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ON THE OSCILLATION OF SECOND ORDER NONLINEAR NEUTRAL DELAY DIFFERENCE EQUATIONS

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Abstract

In this paper sufficient conditions are obtained for oscillation of all solutions of a class of nonlinear neutral delay difference equations of the form

$$\Delta^2(y(n) + p(n)y(n-m)) + q(n)G(y(n-k)) = 0$$

under various ranges of $p(n)$. The nonlinear function $G, G \in C(R, R)$ is either sublinear or superlinear.

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1 Introduction

Recently, a good deal of work has been published on the oscillation theory of difference equations. Most of the works in first and higher order neutral delay difference equations are concerned with the study of the behaviour of the solution which oscillates or tends to zero (see [3], [4], [5], [7]). But very few papers are available on oscillatory higher order nonlinear delay difference equations.

In [7], authors Parhi and Tripathy has considered a class of nonlinear neutral delay difference equations of higher order of the form

$$\Delta^m[y(n)+p(n)y(n-m)]+q(n)G(y(n-k))=0, \tag{E}$$

where $m \geq 2$. They have obtained the results which hold good when G is sub linear only. However, the behaviour of solutions of (E) under the superlinear nature of G is still in progress. In fact, various ranges of $p(n)$ are restricting for all solutions as oscillatory.

In this paper, author has studied the second order nonlinear neutral delay difference equation of the form

$$\Delta^2[y(n)+p(n)y(n-m)]+q(n)G(y(n-k))=0, n \geq 0, \tag{1}$$

where Δ is the forward difference operator defined by $\Delta\ y(n)=y(n+1)-y(n),\ p,\ q$ are real valued functions defined on $N(0)=\{0,1,2,...\}$ such that $q(n)\geq 0, G\ \in\ C(R,R)$ is nondecreasing and $x\ G(x)>0$ for $x\neq 0$ and $m>0,\ k\geq 0$ are integers. Here, an attempt is made to establish sufficient conditions under which every solution of Eq.(1) oscillates.

The motivation of present work has come under two directions. Firstly, due to the work in [6] and second is due to the work in [7], where G is almost sublinear. It is interesting to observe that unlike differential equation, Eq. (1) is converting immediately into a first order difference inequality and hence study of both are interrelated hypothetically. In this regard the work in [8] provides a good input for the completion of the present work.

By a solution of Eq. (1) we mean a real valued function $y(n)$ defined on $N(-r)=\{-r,-r+1,\cdots\}$ which satisfies (1) for $n\geq\ 0$, where $r=max\{k,m\}$. If

$$y(n)=A_n,\ n=-r,-r+1,\cdots,0 \tag{2}$$

are given, then (1) admits a unique solution satisfying the initial condition (2). A solution $y(n)$ of (1) is said to be oscillatory, if for every integer $N>0$, there exists an $n\geq N$ such that $y(n)y(n+1)\leq 0$: otherwise, it is called nonoscillatory.

The following two results are useful for our discussion in the next sections.

Theorem 1.1 [2]. If $q(n)\geq 0$ for $n\geq 0$ and

$$\liminf_{n\to\infty}\sum_{s=n-k}^{n-1}q(s)>(\frac{k}{k+1})^{k+1},$$

then $\Delta\ x(n)+q(n)\ x(n-k)\leq 0,\ n\geq 0$ can not have an eventually positive solution.

Theorem 1.2 [8]. Assume that $\liminf\limits_{n\to\infty} \sum\limits_{s=n-k}^{n-1} q(s) > 0$.

(H) Suppose there are a function $g(u) \in C(R, R^+)$ and a number $\epsilon > 0$ such that

1. $g(u)$ is nondecreasing in R^+,

2. $g(-u) = g(u)$ and $\lim\limits_{n\to 0} g(u) = 0$,

3. $\int\limits_0^\infty g(e^{-u})du < \infty$,

4. $\left|\frac{G(u)}{u} - 1\right| \le g(u), 0 < |u| < \epsilon$.

If

$$\sum_{n=0}^{\infty}\left[\sum_{s=n}^{n+k} q(s)\ell n\left(\sum_{s=n}^{n+k} q(s)\right) - \sum_{s=n+1}^{n+k} q(s)\ell n\left(\sum_{s=n+1}^{n+k} q(s)\right)\right] = \infty,$$

then every solution of $\Delta\, x(n) + q(n)\, G(x(n-k)) = 0$ oscillates.

Remark $\liminf\limits_{n\to\infty} \sum\limits_{s=n-k}^{n-1} q(s) > 0$ implies that $\sum\limits_{n=0}^{\infty} q(n) = \infty$.

Corollary 1.3 If all the conditions of Theorem 1.2 are satisfied, then

$$\Delta x(n) + q(n)G(x(n-k)) \le 0, n \ge 0$$

doesn't possess any eventually positive solution.

The proof follows from the Theorem 1.2.

2 Sublinear Oscillation

This section deals with the sufficient conditions for the oscillation of all solutions of Eq.(1) when G is sublinear. The following conditions are needed for our use in the sequel.

(H_1) $\sum\limits_{n=0}^{\infty} q(n) = \infty$

(H_2) $\sum\limits_{n=m}^{\infty} Q(n) = \infty,$

where $Q(n) = min\{q(n), q(n-m)\}$

(H_3) $G(u)G(v) \geq G(uv)$ for $u > 0, v > 0$

(H_4) $G(-u) = -G(u),\ u \in R$

(H_5) There exists $\lambda > 0$ such that $G(u) + G(v) \geq \lambda\, G(u+v),\ u \in R,\ v \in R$

(H_6) $\int\limits_0^{\pm c} \frac{du}{G(u)} < \infty,\ c > 0$

(H_7) $\liminf\limits_{|x|\to 0} \frac{G(x)}{x} \geq \gamma > 0$

(H_8) $\liminf\limits_{n\to\infty} \sum\limits_{s=n+m-k}^{n-1} q(s) > \frac{b}{\gamma}(\frac{k-m}{k-m+1})^{k-m+1}, b > 0$ and $k > m$.

Theorem 2.1 Let $0 \leq p(n) \leq a < \infty$. Suppose that ($H_2$)-($H_6$) hold. Then Eq.(1) is oscillatory.

Proof Suppose for contrary that $y(n)$ is a nonoscillatory solution of (1). Then there exists $n_1 > 0$ such that $y(n) > 0$ or < 0 for $n \geq n_1$. Let the former hold. Setting

$$z(n) = y(n) + p(n)y(n-m) \tag{3}$$

we have from (1)

$$\Delta^2 z(n) + q(n)G(y(n-k)) = 0, \tag{4}$$

that is,

$$\Delta^2 z(n) = -q(n)G(y(n-k)) \leq 0,$$

for $n \geq n_2 > n_1 + r$. Hence $\Delta z(n)$ is non-increasing. If $\Delta z(n) < 0$, then Eq.(1) becomes

$$\Delta z(n+1) + q(n)G(y(n-k)) = \Delta z(n) < 0, n \geq n_2. \tag{5}$$

Using (5) we get

$$\Delta z(n-m+1) + q(n-m)\, G\,(y(n-k-m)) < 0, \qquad n \geq n_2^*$$

and hence for $n_3 > max\{n_2,\ n_2^*\}$,

$$\begin{aligned} 0 > \Delta z(n+1) + q(n)\, G(y(n-k)) + G(a)\, \Delta z(n-m+1) \\ +G(a)q(n-m)G(y(n-k-m)). \end{aligned} \tag{6}$$

Using (H_2), (H_3) and (H_5), the last inequality implies

$$0 > \Delta z(n+1) + G(a)\Delta z(n-m+1) + \lambda Q(n) G(z(n-k)),$$

where $0 < z(n) \leq y(n) + a\, y(n-m)$. Thus

$$\lambda Q(n) + \frac{\Delta z(n+1)}{G(z(n-k))} + G(a)\frac{\Delta z(n-m+1)}{G(z(n-k))} < 0,$$

that is,

$$\lambda Q(n) + \int\limits_{z(n+1)}^{z(n+2)} \frac{du}{G(u)} + G(a) \int\limits_{z(n-m+1)}^{z(n-m+2)} \frac{dv}{G(v)} < 0,$$

where $z(n+2) < u < z(n+1),\ z(n-m+2) < v < z(n-m+1)$ and $n-k < n+1$. Hence for $n \geq n_3$,

$$\lambda \sum_{s=n_3}^{n} Q(s) + \sum_{s=n_3}^{n} \int\limits_{z(s+1)}^{z(s+2)} \frac{du}{G(u)} + G(a) \sum_{s=n_3}^{n} \int\limits_{z(s-m+1)}^{z(s-m+2)} \frac{dv}{G(v)} < 0,$$

that is,

$$\lambda \sum_{s=n_3}^{n} Q(s) + \int\limits_{z(n_3+1)}^{z(n+2)} \frac{du}{G(u)} + G(a) \int\limits_{z(n_3-m+1)}^{z(n_3-m+2)} \frac{dv}{G(v)} < 0.$$

Since $\lim\limits_{n\to\infty} z(n)$ exists, then the above inequality implies that

$$\sum_{s=n_3}^{\infty} Q(s) < \infty,$$

a contradiction to (H_2). If $\Delta z(n) > 0$ for $n \geq n_2$, then $z(n)$ is nondecreasing and hence there exists a constant $\alpha > 0$ such that $z(n) > \alpha,\ n \geq n^*$. Application of Eq.(4) gives

$$\Delta^2 z(n) + G(a)\Delta^2 z(n-m) + \lambda Q(n) G(z(n-k)) \leq 0.$$

Consequently, for $n \geq n_3 > max\{n_2, n^*\}$,

$$\lambda G(a)Q(n) < -\Delta^2 z(n) - G(a)\Delta^2 z(n-m).$$

Thus,

$$\sum_{n=n_3}^{\infty} Q(n) < \infty,$$

a contradiction to (H_2).

Suppose the later holds. Then setting $x(n) = -y(n) > 0$, for $n \geq n_1$ and using (H_4), Eq.(1) can be written as

$$\Delta^2(x(n) + p(n)x(n-m)) + q(n)G(x(n-k)) = 0. \tag{7}$$

Following the above procedure to Eq.(7), similar contradictions can be obtained. Hence the proof of the theorem is complete.

Remark The prototype of G satisfying (H_3), (H_4) and (H_5) is

$$G(u) = \left(\alpha + \beta\,|u|^{\lambda}\right)|u|^{\mu}\, sqn\ u,$$

where $\alpha \geq 1,\ \beta \geq 1,\ \lambda \geq 0$ and $\mu \geq 0$.

Example Consider

$$\Delta^2[y(n) + (3 + (-1)^n)y(n-1)] + 8\, y^{\frac{1}{3}}(n-2) = 0, n \geq 0.$$

Clearly, the above equation satisfies all the conditions of Theorem 2.1 and hence it is oscillatory. In particular, $y(n) = (-1)^{3n}$ is such an oscillatory solution.

Theorem 2.2 Let $-1 < b \leq p(n) \leq 0$. If (H_1), (H_4), (H_6) and $m < k$ hold, then every solution of (1) oscillates.

Proof Proceeding as in Theorem 2.1, we get the inequality (5), where $\Delta z(n) < 0$ and $z(n) > 0$ for $n \geq n_2$. Since $z(n) \leq y(n)$, then (5) becomes

$$\Delta z(n+1) + q(n)G(z(n-k)) < 0, n \geq n_2.$$

Following the similar steps of Theorem 2.1, we have a contradiction to (H_1). If $z(n) < 0$, for $n \geq n_2$, then $y(n) < y(n-m)$, that is, $y(n)$ is bounded. Consequently, $z(n)$ is bounded, a contradiction to the fact that $\Delta z(n) < 0$ and $z(n) < 0$ for $n \geq n_2$. Hence $\Delta z(n) > 0$ for $n \geq n_2$. If $z(n) > 0$, then using the similar argument as in Theorem 2.1, the contradiction is obtained to (H_1). Suppose that $z(n) < 0$, for $n \geq n_2$. Then $y(n-k) > (1/b)z(n+m-k)$. Thus Eq.(1) becomes

$$\Delta^2 z(n) + q(n)G\left(\frac{1}{b}z(n)\right) < 0$$

due to increasing $z(n)$. Consequently, the last inequality can be made as

$$-\Delta z(n) + q(n)G\left(\frac{1}{b}\ z(n+1)\right) < 0$$

that is,

$$\Delta x(n) - \frac{1}{b}q(n)G(x(n+1) < 0,$$

where $x(n) = \frac{1}{b}z(n) > 0$. Hence

$$-\frac{1}{b}\ q(n) < -\frac{\Delta x(n)}{G(x(n+1))} = -\int\limits_{x(n)}^{x(n+1)} \frac{du}{G(x(n+1))}$$

For $x(n+1) < u < x(n)$, it is immediate to get

$$\frac{1}{b}\sum_{n=n_2}^{N} q(n) < -\sum_{n=n_2}^{N}\int\limits_{x(n)}^{x(n+1)} \frac{du}{G(u)} = -\int\limits_{x(n_2)}^{x(N+1)} \frac{du}{G(u)}$$

that is,

$$-\frac{1}{b}\sum_{n=n_2}^{\infty} q(n) < -\lim_{N\to\infty}\int\limits_{x(n_2)}^{x(N+1)} \frac{du}{G(u)} < \infty,$$

a contradiction.

The case $y(n) < 0$ for $n \geq n_1$ is similar. This completes the proof of the theorem.

Theorem 2.3 Let $-\infty < -b \leq p(n) < -1, b > 0$. If ($H_1$), ($H_4$), ($H_7$) and ($H_8$) hold, then every bounded solution of (1) oscillates.

Proof Let $y(n)$ be a bounded nonoscillatory solution of (1). Then there exists $n_1 > 0$ such that $y(n) > 0$ for $n \geq n_1$. The case $y(n) < 0$ for $n \geq n_1$ can similarly be dealt with. Setting z(n) as in (3), we get (4). Hence $\Delta^2 z(n) \leq 0$, for $n \leq n_2 > n_1 + r$ implies that $\Delta z(n)$ in nonincreasing. Let $\Delta z(n) < 0$. Then $z(n) > 0$ or < 0, for $n \geq n_2$. Suppose the former holds. Then $\lim\limits_{n\to\infty} z(n)$ exists. On the other hand, $\Delta^2 z(n) < 0$ and $\Delta z(n) < 0$ implies that there exists $L > 0$ such that $\Delta z(n) < -L$, for $n \geq n^*$ that is, $z(n) < -L_1(n)$, where $L_1 > 0$

is a constant. Hence $\lim_{n\to\infty} z(n) < -\infty$, a contradiction. Consequently, the later holds and hence $\lim_{n\to\infty} z(n) = -\infty$, a contradiction to the fact that $z(n+m-k) \geq -by(n-k)$ which is bounded. Assume that $\Delta z(n) > 0$, for $n \geq n_2$. If $z(n) > 0$, then there exists a constant $\alpha > 0$ such that $z(n) > \alpha$, for $n \geq n^*$. Thus $z(n) = y(n) + p(n)y(n-m) < y(n)$ implies that

$$\Delta^2 z(n) + G(\alpha)q(n) < 0,$$

for $n \geq n_3 > max\{n_2, n^*\}$. Hence

$$\sum_{n=n_3}^{\infty} Q(n) < \infty,$$

a contradiction to (H_1). Ultimately, $z(n) < 0$, for $n \geq n_2$. In this case $\lim_{n\to\infty} z(n)$ exists. Let it be $\beta,\ 0 \leq \beta < \infty$. Suppose that $\beta = 0$. Due to $\Delta z(n) > 0$, Eq.(1) becomes

$$-\Delta z(n) + q(n)G\left(-\frac{1}{b}z(n+m-k)\right) < 0$$

where $y(n-k) \geq -\frac{1}{b}z(n+m-k)$. Setting $-\frac{1}{b}z(n) = x(n)$, the last inequality can be written as

$$\Delta x(n) + \left(\frac{1}{b}\right) q(n)G(x(n+m-k)) < 0$$

for $n \geq n_3 > n_2$ and hence using (H_7), we get

$$\Delta x(n) + \left(\frac{\gamma}{b}\right) q(n)x(n+m-k) < 0$$

which has no positive solution due to (H_8), a contradiction to the fact that $x(n) > 0$ is a solution. If $0 < \beta < \infty$, then the contradiction is obivious due to (H_1). Hence the theorem is proved.

3 Superlinear Oscillation

This section deals with the oscillation of all solutions of Equation (1) such that G is superlinear.

Theorem 3.1 Let $0 \leq p(n) \leq a < \infty$. Assume that (H), ($H_3$)-($H_5$) and the following conditions

(H_9) $$\liminf_{n\to\infty} \sum_{s=n-k+1}^{n} Q(s) > 0,$$

where $Q(n+1) = min\{q(n), q(n-m)\}$.

$$(\mathrm{H}_{10}) \qquad \sum_{n-0}^{\infty}\left[\sum_{s=n+1}^{n+k+1} Q(s)\ell n\left(\sum_{s=n+1}^{n+k+1}\frac{\lambda Q(s)}{(1+G(a))}\right)-\sum_{s=n+2}^{n+k+1} q(s)\ell n\left(\sum_{s=n+2}^{n+k+1}\frac{\lambda Q(s)}{(1+G(a))}\right)\right]=\infty$$

hold. Then Eq.(1) is oscillatory.

Proof Proceeding as in Theorem 2.1, we get the inequality (6). Using the fact that both $z(n)$ and $\Delta z(n)$ are non-increasing, inequality (6) can be written as

$$0 > (1+G(a))\Delta z(n+1) + \lambda Q(n+1)G(z(n+1-k))$$

that is,

$$\Delta z(n+1) + \frac{\lambda}{(1+G(a))}Q(n+1)G(z(n+1-k)) < 0$$

for $n \geq n_3 > n_2$. In view of (H_{10}) and Theorem 1.2, the last inequality has no positive solution, a contradiction. If $\Delta z(n) > 0$ for $n \geq n_2$, then proceeding as in Theorem 2.1, we get

$$\sum_{n=n_3}^{\infty} Q(n+1) < \infty.$$

On the otherhand (H_9) implies that

$$\sum_{n=0}^{\infty} Q(n+1) = \infty,$$

a contradiction. Hence the theorem is proved.

Theorem 3.2 Let $-1 < b \leq p(n) \leq 0$. Assume that (H_1), (H_4) and the following conditions

$$(\mathrm{H}_{11}) \qquad \liminf_{n\to\infty}\sum_{s=n+m-k}^{n-1} q(s) > 0, k > m$$

$$(\mathrm{H}_{12}) \qquad \sum_{n=0}^{\infty}\left[\sum_{s=n}^{n+k-m} q(s)\ell n\left(\sum_{s=n}^{n+k-m}\left(\frac{-1}{b}\right)q(s)\right)-\sum_{s=n+1}^{n+k-m} q(s)\ell n\left(\sum_{s=n+1}^{n+k-m}\left(\frac{-1}{b}\right)q(s)\right)\right]=\infty$$

hold. Then every solution of (1) oscillates.

Proof Clearly, (H_{11}) implies that $\sum_{n=0}^{\infty} q(n) = \infty$. Let $y(n)$ be a non-oscillatory solution of (1) such that $y(n) > 0$ for $n \geq n_0$. Proceeding as in Theorem 2.1, we get the inequality (4). Consequently, $\Delta^2 z(n) \leq 0$ for $n \geq n_1 > n_0 + r$ shows that there are four possible cases:

1. $z(n) > 0, \Delta z(n) < 0,$

2. $z(n) < 0, \Delta z(n) < 0,$

3. $z(n) > 0, \Delta z(n) > 0,$

4. $z(n) < 0, \Delta z(n) > 0,$

Using the same type of reasoning as in the proof of the Theorem 2.2 we may obtain respective contradictions for the cases (1), (2) and (3). Consider the case (4). Since $z(n) < 0$. Then substituting $y(n-k) > (\frac{1}{b})z(n+m-k)$ for $n \geq n_2$. Eq.(1) can be written as

$$\Delta x(n) - \frac{1}{b} q(n) G(x(n+m-k)) < 0,$$

where $x(n) = \left(\frac{1}{b} z(n)\right) > 0$. In view of Theorem 1.2 and (H_{12}), the last inequality has no positive solution, a contradiction. This completes the proof of the theorem.

Theorem 3.3 Let $-\infty < b \leq p(n) < -1$. Assume that all the conditions of Theorem 3.2 hold. Then every bounded solution of (1) oscillates.

Proof Let $y(n)$ be a bounded solution of (1) such that $y(n) > 0$ for $n \geq n_0$. Then proceeding as in the proof of the Theorem 3.2, we have four cases, cases (1), (2) and (3) follows from the Theorem 2.3 and case (4) follows from the Theorem 3.2. Hence the details are omitted. This completes the proof of the theorem.

Theorem 3.4 Let $-\infty < b \leq p(n) < -1$. Assume that $m \geq k+1$ and the following conditions

(H_{13}) $\int\limits_0^\infty \frac{dx}{G(x)} < \infty$

(H_{14}) $\sum\limits_{j=0}^{\infty} q(n_j) = \infty$ for every sequence $\{n_j\}$ of $\{n\}$

hold. Then every unbounded solution of (1) oscillates.

Proof Let $y(n)$ be an unbounded solution of (1) such that $y(n) > 0$ for $n \geq n_0$. The case $y(n) < 0$ for $n \geq n_0$ is similar. Using the same type of reasoning as in the proof of Theorem 3.2, we consider the four cases (1), (2), (3) and (4). For the cases (1) and (3), the discussion is same which can be followed from the Theorem 3.2 directly. Consider the case (2). Here Eq.(1) reducess to

$$\Delta z(n+1) + q(n) G\left(\frac{1}{b} z(n+1)\right) = \Delta z(n) < 0$$

that is,

$$q(n) < -\frac{\Delta z(n+1)}{G\left(\frac{1}{b}z(n+1)\right)} = -\int\limits_{z(n+1)}^{z(n+2)} \frac{du}{G\left(\frac{1}{b}z(n+1)\right)}$$
$$< -\int\limits_{z(n+1)}^{z(n+2)} \frac{du}{G\left(\frac{1}{b}u\right)} = -b\int\limits_{z(n+1)}^{z(n+2)} \frac{d\left(\frac{1}{b}u\right)}{G\left(\frac{1}{b}u\right)}$$

where $z(n+2) < u < z(n+1)$. Hence

$$\sum_{n=n_2}^{N} q(n) < -b\sum_{n=n_2}^{N} \int\limits_{z(n+1)}^{z(n+2)} \frac{d\left(\frac{1}{b}u\right)}{G\left(\frac{1}{b}u\right)},$$
$$= -b\int\limits_{z(n_2+1)}^{z(N+2)} \frac{d\left(\frac{1}{b}u\right)}{G\left(\frac{1}{b}u\right)}.$$

Consequently,

$$\sum_{n=n_2}^{\infty} q(n) < -b\lim_{N\to\infty} \int\limits_{z(n_2+1)}^{z(N+2)} \frac{d\left(\frac{1}{b}u\right)}{G\left(\frac{1}{b}u\right)} < \infty,$$

a contradiction to (H_{14}). This is because (H_{14}) implies that (H_1) hold. Next, we consider case (4). Since $y(n)$ is unbounded, there exists a sequence $\{n_j\}$ of $\{n\}$ such that $y(n_j) \to \infty$ as $j \to \infty$. Hence for every $M > 0$, it is possible to find $n_2 > n_1$ such that $n_j \geq n_2$ implies $y(n_j) > M$. Let $n_3 \geq n_2 + K$. Then Eq.(1) yields

$$\sum_{n_j=n_3}^{s-1} q(n_j)G(y(n_j-k)) = -\sum_{n_j=n_3}^{s-1} \Delta^2 z(n_j) = -\Delta z(s) + \Delta z(n_3).$$

Thus

$$G(M)\sum_{n_j=n_3}^{\infty} q(n_j) < \sum_{n_j=n_3}^{\infty} q(n_j)G(y(n_j-k)) < \infty,$$

a contradiction to (H_{14}). Hence the proof of the theorem is complete.

Remark The prototype of G satisfying the Theorems 3.1 - 3.3 may be of the form

$$G(u) = \begin{cases} u\left[1+(a+\ell n^2|u|)^{-1}\right], u \neq 0 \\ 0, u = 0 \end{cases}$$

and

$$g(u) = \begin{cases} a, \ |u| > 1 \\ \\ (a + \ell n^2|u|)^{-1}, \ 0 < |u| \leq 1 \\ \\ 0, \ u = 0. \end{cases}$$

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