that for a.e. $t \in J$

$$f(t) \in \bigcap_{j\geq 1} \overline{\{z_n(t) : n \geq j\}} \subseteq \bigcap_{j\geq 1} \overline{\mathrm{Conv}}\{f_n\ (t) : n \geq j\},$$

and z_n converges strongly to $f \in L^1(J,E)$. Then, for a.e. $t \in J$

$$f(t) \in \bigcap_{j\geq 1} \overline{\{z_n(t) : n \geq j\}} \subseteq \bigcap_{j\geq 1} \overline{\mathrm{Conv}}\{f_n(t) : n \geq j\} \subseteq \bigcap_{j\geq 1} \overline{\mathrm{Conv}} \cup \{F(t, x_n(t)) : n \geq j\}.$$

But from the upper semicontinuity of $F(t,\cdot)$ with Lemma 2.12, we get

$$\bigcap_{j\geq 1} \overline{\mathrm{Conv}} \cup \{F(t, x_n(t)) : n \geq j\} \subseteq F(t, x(t)).$$

Then $f(t) \in F(t,x(t))$ for a.e. $t \in J$. Now, for any $n \geq 1$ we define

$$\begin{aligned} g_n(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s) z_n(s)\,ds - \frac{1}{2\Gamma(\alpha)} \int_0^b (b-s)^{\alpha-1} z_n(s)\,ds \\ + \frac{(b-2t)}{4\Gamma(\alpha-1)} \int_0^b (b-s)^{\alpha-2} z_n(s)\,ds \\ + \frac{t(b-t)}{4\Gamma(\alpha-2)} \int_0^b (b-s)^{\alpha-3} z_n(s)\,ds. \end{aligned} \tag{4.16}$$

Obviously, for a.e. $t \in J$, the sequence $g_n(t)$ converges to $y(t)$. Since F takes convex values, $z_j(t) \in F(t, x(t))$, for a.e. $t \in J$. So, for every $n \geq 1$

$$
\begin{aligned}
\|z_n(t)\| &\leq \varphi(t)\Omega(\|x(t)\| \\
&\leq \varphi(t)\Omega(\|x\|_{C(J,E)}), \quad \text{for a.e. } t \in J.
\end{aligned}
$$

Hence, for every $t \in J$ and for a.e. $s \in (0, t]$

$$
\|(t-s)^{\alpha-1} z_n(s)\| \leq |t-s|^{\alpha-1}\varphi(t)\Omega(\|x\|_{C(J,E)}).
$$

Therefore, by passing to the limit as $n \to \infty$ in (4.16),we obtain from the Lebesgue dominated convergence theorem that, for every $t \in J$

$$
\begin{aligned}
y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds - \frac{1}{2\Gamma(\alpha)} \int_0^b (b-s)^{\alpha-1} f(s)\, ds \\
+ \frac{(b-2t)}{4\Gamma(\alpha-1)} \int_0^b (b-s)^{\alpha-2} f(s)\, ds + \frac{t(b-t)}{4\Gamma(\alpha-2)} \int_0^b (b-s)^{\alpha-3} f(s)\, ds.
\end{aligned}
$$

This shows that the graph of R is closed.

Observe that, by repeating the same procedure in the previous step we can deduce that the values of R is closed.

As a result of the Steps 1–4, the multivalued $R|_D \colon D \to 2^D$ is an *u.s.c.* multifunction with nonempty convex compact values. By applying Lemma 2.21 there is $x \in D$ and $x \in R(x)$. $\square$

Remark 4.7. The preceding theorem extends Theorem 3.2 in [12] to infinite dimensional spaces. Moreover, it gives a correct formula for the solutions.

In the following corollary we simplify the condition (4.9).

Corollary 4.8. *Assume that the assumptions (H_2)–(H_4) are satisfied. If the function φ in (H_2) is constant, that is, there exists a positive constant μ such that $\varphi(t) = \mu$ for all $t \in J$, then the problem* (1.2) *has a solution provided that there is an $r > 0$ such that*

$$
\frac{\mu\Omega(r)b^\alpha}{2\Gamma(\alpha+1)}\left[3 + \frac{\alpha^2}{2}\right] \leq r. \tag{4.17}
$$

Proof. We need only to check that $R(D_0) \subseteq D_0$. Let $x \in D_0$ and $y \in R(x)$. As in Step 2 of Theorem 4.6, for any $t \in J$

$$
\begin{aligned}
\|y(t)\| &\leq \frac{\mu\Omega(r)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\, ds + \frac{\mu\Omega(r)}{2\Gamma(\alpha)} \int_0^b (b-s)^{\alpha-1}\, ds \\
&\quad + \frac{\mu b\, \Omega(r)}{4\Gamma(\alpha-1)} \int_0^b (b-s)^{\alpha-2} ds + \frac{\mu b^2\Omega(r)}{4\Gamma(\alpha-2)} \int_0^b (b-s)^{\alpha-3}\, ds \\
&\leq \frac{\mu\Omega(r)b^\alpha}{\Gamma(\alpha+1)} + \frac{\mu\Omega(r)b^\alpha}{2\Gamma(\alpha+1)} + \frac{\mu\Omega(r)b^\alpha}{4\Gamma(\alpha)} + \frac{\mu\Omega(r)b^\alpha}{4\Gamma(\alpha-1)} \\
&\leq \frac{\mu\Omega(r)b^\alpha}{\Gamma(\alpha+1)}\left[\frac{3}{2} + \frac{\alpha}{4} + \frac{\alpha(\alpha-1)}{4}\right] \\
&\leq \frac{\mu\Omega(r)b^\alpha}{2\Gamma(\alpha+1)}[3 + \frac{\alpha^2}{2}] \\
&\leq r.
\end{aligned}
$$

$\square$

In the following theorem the compactness of the solution set of (1.2) is established.

Theorem 4.9. *If the function Ω in (H_3) is given of the form $\Omega(t) = t + 1$, then under the assumptions of Theorem 4.6 the set of solutions of (1.2) is compact in $C(J,E)$ provided that*

$$\frac{(\alpha+5)b^{\alpha-1}}{4\Gamma(\alpha)}\|\varphi\|_{L^1(J,\mathbb{R}^+)} + \frac{1}{4\Gamma(\alpha-2)}\|\varphi\|_{L^{\frac{3}{q}}(J,\mathbb{R}^+)}\frac{b^{\alpha-\frac{q}{3}}}{\eta^{1-\frac{q}{3}}} < 1. \tag{4.18}$$

Proof. Note that by Theorem 4.6 the set of solutions of (1.1) is nonempty. In fact, by (4.18) we can take

$$r = \frac{\frac{(\alpha+5)b^{\alpha-1}}{4\Gamma(\alpha)}\|\varphi\|_{L^1(J,\mathbb{R}^+)} + \frac{1}{4\Gamma(\alpha-2)}\|\varphi\|_{L^{\frac{3}{q}}(J,\mathbb{R}^+)}\frac{b^{\alpha-\frac{q}{3}}}{\eta^{1-\frac{q}{3}}}}{1-\left[\frac{(\alpha+5)b^{\alpha-1}}{4\Gamma(\alpha)}\|\varphi\|_{L^1(J,\mathbb{R}^+)} + \frac{1}{4\Gamma(\alpha-2)}\|\varphi\|_{L^{\frac{3}{q}}(J,\mathbb{R}^+)}\frac{b^{\alpha-\frac{q}{3}}}{\eta^{1-\frac{q}{3}}}\right]}$$

in (4.9). So we have a solution in D_0. Note that, from Steps 2 and 3 of the proof of Theorem 4.6, the multivalued function R is completely continuous. According to Lemma 2.22 , in order to show that the set of solutions of (1.2) is compact, it suffices to prove that the set of fixed points of the multivalued function R is bounded. So, let x be a mild solution for (1.2). Then there is an integrable selection f for $F(\cdot,x(\cdot))$ such that for every $t \in J$

$$\begin{aligned} x(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)\,ds - \frac{1}{2\Gamma(\alpha)}\int_0^b (b-s)^{\alpha-1}f(s)\,ds \\ &\quad + \frac{(b-2t)}{4\Gamma(\alpha-1)}\int_0^b (b-s)^{\alpha-2}f(s)\,ds + \frac{t(b-t)}{4\Gamma(\alpha-2)}\int_0^b (b-s)^{\alpha-3}f(s)\,ds. \end{aligned}$$

By arguing as in Step 1 of Theorem 4.6 we get

$$\|x(t)\| \leq (1+\|x\|)\left[\frac{(\alpha+5)b^{\alpha-1}}{4\Gamma(\alpha)}\|\varphi\|_{L^1(J,\mathbb{R}^+)} + \frac{1}{4\Gamma(\alpha-2)}\|\varphi\|_{L^{\frac{3}{q}}(J,\mathbb{R}^+)}\frac{b^{\alpha-\frac{q}{3}}}{\eta^{1-\frac{q}{3}}}\right].$$

Therefore,

$$\|x\|_{C(J,E)} \leq \frac{\frac{(\alpha+5)b^{\alpha-1}}{4\Gamma(\alpha)}\|\varphi\|_{L^1(J,\mathbb{R}^+)} + \frac{1}{4\Gamma(\alpha-2)}\|\varphi\|_{L^{\frac{3}{q}}(J,\mathbb{R}^+)}\frac{b^{\alpha-\frac{q}{3}}}{\eta^{1-\frac{q}{3}}}}{1-\left[\frac{(\alpha+5)b^{\alpha-1}}{4\Gamma(\alpha)}\|\varphi\|_{L^1(J,\mathbb{R}^+)} + \frac{1}{4\Gamma(\alpha-2)}\|\varphi\|_{L^{\frac{3}{q}}(J,\mathbb{R}^+)}\frac{b^{\alpha-\frac{q}{3}}}{\eta^{1-\frac{q}{3}}}\right]} = r.$$

Then, the set of fixed points of the multivalued function R is bounded of $C(J,E)$. Hence, by Lemma 2.22, the set of mild solutions of (1.2) is compact. $\square$

In the following theorem we give another version for Theorem 4.6.

Theorem 4.10. *Let $F\colon J\times E \to P_{ck}(E)$ be a multifunction. We suppose the following assumptions:*

(H_5) *For every $x \in E$, $t \longrightarrow F(t,x)$ is measurable.*

(H_6) *There is a function $\varsigma \in L^{\frac{3}{q}}(J,\mathbb{R}^+)$, $(0 < q < 3\alpha - 6)$ such that for every $x, y \in E$*

$$h(F(t,x),F(t,y)) \leq \varsigma(t)\,\|x-y\|, \quad \text{for a.e. } t \in J,$$

and

$$\sup\{\|x\| : x \in F(t,0)\} \leq \varsigma(t), \quad \text{for a.e. } t \in J.$$

Then the problem (1.2) *has a solution provided that*

$$\frac{(\alpha+5)b^{\alpha-1}}{4\Gamma(\alpha)}\|\varsigma\|_{L^1(J,\mathbb{R}^+)}+\frac{b^{\alpha-\frac{q}{3}}}{4\Gamma(\alpha-2)\eta^{1-\frac{q}{3}}}\|\varsigma\|_{L^{\frac{3}{q}}(J,\mathbb{R}^+)}< 1. \tag{4.19}$$

Proof. By(H_5) and (H_6) we conclude, from Lemma 2.2, that for any $x\in C(J,E)$ the set $S^1_{F(\cdot,x(\cdot))}$ is nonempty. Then we can consider a multifunction map $R\colon C(J,E)\to 2^{C(J,E)}$ defined as in Theorem 4.6. We shall show that R satisfies the assumptions of Lemma 2.23. The proof will be given in two steps.

Step 1. The values of R are nonempty and closed. Since $S^1_{F(\cdot,x(\cdot))}$ is nonempty, the values of R are nonempty. In order to prove the values of R are closed, let $x\in C(J,E)$ and $(y_n), n\geq 1$ be a sequence in $R(x)$ such that $y_n\to y$ in $C(J,E)$. Then, according to the definition of R, there is a sequence $(f_n), n\geq 1$ in $S^1_{F(\cdot,x(\cdot))}$ such that for any $t\in J$

$$\begin{aligned} y_n(t)&=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f_n(s)\,ds-\frac{1}{2\Gamma(\alpha)}\int_0^b(b-s)^{\alpha-1}f_n(s)\,ds\\ &\quad+\frac{(b-2t)}{4\Gamma(\alpha-1)}\int_0^b(b-s)^{\alpha-2}f_n(s)\,ds+\frac{t(b-t)}{4\Gamma(\alpha-2)}\int_0^b(b-s)^{\alpha-3}f_n(s)\,ds.\end{aligned}$$

Let $t\in J$ be a fixed. In view of (H_5), for every $n\geq 1$, and for a.e. $t\in J$

$$\begin{aligned}\|F(t,x)\|&=H(F(t,x(t)),\{0\})\\ &\leq H(F(t,x(t)),F(t,0))+H(F(t,0),\{0\})\\ &\leq \varsigma(t)\|x(t)\|+\varsigma(t)\\ &\leq \varsigma(t)\left(1+\|x\|_{C(J,E)}\right).\end{aligned}$$

Then, for every $n\geq 1$, and for a.e. $t\in J$, $\|f_n(t)\|\leq\varsigma(t)(1+\|x\|_{C(J,E)})$. This shows that the set $\{f_n: n\geq 1\}$ is integrably bounded. Arguing as in Step 4 in the proof of Theorem 4.6, we can show that the values of R are closed.

Step 2. R is a contraction.

Let $z_1,z_2\in C(J,E)$ and $y_1\in R(z_1)$. Then there is $f\in S^1_{F(\cdot,z_1(\cdot))}$ such that for any $t\in J$

$$\begin{aligned} y_1(t)&=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f_1(s)\,ds-\frac{1}{2\Gamma(\alpha)}\int_0^b(b-s)^{\alpha-1}f_1(s)\,ds\\ &\quad+\frac{(b-2t)}{4\Gamma(\alpha-1)}\int_0^b(b-s)^{\alpha-2}f_1(s)\,ds+\frac{t(b-t)}{4\Gamma(\alpha-2)}\int_0^b(b-s)^{\alpha-3}f_1(s)\,ds.\end{aligned}$$

Consider the multifunction $Z\colon J\to 2^E$ defined by

$$Z(t)=\{u\in E:\|f(t)-u\|\leq\varsigma(t)\,\|z_1(t)-z_2(t)\|\}.$$

For each $t\in J$, $Z(t)\cap F(t,z_2(t))$ is nonempty. Indeed, let $t\in J$. From (H_6), we have $h(F(t,z_2(t)),F(t,z_1(t)))\leq\varsigma(t)\,\|z_1(t)-z_2(t)\|$. Hence, there exists $u_t\in F(t,z_2(t))$ such that

$$\|u_t-f(t)\|\leq\varsigma(t)\,\|z_1(t)-z_2(t)\|.$$

Moreover, since the functions ς, z_1, z_2 and f are measurable [11, Proposition III. 4], the multifunction $V\colon t\to Z(t)\cap F(t,z_1(t))$ is measurable. Then there is $h\in S^1_{F(\cdot,z_2(\cdot))}$ with

$$\|h(t)-f(t)\|\leq\varsigma(t)\,\|z_1(t)-z_2(t)\|,\quad \text{a.e. } t\in J. \tag{4.20}$$

Let us define

$$y_2(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds - \frac{1}{2\Gamma(\alpha)} \int_0^b (b-s)^{\alpha-1} h(s)\, ds \\ + \frac{(b-2t)}{4\Gamma(\alpha-1)} \int_0^b (b-s)^{\alpha-2} h(s)\, ds + \frac{t(b-t)}{4\Gamma(\alpha-2)} \int_0^b (b-s)^{\alpha-3} h(s)\, ds.$$

Obviously $y_2 \in R(z_2)$. Furthermore, we get from the definitions of y_1 and y_2, (4.19), (4.20) and Hölder's inequality

$$\begin{aligned} \|y_2(t) - y_1(t)\| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \varsigma(s) \|z_1(s) - z_2(s)\|\, ds \\ &\quad + \frac{1}{2\Gamma(\alpha)} \int_0^b (b-s)^{\alpha-1} \varsigma(s)\, \|z_1(s) - z_2(s)\|\, ds \\ &\quad + \frac{|b-2t|}{4\Gamma(\alpha-1)} \int_0^b (b-s)^{\alpha-2} \varsigma(s) \|z_1(s) - z_2(s)\|\, ds \\ &\quad + \frac{t(b-t)}{4\Gamma(\alpha-2)} \int_0^b (b-s)^{\alpha-3} \varsigma(s) \|z_1(s) - z_2(s)\|\, ds \\ &\leq \left(\frac{3b^{\alpha-1}}{2\Gamma(\alpha)} + \frac{b^{\alpha-1}}{4\Gamma(\alpha-1)} \right) \|z_1 - z_2\|_{C(J,E)}\, \|\varsigma\|_{L^1(J,\mathbb{R}^+)} \\ &\quad + \frac{b^{\alpha-\frac{q}{3}}}{4\Gamma(\alpha-2)\eta^{1-\frac{q}{3}}} \|\varsigma\|_{L^{\frac{3}{q}}(J,\mathbb{R}^+)} \|z_1 - z_2\|_{C(J,E)} \\ &\leq \|z_1 - z_2\|_{C(J,E)} \left[\left(\frac{3}{2} + \frac{\alpha-1}{4} \right) \frac{b^{\alpha-1}}{\Gamma(\alpha)} \|\varsigma\|_{L^1(J,\mathbb{R}^+)} \right. \\ &\quad \left. + \frac{b^{\alpha-\frac{q}{3}}}{4\Gamma(\alpha-2)\eta^{1-\frac{q}{3}}} \|\varsigma\|_{L^{\frac{3}{q}}(J,\mathbb{R}^+)} \right] \\ &< \|z_1 - z_2\|_{C(J,E)}. \end{aligned}$$

By interchanging the role of y_2 and y_1 we obtain

$$\|R(z_1) - R(z_2)\| \leq \|z_1 - z_2\|_{C(J,E)}.$$

Therefore, the multivalued function R is a contraction and thus, by Lemma 2.23, R has a fixed point which is a solution for (1.2). □

In the following corollary we simplify the condition (4.19).

Remark 4.11. The previous corollary extends Theorem 3.1 in [2] to a multivalued version and Theorem 3.7 in [12] to infinite dimensional Banach spaces. In addition, it gives a correct formula for the solutions.

Corollary 4.12. *Assume that the assumptions (H_5) and (H_6) are satisfied. If there exists a positive constant v such that $\varsigma(t) = v$, for all $t \in J$, then the problem (1.2) has a solution provided that*

$$\frac{v b^{\alpha}}{2\Gamma(\alpha+1)} \left(3 + \frac{\alpha^2}{2} \right) < 1. \tag{4.21}$$

Proof. Let $t \in J$, we get from (4.19), (4.20) and (4.21)

$$
\begin{aligned}
\|y_2(t) - y_1(t)\| \leq & \frac{\nu}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|z_1(s) - z_2(s)\| \, ds \\
& + \frac{v}{2\Gamma(\alpha)} \int_0^b (b-s)^{\alpha-1} \|z_1(s) - z_2(s)\| \, ds \\
& + \frac{v|b-2t|}{4\Gamma(\alpha-1)} \int_0^b (b-s)^{\alpha-2} \, \|z_1(s) - z_2(s)\| \, ds \\
& + \frac{vt(b-t)}{4\Gamma(\alpha-2)} \int_0^b (b-s)^{\alpha-3} \, \|z_1(s) - z_2(s)\| \, ds \\
\leq & \, v \left(\frac{3b^\alpha}{2\Gamma(\alpha+1)} + \frac{b^\alpha}{4\Gamma(\alpha)} + \frac{b^\alpha}{4\Gamma(\alpha-1)} \right) \|z_1 - z_2\|_{C(J,E)} \\
\leq & \, v \left(\frac{3b^\alpha}{2\Gamma(\alpha+1)} + \frac{\alpha b^\alpha}{4\Gamma(\alpha+1)} + \frac{b^\alpha \alpha(\alpha-1)}{4\Gamma(\alpha+1)} \right) \|z_1 - z_2\|_{C(J,E)} \\
\leq & \frac{vb^\alpha}{2\Gamma(\alpha+1)} \left(3 + \frac{\alpha}{2} + \frac{\alpha(\alpha-1)}{2} \right) \|z_1 - z_2\|_{C(J,E)} \\
\leq & \frac{vb^\alpha}{2\Gamma(\alpha+1)} \left(3 + \frac{\alpha^2}{2} \right) \|z_1 - z_2\|_{C(J,E)} \\
< & \|z_1 - z_2\|_{C(J,E)} \, .
\end{aligned}
$$

Then R is a contraction. □

4.2 Nonconvex case

Now we present an existence result for the problem (1.1) when the values of the multivalued function are not necessarily convex. The proof is based on a selection theorem due to Bressan and Colombo [9] for lower semicontinuous maps with decomposable values. Our hypothesis on the orient field is the following:

(H_7) $F \colon J \times E \to P_{cl}\ (E)$ is a multifunction such that

- **(i)** $(t, x) \to F(t, x)$ is graph measurable and $x \to F(t, x)$ is lower semicontinuous.
- **(ii)** There exists a function $\varphi \in L^1(J, \mathbb{R}^+)$, such that for any $x \in E$

$$\|F(t, x)\| \leq \varphi(t), \quad \text{a.e. } t \in J.$$

Theorem 4.13. *If the hypotheses $(H_4), (H_5)$ and (H_7) hold, then the problem (1.2) has a solution provided that there is $r > 0$ such that the condition (4.9) is satisfied.*

Proof. Consider the multivalued Nemitsky operator $N \colon C(J, E) \to 2^{L^1(J,E)}$, defined by

$$N(x) = S^1_{F(\cdot, x(\cdot))} = \left\{ f \in L^1(J, E) : f(t) \in F(t, x(t))), \text{ a.e. } t \in J \right\}.$$

We will prove that N has a nonempty closed decomposable value and *l.s.c.* Since F has closed values, S^1_F is closed. Because F is integrably bounded, S^1_F is nonempty. It is readily verified, S^1_F is decomposable. To check the lower semicontinuity of N, we need to show that, for every $u \in L^1(J,E)$, $x \to d(u, N(x))$ is upper semicontinuous. To this end from Theorem 2.2 [21] we have

$$\begin{aligned} d(u, N(x)) &= \inf_{v \in N(x)} \|u - v\|_{L^1} \\ &= \inf_{v \in N(x)} \int_0^b \|u(t) - v(t)\|\, dt \\ &= \int_0^b \inf_{z \in F(t,x(t))} \|u(t) - z\|\, dt \\ &= \int_0^b d(u(t), F(t, x(t)))\, dt. \end{aligned} \tag{4.22}$$

We shall show that, for any $\lambda \geq 0$, the set

$$u_\lambda = \{x \in C(J,E) : d(u, N(x)) \geq \lambda\}$$

is closed. For this purpose, let (x_n) be a sequence in u_λ such that $x_n \to x$ in $C(J,E)$. Then, for all $t \in J$, $x_n(t) \to x(t)$ in E. By virtue of $(H_7)(i)$ the function $z \to d(u(t), F(t,z))$ is upper semicontinuous. So, via Fatou's lemma and (4.22) we have

$$\begin{aligned} \lambda &\leq \limsup_{n\to\infty} d(u, N(x_n)) \\ &= \limsup_{n\to\infty} \int_0^b d(u(t), F(t, x_n(t)))\, dt \\ &\leq \int_0^b \limsup_{n\to\infty} d(u(t), F(t, x_n(t)))\, dt \\ &\leq \int_0^b d(u(t), F(t, x(t)))\, dt \\ &= d(u, N(x)). \end{aligned}$$

Therefore $x \in u_\lambda$ and hence N is lower semicontinuous. By applying Theorem 3 of [9], there is a continuous map $Z\colon C(J,E) \to L^1(J,E)$ such that $Z(x) \in N(x)$, for every $x \in C(J,E)$. Then, $Z(x)(s) \in F(s, x(s))$, a.e. $s \in J$. Consider a map $\pi\colon C(J,E) \to C(J,E)$ defined by

$$\begin{aligned} (\pi x)(t) = {}& \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} Z(x)(s)\, ds - \frac{1}{2\Gamma(\alpha)} \int_0^b (b-s)^{\alpha-1} Z(x)(s)\, ds \\ & + \frac{(b-2t)}{4\Gamma(\alpha-1)} \int_0^b (b-s)^{\alpha-2} Z(x)(s)\, ds \\ & + \frac{t(b-t)}{4\Gamma(\alpha-2)} \int_0^b (b-s)^{\alpha-3} Z(x)(s)\, ds. \end{aligned}$$

Arguing as in the proof of Theorem 4.6, we can show that π satisfies all the conditions of Schauder's fixed point theorem. Thus, there is $x \in C(J,E)$ such that $x(t) = (\pi x)(t)$. This means that x is a solution for (1.2). $\square$

5 Examples

The following examples illustrate the feasibility of our assumptions.

Example 5.1. Let E be a separable Banach space and $f\colon [0,1]\times E\to E$, be a function defined by

$$f(t,x)=\frac{t\,x_0}{20\|x_0\|}+\frac{x}{20}, \tag{5.1}$$

where $x_0\in E\setminus\{0\}$. Clearly

$$\|f(t,x)-f(s,y)\|\le\frac{1}{20}\max\{|t-s|,\|x-y\|\}.$$

Moreover, the inequality

$$\frac{(2\alpha^2-\alpha+6)}{\Gamma(\alpha+1)}<40$$

is always true for any $\alpha\in(2,3)$. Then, by Corollary 3.2, the problem (1.1), where f is given by (5.1), has a solution.

Example 5.2. Let $J=[0,1]$, E be a separable Banach space and K a nonempty convex compact subset of E. Let $F\colon J\times E\to P_{ck}(E)$ be a multivalued function defined by

$$F(t,x)=\frac{\|x\|}{\lambda(10+e^t)(1+\|x\|)}K, \tag{5.2}$$

where λ is a positive constant such that $\sup\{\|z\|:z\in K\}\le\lambda$.

Our aim is to prove the assumptions of Corollary 4.12 are satisfied. Obviously the assumption (H_5) is satisfied. In order to show that (H_6) is satisfied. Furthermore, for $t\in J$, we have

$$\begin{aligned}h(F(t,x),F(t,y))&\le\frac{1}{(10+e^t)}\left|\frac{\|x\|}{(1+\|x\|)}-\frac{\|y\|}{(1+\|y\|)}\right|\\&\le\frac{1}{(10+e^t)}\|x-y\|\\&\le\frac{1}{10}\|x-y\|.\end{aligned}$$

Note that $F(t,0)=\{0\}$. Hence, the assumption (H_6) holds with $\varsigma(t)=v=\frac{1}{10}$. We shall check that condition (4.21) is satisfied with $\nu=\frac{1}{10}$ and $b=1$. Indeed, it is easy to show that the inequality

$$\frac{1}{20\Gamma(\alpha+1)}\left(3+\frac{\alpha^2}{2}\right)<1$$

is verified for any $\alpha\in(2,3)$. Therefore the condition (4.21) is satisfied. Then by Corollary 4.12, the problem (1.2), where F is given by (5.2), has a solution.

6 Conclusion

In this paper, existence problems for fractional differential inclusions with anti-periodic boundary conditions have been considered in infinite dimensional Banach spaces. Some sufficient conditions have been obtained, as pointed in the first section, these conditions are strictly weaker than the most of the existing ones. We have considered the convex as well as the nonconvex case. The obtained results extend those of [3, 12] to infinite dimensional Banach spaces. Moreover, our technique allows to consider many boundary value problems in infinite dimensional Banach spaces.

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