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A THIRD-ORDER 3-POINT BVP. APPLYING KRASNOSEL'SKIĬ'S THEOREM ON THE PLANE WITHOUT A GREEN'S FUNCTION.

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Abstract. Consider the three-point boundary value problem for the 3^{rd} order differential equation:

$$\begin{cases} x^{'''}(t) = \alpha(t) f(t, x(t), x'(t), x''(t)), \quad 0 < t < 1, \\ x(0) = x'(\eta) = x''(1) = 0, \end{cases}$$

under positivity of the nonlinearity. Existence results for a positive and concave solution $x(t)$, $0 \le t \le 1$ are given, for any $1/2 < \eta < 1$. In addition, without any monotonicity assumption on the nonlinearity, we prove the existence of a sequence of such solutions with

$$\lim_{n\to\infty} ||x_n|| = 0.$$

Our principal tool is *a very simple applications on a new cone of the plane* of the well-known Krasnosel'skiĭ's fixed point theorem. The main feature of this aproach is that, we do not use at all the associated Green's function, the necessary positivity of which yields the restriction $\eta \in (1/2, 1)$. Our method still guarantees that the solution we obtain is positive.

1. Introduction

Ma in [21] proved the existence of a positive solution to the three-point nonlinear boundary-value problem

$$-u''(t) = q(t)f(u(t)), \quad 0 < t < 1,$$
$$u(0) = 0, \quad \alpha u(\eta) = u(1),$$

where $\alpha > 0$, $0 < \eta < 1$ and $\alpha\eta < 1$. Later Webb and Infante [14] studied the three-point nonlinear boundary-value problem

$$-u''(t) = q(t)f(u(t)), \quad u'(0) = 0, \alpha u'(1) + u(\eta) = 0$$

and mainly the loss of positivity of its solutions, as α decreases. The results of Ma were complemented in the works of Kaufmann [15] and Kaufmann and Raffoul [16].

In the above papers there are no assumptions for singularity of the nonlinearity f at the point $u = 0$. Zhang and Wang [29] and recently Liu [18] obtained some existence results for a singular nonlinear second order 3-point boundary-value problem, for the case where only singularity of $q(t)$ at $t = 0$ or $t = 1$ is permitted. Other applications of Krasnosel'skiĭ's fixed point theorem to semipositone problems can, for example, be found in [1]. Further recently interesting results have been proved in [4], [11], or [26].

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Anderson and Avery [2] and Anderson [3], proved that there exist at least three positive solutions to the BVP (1.1) (below) and the analogous discrete one respectively, by using the Leggett-Williams fixed point theorem. Yao in [28] and Haiyan and Liu in [10], using the Krasnosel'skiĭ's fixed point theorem showed the existence of multiple solutions to the BVP (1.1). More similar results can be found in Du et al [6] and also in Feng and Webb [7].

Recently, Du et al [5] via the coincidence degree of Mawhin, proved existence for the BVP

$$\begin{cases} x^{'''}(t) = f(t, x(t), x'(t), x''(t)), \quad 0 < t < 1, \\ x(0) = \alpha x(\xi), \ x''(0) = 0, \ x'(1) = \sum_{j=1}^{m-2} \beta_j x'(\eta_j), \end{cases}$$

at the resonance case. In an also recent paper Sun [25], obtained existence of infinitely many positive solutions to the BVP

$$\begin{cases} u^{'''}(t) = \lambda\alpha(t) f(t, u(t)), \quad 0 < t < 1, \\ u(0) = u'(\eta) = u''(1) = 0, \ \eta \in (1/2, 1) \end{cases} \tag{1.1}$$

mainly under superlinearity on the nonlinearity f of the type

There exist two positive constants $\theta,\ R \neq r$ such that

$$\begin{cases} f(t,x) \leq \frac{r}{\lambda M}, \quad \forall (t,x) \in [0,1] \times [0,r]; \\ f(t,x) \geq \frac{R}{\lambda N}, \quad \forall (t,x) \in [0,1] \times [\theta R, R], \end{cases}$$

where M and N are also constants. Sun, in order to obtain his existence results applied the classical Krasnosel'skii fixed-point theorem on cone expansion-compression type and furthermore to prove his multiplicity results he assumed monotonicity of the nonlinearity with respect the second variable.

Very recently there have been several papers on third-order boundary value problems. Hopkins and Kosmatov [12], Li [17], Liu et al [19, 20], Guo et al [9] and Kang et al [22] have all considered third-order problems. Graef and Yang [8] and Wong [27] consider three-point focal problems, while Palamides and Smyrlis [23] consider the three-point boundary conditions

$$u'''(t) = a(t) f(t, u(t)), \quad x(0) = x''(\eta) = x(1) = 0.$$

In this work, motivated by the above mentioned papers and especially the ones of Sun [25] and Palamides and Smyrlis [23], we suppose a superlinearity-type growth rate of $f(t, u, u', u'')$ at both the origin $u = 0$ and $u = +\infty$. The emphasis in this paper is mainly to apply the well-known Krasnosel'skiĭ's fixed point theorem *just on the plane, using in this way an alternative to the classical methodologies, in which as it is common, a Banach space of functions is used.* We combine the above Krasnosel'skii's theorem with properties of the associating vector field, defined on the phase plane and this results in the use of similar quite natural hypothesis.

Furthermore we prove existence of infinitely many positive solutions for the more general boundary value problem

$$\begin{cases} x^{'''}(t) = \alpha(t) F(t, x(t), x'(t), x''(t)), \quad 0 < t < 1, \\ x(0) = x'(\eta) = x''(1) = 0, \end{cases} \tag{E}$$

and at the same time, we eliminate at all the related monotonicity assumption on the nonlinearity in [25].

2. Preliminaries

Consider the third-order nonlinear boundary value problem (E), where we assume (within this paper) that $\eta \in (1/2, 1)$, the continuous functions $\alpha(t)$, $t \in (0,1)$ and $F \in C(\Omega, [0,+\infty))$ are nonnegative and $\Omega = [0,1] \times [0,+\infty) \times \mathbb{R} \times (-\infty, 0]$.

Then, a vector field is defined with crucial properties for our study. More precisely, considering the (x', x'') phase semi-plane $(x' > 0)$, we easily check that $x''' = \alpha(t) F(t, x, x', x'') \geq 0$. Thus, any trajectory $(x'(t), x''(t))$, $t \geq 0$, emanating from any point in the fourth quadrant:

$$\{(x', x'') : x' > 0,\ x'' < 0\}$$

"evolutes" in a natural way, when $x'(t) > 0$, toward the negative $x''-$semi-axis. Then, when $x'(t) \leq 0$, the trajectory "evolutes" toward the negative $x'-$semi-axis and finally it stays asymptotically in the second quadrant. As a result, assuming a certain growth rate on f (e.g. a superlinearity), we can control the vector field in a way that assures the existence of a trajectory satisfying the given boundary conditions. These properties, which will be referred as *"the nature of the vector field"*, combined with the Krasnosel'skii's principle, are the main tools that we will employ in our study.

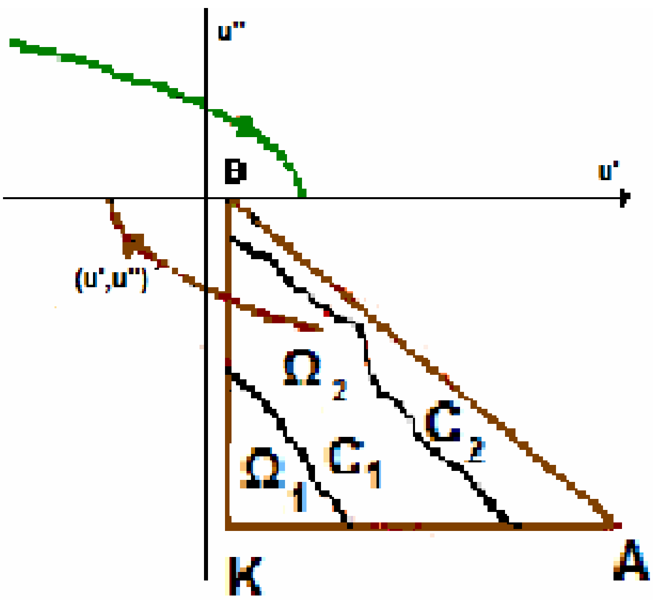

Fig 1.

In this paper, we employ a simple cone on the phase plane. First we recall the next definition:

Definition 1. *Let E be a Banach space. A nonempty closed convex set $K^* \subset E$ is called a cone of E, iff*

(1) $x \in K^*,\ \lambda > 0 \quad \Rightarrow \quad \lambda x \in K^*;$

(2) $x \in K^*, \quad -x \in K^* \quad \Rightarrow x = 0.$

For example, the above fourth quadrant

$$\tilde{K} = \{(x', x'') \in \mathbb{R}^2 : x' \geq 0,\ x'' \geq 0\}$$

on the plane $\mathbb{R}^2$ is a cone.

We need a preliminary result from the fixed point theory, which will be our base for all results in this paper.

Precisely will apply the well known Krasnosel'skiĭ's fixed point theorem in cones.

Lemma 1. *Let E be a Banach space and $K^* \subset E$ a cone in E. Assume that Ω_1 and Ω_2 are open subsets of E with $0 \in \Omega_1$ and $\bar{\Omega}_1 \subset \Omega_2$. Let*

$$T : K^* \cap \left(\bar{\Omega}_2 \backslash \Omega_1\right) \to K^*$$

be a completely continuous operator. We assume furthermore either

(A) $\quad ||Tu|| \leq ||u||, \quad \forall u \in K^* \cap \partial\Omega_1$ *and* $||Tu|| \geq ||u||, \quad \forall u \in K^* \cap \partial\Omega_2$ *or*

(B) $\quad ||Tu|| \geq ||u||, \quad \forall u \in K^* \cap \partial\Omega_2$ *and* $||Tu|| \leq ||u||, \quad \forall u \in K^* \cap \partial\Omega_1.$

Then T has a fixed point in $K^* \cap (\Omega_2 \backslash \Omega_1)$.

3. Existence Results.

Consider the third-order nonlinear three-point boundary value problem:

$$u''' = \alpha(t) f(t, u, u', u''), \quad 0 < t < 1, \tag{3.1}$$

$$u(0) = u'(\eta) = u''(1) = 0. \tag{3.2}$$

where f is a continuous extension of F, i.e.

$$f(t, u, u', u'') = \begin{cases} F(t, u, u', u''), & u \geq 0,\ u'' \leq 0; \\ F(t, u, u', 0), & u \geq 0,\ u'' \geq 0; \\ F(t, 0, u', 0), & u < 0, \quad u'' > 0; \\ F(t, 0, u', u''), & u < 0, \quad u'' < 0. \end{cases}$$

Remark 1. *By the sign property of F, it follows that*

$$f(t, u, u', u'') \geq 0, \quad (t, u, u', u'') \in [0,1] \times \mathbb{R}^3.$$

Lemma 2. *Let $u = u(t),\ t \in [0,1]$ be a solution of the boundary value problem (E) such that*

$$u(0) = 0,\ u'(0) = u_0' > 0 \quad \textit{and} \quad u''(0) = u_0'' < 0. \tag{3.3}$$

Then

$$u(t) \geq 0, \quad t \in [0,1]$$

for any initial value (u_0', u_0'') with $u_0'' \geq -2u_0'$.

Proof. By the Taylor's Formula

$$u(t) = tu_0' + \frac{t^2}{2}u_0'' + \frac{t^3}{2}\int_0^1 (1-s)^2 \alpha(st) F[st, u(ts), u'(ts), u''(ts)]ds, \ t \in [0,1].$$

and (3.3), we get $u(t) > 0$ for all t in a (right) neighborhood of $t = 0$. Assume that there exists a $t^* \in (0,1)$ such that

$$u(t^*) = 0 \ \text{ and } \ u(t) \geq 0, \quad t \in [0, t^*].$$

Given that $u_0'' \geq -2u_0'$, we get, noticing the sign of the nonlinearity

$$\frac{t^*}{2}(2u_0' + t^* u_0'') \leq 0 \quad \Leftrightarrow \quad t^* \geq -\frac{2u_0'}{u_0''} \geq 1,$$

a contradiction. ∎

Assume throughout of this paper, that $0 < \theta < 1/2$ and there exist positive constants r_0 and R_0 with

$$r_0\left(1+\eta^2\right) \leq \eta R_0,$$

such that for every $0 < r \leq r_0$ and any $R \geq R_0$,

$$(A_1) \qquad \begin{cases} f(t,x,y,z) < \frac{r}{M}, \ (t,x,y,z) \in \Delta_1, \quad \text{with} \\ \Delta_1 = [0,1] \times [0,r] \times [-r_0\frac{1+\eta^2}{\eta}, \frac{(1+\eta^2)r_0 + R_0}{\eta}] \times [-\frac{(1+\eta^2)r_0}{\eta}, 0]; \end{cases}$$

$$(A_2) \qquad \begin{cases} f(t,x,y,z) > \frac{R}{N}, \ (t,x,y,z) \in \Delta_2, \quad \text{with} \\ \Delta_2 = [0,1] \times [\theta R, +\infty) \times [-r_0\frac{1+\eta^2}{\eta}, \frac{(1+\eta^2)r_0 + R_0}{\eta}] \times [-\frac{(1+\eta^2)r_0}{\eta}, +\infty], \end{cases}$$

where

$$M = \int_0^1 \alpha(s)\,ds > 0 \quad \text{and} \quad N = \int_\theta^{1-\theta} \alpha(s)\,ds > 0$$

Proposition 1. *For every initial value* (u_0', u_0''), *with* $u_0'' \leq -r_0\frac{1+\eta^2}{\eta} < -r_0 \leq -u_0'$, *any solution* $u = u(t)$ *of the initial value problem (3.1),(3.3) satisfies*

$$u'(\eta) < 0, \quad \textit{and} \quad u''(t) < 0, \quad t \in [0,1].$$

Proof. We choose (without loss of generality)

$$u_0' = r_0 \quad \text{and} \quad u_0'' = -r_0\frac{1+\eta^2}{\eta} \tag{3.4}$$

(then $u_0'' + 2u_0' \leq 0$) and assume that $u''(1) > 0$. Since by Remark 1 it follows that $u'''(t) > 0$, the function $u''(t)$, $t \in [0,1]$ is nondecreasing. Hence there exists a $t^* \in (0,1)$ such that

$$-r_0\frac{1+\eta^2}{\eta} \leq u''(t) < 0, \ t \in [0, t^*) \ \text{ and } \ u''(t^*) = 0.$$

Furthermore,

$$u'(t) \geq -r_0\frac{1+\eta^2}{\eta}t \geq -r_0\frac{1+\eta^2}{\eta}, \ t \in [0, t^*).$$

Thus by the mean value theorem,

$$0 = u''(t^*) = u_0'' + t^* \int_0^1 \alpha(st^*) f[st^*, u(st^*), u'(st^*), u''(st^*)]ds.$$

Now since the derivative $u'(t)$, $t \in [0, t^*)$ is decreasing, we obtain $u'(t) \le u_0'$, $t \in [0, t^*)$. Hence $u(t) < t^* u_0' \le u_0' = r_0$, $t \in [0, t^*)$. Consequently in view of the Remark 1 and the assumption (A_1), we obtain the contradiction

$$u''(t^*) \le u_0'' + t^* \frac{r_0}{M} \int_0^1 \alpha(st^*) \, ds \le u_0'' + t^* r_0 < u_0'' + r_0 \le 0.$$

On the other hand, again by Taylor's formula and condition (A_1),

$$\begin{aligned} u'(\eta) &= u_0' + \eta u_0'' + \eta^2 \textstyle\int_0^1 (1-s) \alpha(s\eta) f[s\eta, u(s\eta), u'(s\eta), u''(s\eta)]ds \\ &< u_0' + \eta u_0'' + \eta^2 r_0 = 0. \end{aligned}$$

∎

We recall choices (3.4) and $r_0 \left(1+\eta^2\right) \le \eta R_0$ and fix the obtained initial point $K = (u_0', u_0'')$. Furthermore consider the simplex $S = [K, A, B]$, where the vertices $A = (u_A', u_0'')$ and $B = (u_0', 0)$ are chosen so that

$$u_A' + u_0'' = \eta^{-1} R_0 > 0 \;\text{ i.e. }\; u_A' = \frac{\left(1+\eta^2\right) r_0 + R_0}{\eta}. \tag{3.5}$$

Proposition 2. *The derivative of every solution $u = u(t)$ of (3.1) emanating from any initial point $P_1 = (u_1', u_1'') \in [A, B]$ (we denote in the sequel such a choice by $u \in \mathcal{X}(P_1)$) satisfies*

$$u'(t) > 0, \quad 0 \le t \le \eta.$$

Proof. We assume on the contrary that $u'(\eta) \le 0$ and notice that

$$u_1' + \eta u_1'' > 0, \tag{3.6}$$

for every $P = (u_1', u_1'') \in [A, B]$. Indeed, since

$$u_1'' = \frac{r_0\left(1+\eta^2\right)(u_1' - r_0)}{r_0\eta - r_0(1+\eta^2) - R_0}, \; r_0 \le u_1' \le \frac{r_0\left(1+\eta^2\right) + R_0}{\eta},$$

it follows that

$$\begin{aligned} u_1' + \eta u_1'' &= u_1' \left[1 + \frac{\eta r_0 \left(1+\eta^2\right)}{r_0\eta - r_0(1+\eta^2) - R_0}\right] - \frac{\eta r_0^2 \left(1+\eta^2\right)}{r_0\eta - r_0(1+\eta^2) - R_0} \\ &= r_0 \left[1 + \frac{\eta r_0 \left(1+\eta^2\right)}{r_0\eta - r_0(1+\eta^2) - R_0}\right] - \frac{\eta r_0^2 \left(1+\eta^2\right)}{r_0\eta - r_0(1+\eta^2) - R_0} = r_0 > 0. \end{aligned}$$

Consider now the two possible cases:

- Let $u''(1) < 0$. Since obviously $u''(t) < 0$, $0 \le t \le 1$, the map $u'(t)$ is decreasing and thus there is a point $t^* \in (0, \eta]$ such that

$$u'(t^*) = 0 \text{ and } u'(t) \ge 0, \quad 0 \le t \le t^*.$$

This clearly implies that $u(t) \geq 0, \quad 0 \leq t \leq t^*$ and furthermore we have $f[t, u(t), u'(t), u''(t)] \geq 0$. In view of (3.6) and Taylor's formula, we get the contradiction

$$\begin{aligned} u'(t^*) &= u_1' + t^* u_1'' + t^{*2} \int_0^1 (1-s)\, \alpha(st^*)\, f[st^*, u(st^*), u'(st^*), u''(st^*)] ds \\ &> u_1' + \eta u_1'' > 0. \end{aligned}$$

- Let us assume now that $u''(1) \geq 0$. Then there exists a $\hat{t} \in (0,1]$ with

$$u''\left(\hat{t}\right) = 0 \text{ and } u''(t) \leq 0, \ 0 \leq t \leq \hat{t}.$$

As above we conclude immediately that the function $u'(t), \ 0 \leq t \leq \hat{t}$ is decreasing. If $u'\left(\hat{t}\right) > 0$, then, in view of the nature of vector field, we obtain $u'(t) > 0, \ 0 \leq t \leq 1$, a contradiction to $u'(\eta) \leq 0$. Hence $u'\left(\hat{t}\right) \leq 0$ and thus we get a point $t^* \leq \hat{t}$ such that

$$u'(t^*) = 0 \text{ and } u'(t) \geq 0, \quad 0 \leq t \leq t^*.$$

Then as above, Taylor's formula also leads to another contradiction $u'(t^*) > 0$.

∎

Lemma 3. *Consider a function $y \in C^{(3)}[(0,1), [0, +\infty)]$ such that*

$$\begin{cases} y(0) = 0, \quad y'(0) > 0 \text{ and } y''(0) < 0 \text{ and} \\ y'''(t) \geq 0, \ 0 < t < 1, \quad y'(\eta) \leq 0 \text{ and } y''(1) \leq 0. \end{cases}$$

Then

$$\min_{\theta \leq t \leq 1-\theta} y(t) \geq \theta ||y||,$$

where $||y|| = \max_{0 \leq t \leq 1} y(t)$.

Proof. Since $y'''(t) \geq 0$, the function $y''(t)$ is nondecreasing. So noticing $y''(1) \leq 0$, this implies that

$$y''(t) \leq 0, \ 0 < t < 1.$$

Now due to the concavity of $y(t)$, for any $\mu, \ t_1$ and t_2 in $[0,1]$, we have

$$y(\mu t_1 + (1-\mu) t_2) \geq \mu y(t_1) + (1-\mu) y(t_2).$$

Moreover using the assumption $y'(\eta) \leq 0$, we conclude that there is a $t^* \in (0, \eta)$ such that $y'(t^*) = 0$ and $||y|| = y(t^*)$. Therefore

$$y(t) \geq ||y|| \min_{\theta \leq t \leq 1-\theta} \left\{ \frac{t}{t^*}, \frac{1-t}{1-t^*} \right\} \geq ||y|| \min_{\theta \leq t \leq 1-\theta} \{t, 1-t\} = \theta ||y||.$$

∎

The next result is crucial for the sequence of our theory.

Lemma 4. *Assume that a solution $u = u(t)$ of a BVP (3.1),(3.2) satisfies moreover the inequalities*

$$u'(t) > 0, \ 0 \leq t < \eta \text{ and } u''(t) < 0, \ 0 \leq t < 1.$$

Then

$$u(t) \geq 0, \ 0 \leq t \leq 1.$$

Proof. Suppose that there is a $T \in (\eta, 1)$ such that

$$u(t) > 0,\ t \in (0, T),\ u(T) = 0 \text{ and } u(t) < 0,\ t \in (T, 1].$$

Since $\eta \in (1/2, 1)$, we get $2\eta - T \geq 0$. Consider then, two symmetric with respect to η, partitions

$$\{2\eta - T = r_0 < r_1 < ... < r_k = \eta\} \quad \text{and} \quad \{\eta = t_0 < t_1 < ... < t_k = T\}$$

of $[2\eta - T, \eta]$ and $[\eta, T]$ respectively, i.e.

$$r_k - r_{k-1} = t_1 - t_0,\ r_{k-1} - r_{k-2} = t_2 - t_1, ...,\ r_1 - r_0 = t_k - t_{k-1}.$$

The map $u = u''(t)$, $t \in [0, 1]$ is nondecreasing and thus we get

$$u'(r_i) > -u'(t_{k-i}), \quad (i = 0, 1, ..., k-1).$$

So

$$-(t_{k-i+1} - t_{k-i})u'(t_{k-i}) < (r_{i+1} - r_i)u'(r_i), \quad (i = 1, 2, ..., k),$$

reduces to

$$-\sum_{i=1}^{k}(t_{k-i+1} - t_{k-i})u'(t_{k-i}) < \sum_{i=1}^{k}(r_{i+1} - r_i)u'(r_i). \tag{3.7}$$

In addition, since the map $u' = u'(t)$, $0 \leq r \leq T$ is continuous (and bounded), we can choose the $\max\{r_i - r_{i-1} : i = 1, 2, ..., k\}$ small enough and given that $2\eta - T \geq 0$, we obtain

$$\int_0^\eta u'(t)\,dt \geq \int_{2\eta - T}^{\eta} u'(t)\,dt > -\int_\eta^T u'(r)\,dr.$$

Consequently

$$u(T) = \int_0^\eta u'(t)\,dt + \int_\eta^T u'(r)\,dr > 0,$$

a contradiction. ∎

Remark 2. *The restriction $\eta \in \left(\frac{1}{2}, 1\right)$ is necessary for the validity of the above Lemma 4. Indeed, for $\eta = 1/3$ and $f(t, u) = 1$, the function $u(t) = \left(t^3/6\right) - \left(t^2/2\right) + (5t/18)$ is a solution of the BVP (3.1)-(3.2), which satisfies the assumptions of Lemma. But $u(1) = -1/18 < 0$.*

Proposition 3. *Any solution $u = u(t)$ of (3.1) emanating from the above initial point $A = (u'_A, u''_0)$ (with (3.5) to hold) satisfies*

$$||u|| \geq \theta R_0,\quad u'(\eta) > 0 \quad \text{and} \quad u''(1) \geq 0.$$

Proof. We will show (extending partially the conclusion of previous Proposition 2) first that

$$u'(t) > \eta^{-1}R_0, \quad 0 \leq t \leq 1.$$

If not, then proceeding as in the proof of Proposition 2, we have $u'(t^*) = \eta^{-1}R_0$ for some $t^* \in (0, 1]$, $u'(t) \geq \eta^{-1}R_0$, $t \in (0, t^*)$. Then we get the contradiction (see

(3.5))

$$\begin{aligned} u'(t^*) &= u'_A + t^* u''_0 + t^{*2} \int_0^1 (1-s)\,\alpha(st^*)\, f[st^*, u(st^*), u'(st^*), u''(st^*)]ds \\ &> u'_A + u''_0 = \eta^{-1} R_0. \end{aligned}$$

Hence, given that $u'(t) \leq u'_A,\ 0 \leq t \leq 1$, we obtain

$$\eta^{-1} R_0 \leq u'(t) \leq \frac{(1+\eta^2)\, r_0 + R_0}{\eta} \quad \text{and} \quad u(t) > 0, \quad 0 \leq t \leq 1$$

and this yields

$$u(t) = \int_0^t u'(s)\, ds \geq \eta^{-1} t R_0.$$

Moreover, since the map $u = u(t),\ \ 0 \leq t \leq 1$ is nondecreasing, we obtain

$$\min_{\theta \leq t \leq 1-\theta} u(t) = u(\theta) \geq \eta^{-1} \theta R_0 \geq \theta R_0.$$

Consequently, since $u(t) \geq \theta R_0$ and $u''(t) \geq\ u''_0 = -r_0 \frac{1+\eta^2}{\eta},\ \theta \leq t \leq 1-\theta$, in view of the assumption (A_2),

$$\begin{aligned} u''(1) &= u''_0 + \int_0^1 \alpha(s)\, f[s, u(s), u'(s), u''(s)]ds \\ &> u''_0 + \int_\theta^{1-\theta} \alpha(s)\, f[s, u(s), u'(s), u''(s)]ds \geq u''_0 + R_0 \geq 0. \end{aligned}$$

∎

Remark 3. *We need some concepts, in the sequel, concerning the case where initial value problems have not a unique solution. Consider a set-valued mapping $\mathcal{F}$, which maps the points of a topological space X into compact subsets of another one Y. $\mathcal{F}$ is* upper semi-continuous *(usc) at $x_0 \in X$ iff for any open subset V in Y with $\mathcal{F}(x_0) \subseteq V$, there exists a neighborhood U of x_0 such that $\mathcal{F}(x) \subseteq V$, for every $x \in U$. Let P be any initial point such that every solution $u \in \mathcal{X}(P)$ is defined on the interval $[0, \eta]$. Then, by the well-known Knesser's property (see* [13, 24]*), the* cross-section

$$\mathcal{X}(\eta; P) = \{u(\eta), u'(\eta), u''(\eta)) : u \in \mathcal{X}(P)\}$$

is a continuum (compact and connected set) in $\mathbb{R}^3$, the same being its projections $\{u'(\eta) : u \in \mathcal{X}(P)\}$ and $\{u''(1) : u \in \mathcal{X}(P)\}$. Furthermore the image of a continuum under an upper semi-continuous map $\mathcal{K}$ is again a continuum. Also considering the set-valued mapping

$$\mathcal{K} : \Omega \to \mathbb{R}, \quad \mathcal{K}(P) = \{u'(\eta) : u \in \mathcal{X}(P)\}$$

we notice (see [24]*) that it is an upper semi-continuous mapping. Obviously, if an IVP has a unique solution, then this map is simply continuous.*

Remark 4. *By Propositions 1 and 3, there always exist points $P_1, P_2 \in [K, A]$ such that $u'(\eta) = 0,\ u \in \mathcal{X}(P_1)$ and $u''(1) = 0,\ u \in \mathcal{X}(P_2)$ respectively. That conclusion follows immediately by the Remark 3.*

In this way, we consider the three-dimensional simplex (triangle) S with vertices $K = (u_0', u_0'')$, $A = (u_A', u_0'')$ and $B = (u_0', 0)$, under the choices (3.4)-(3.5). The next result is also of central importance for the sequel.

Lemma 5. *Let $P_1 = (u_0', u_1'')$ be a point in the face $[K, B]$ such that $u'(\eta) = 0$, for some solution $u = u(t)$ emanating from the initial point P_1 i.e. $u \in \mathcal{X}(P_1)$. Then*

$$u''(1) \leq 0.$$

Proof. We notice firstly that such a point P_1 always exists, because of Propositions 1 and since by the sign of the nonlinearity, $u(\eta) > 0$ for every $u \in \mathcal{X}(B)$. Indeed in view of the Remark 3, the image of the segment $[K, B]$ under the map $\mathcal{X}$, that is

$$\mathcal{X}(\eta; [K, B]) = \cup \{\mathcal{X}(\eta; P) : P \in [K, B]\}$$

is a continuum. Hence its projection $\{u'(\eta) : u \in \mathcal{X}(P) : P \in [K, B]\}$ crosses the negative $u'-$semi axis of the phase-plane.

Next we shall show (following the proof of Proposition 1 and improving partially its conclusion) that, if

$$u_0' = r_0 \quad \text{and} \quad u_1'' \leq -r_0, \;\; (P_1 = (u_0', u_1''))$$

then (if $u''(1) = 0$, we have nothing to prove)

$$u''(1) < 0.$$

Indeed, by the definition of the modification f, it follows (see Remark 1) that $u'''(t) > 0$ and so the function $u''(t)$ $t \in [0, 1]$ is nondecreasing. Assume now, on the contrary, that $u \in \mathcal{X}(P_1)$ is a solution of the differential equation (3.1) such, that $u''(1) > 0$ (and $u'(\eta) = 0$). Hence there exists a $t^* \in (0, 1)$ such that

$$-r_0 \frac{1+\eta^2}{\eta} \leq u''(t) < 0, \;\; t \in [0, t^*) \;\; \text{and} \;\; u''(t^*) = 0.$$

Furthermore

$$u'(t) \geq -r_0 \frac{1+\eta^2}{\eta} t > -r_0 \frac{1+\eta^2}{\eta}, \;\; t \in [0, t^*).$$

Also, since the derivative $u'(t)$, $t \in [0, t^*)$ is decreasing, we obtain $u'(t) \leq u_0'$, $t \in [0, t^*)$ and so $u(t) < t^* u_0' \leq u_0' = r_0$, $t \in [0, t^*)$. Thus by the mean value theorem,

$$0 = u''(t^*) = u_1'' + t^* \int_0^1 \alpha(st^*) f[st^*, u(st^*), u'(st^*), u''(st^*)] ds$$

and in view of the assumption (A_1), we obtain the contradiction

$$u''(t^*) \leq u_1'' + t^* \frac{r_0}{M} \int_0^1 \alpha(st^*) ds \leq u_1'' + t^* r_0 < u_1'' + r_0 \leq 0.$$

Assuming now that $u \in \mathcal{X}(P_1)$ implies that $u''(1) > 0$, we must have

$$u_0' + u_1'' \geq 0 \;\; \text{and (recall)} \;\; u_0' = r_0 \tag{3.8}$$

(since, the inequality $u_1'' < -u_0'$, yields $u''(1) < 0$). Also, given that $u'(\eta) = 0$, by the nature of the vector field (sign of f), it follows that

$$u'(t) \geq 0 \quad \text{and} \quad u(t) \geq 0, \quad 0 \leq t \leq \eta.$$

Consequently by the Taylor's formula, (3.8) and Lemma 2, we get the final contradiction

$$\begin{aligned} u'(\eta) &= u_0' + \eta u_1'' + \eta^2 \int_0^1 (1-s)\,\alpha(s\eta)\, f[s\eta, u(s\eta), u'(s\eta), u''(s\eta)]ds \\ &> u_0' + \eta u_1'' \geq u_0' + u_1'' \geq 0, \end{aligned}$$

due to the properties of the nonlinearity and the assumption $\eta \in (1/2, 1)$. ∎

Consider now the cone in $\mathbb{R}^2$,

$$K^* = \left\{(u', u'') \in \mathbb{R}^2 : \ u' \geq 0, \ u'' \geq 0\right\}$$

and define the sets (see Fig 1)

$$\begin{aligned} \Omega_1 &= \{P = (u_1', u_1'') \in K^* : u'(t) < 0, \ \eta \leq t \leq 1 \ \text{ and } u''(1) < 0, \ \forall u \in \mathcal{X}(P+K)\}, \\ C_1 &= \{P = (u_1', u_1'') \in cl\Omega_1 : \exists\, u \in \mathcal{X}(P+K) \ \text{ with } \ u'(\eta) = 0 \ \text{ and } \ u''(1) \leq 0\ \}. \end{aligned}$$

and also $= \partial\Omega_1 \quad C_2 = \partial\Omega_2$

$$\begin{aligned} \Omega_2 &= \{P = (u_1', u_1'') \in K^* : \ u''(t) < 0, \ 0 \leq t \leq 1, \ \ \forall u \in \mathcal{X}(P+K)\ \} \quad \text{ and} \\ C_2 &= \{P = (u_1', u_1'') \in cl\Omega_2 : \exists\, u \in \mathcal{X}(P+K) \ \text{ with } u'(\eta) \geq 0 \ \text{ and } \ u''(1) = 0\ \}, \end{aligned}$$

where we recall once again that $K = (u_0', u_0'') = \left(r_0, -r_0\frac{1+\eta^2}{\eta}\right)$.

Recalling the Remark 3, we state the next

Proposition 4. *The set Ω_i is open and $\partial\Omega_i \subseteq C_i \quad (i = 1, 2)$.*

Proof. Assume that the set Ω_1 is not open and consider any $P_0 \in \Omega_1 \cap \partial\Omega_1$. Then, noticing the definition of Ω_1, it follows that $u'(t) < 0, \quad \eta \leq t \leq 1$, for every $u \in \mathcal{X}(P_0 + K)$, thus

$$\mathcal{K}(P_0) = \{u'(\eta) : u \in \mathcal{X}(P_0)\} \subset (-\infty, 0) = V.$$

The upper semicontinuity of the map $\mathcal{K}$ yields the existence of an open ball $U(P_0)$ centering at P_0, such that for all $P \in U(P_0)$

$$\mathcal{K}(P) = \{u'(\eta) : u \in \mathcal{X}(P)\} \subset (-\infty, 0) = V.$$

But this clearly means that

$$u'(\eta) < 0, \quad \forall u \in \mathcal{X}(U(P)). \tag{3.9}$$

Hence P_0 is an interior point of Ω_1, that is Ω_1 is an open set, a contradiction. Similarly someone can prove that Ω_2 is also open.

On the other hand, if $P_0 \in \partial\Omega_1$ and $P_0 \notin C_1$, then any solution $u \in \mathcal{X}(P_0 + K)$ yields $u'(\eta) \neq 0$. To be definite, let $u'(\eta) < 0$. Then, as we demonstrated above, (3.9) remains true and hence $U(P) \subseteq \Omega_1$. Consequently $P_0 \notin \partial\Omega_1$, a contradiction. We may study the case $u'(\eta) > 0$, in the same manner indicated above. ∎

Remark 5. *By their definition, it is clear that $\bar{\Omega}_1 \subseteq \Omega_2$. Furthermore, Proposition 1 and the choice of the point K yields $0 \in \Omega_1$. Under the assumption $C_1 \cap C_2 \neq \varnothing$ and noticing Lemma 5 and Remark 4, we get $\varnothing \neq C_1 \subseteq \Omega_2$ and $C_2 \neq \varnothing$ and hence Proposition 4 yields $\bar{\Omega}_2 \backslash \Omega_1 \neq \varnothing$. We finally remark that the sets C_1 and C_2 may not be so simple as in Fig 1.*

Theorem 1. *Under assumptions (A_1) and (A_2), the boundary value problem (E) admits at least one positive and concave solution.*

Proof. We notice first that, if $C_1 \cap C_2 \neq \varnothing$ the BVP (3.1),(3.2) clearly accepts a solution. So assume $C_1 \cap C_2 = \varnothing$. Since Ω_1 and Ω_2 are open, by Lemma 5, it follows that $\bar{\Omega}_1 \subseteq \Omega_2$. Now for any point $P = (u_1', u_1'')$, we define the map

$$T : \tilde{K} \cap \left(\bar{\Omega}_2 \backslash \Omega_1\right) \to \tilde{K}, \quad T\left(P\right) = \left(-u'\left(\eta\right) + u_1', u''\left(1\right) + u_1''\right),$$

where the solution $u = u\left(t\right)$ has its initial value at the point $P + K$, i.e. $u \in \mathcal{X}\left(P + K\right)$. Also recall that

$$\tilde{K} = \left\{(u', u'') \in \mathbb{R}^2 : u' \geq 0 \text{ and } u'' \geq 0\right\}$$

denotes the usual cone in $\mathbb{R}^2$. The map T is well defined, that is $T\left(P\right) \in \tilde{K}$, since $P \in \tilde{K} \cap \left(\bar{\Omega}_2 \backslash \Omega_1\right)$ implies that $u''\left(1\right) \leq 0$ and hence $u'\left(\eta\right) \leq u'\left(0\right) = u_1'$, i.e.

$$-u'\left(\eta\right) + u_1' \geq 0. \tag{3.10}$$

Considering now a point $P \in \partial\Omega_1 \subseteq C_1$, we have

$$\left|\left|T\left(P\right)\right|\right| = \left| - u'\left(\eta\right) + u_1'\right| + \left|u''\left(1\right) + u_1''\right| \geq \left|u_1'\right| + \left|u_1''\right| = \left|\left|P\right|\right|,$$

due to the facts that $u'\left(\eta\right) = 0,\ \ u''\left(1\right) \leq 0$ and $u_1'' \leq 0$.

Similarly, if $P \in \partial\Omega_2 \subseteq C_2$, we obtain

$$\left|\left|T\left(P\right)\right|\right| = \left| - u'\left(\eta\right) + u_1\right| + \left|u''\left(1\right) + u_1''\right| \leq \left|u_1'\right| + \left|u_1''\right| = \left|\left|P\right|\right|,$$

due to the fact that $u'\left(\eta\right) \geq 0$ and (3.10).

Finally, by an application of the Lemma 1, we obtain a fixed point of T in $\tilde{K} \cap \left(\Omega_2 \backslash \Omega_1\right)$, that is a solution of the BVP (3.1),(3.2). But this solution, by Lemma 4, is a positive one and noticing the modification f of the nonlinearity, it follows that it is actually a solution of the original equation (E) ∎

Corollary 1. *Suppose that*

$$\lim_{x \to 0+} \max_{0 \leq t \leq 1} \frac{f\left(t, x, y, z\right)}{x} = 0 \quad \textit{and} \quad \lim_{x \to +\infty} \min_{0 \leq t \leq 1} \frac{f\left(t, x, y, z\right)}{x} = +\infty.$$

for all (y, z) in any compact subset of $\mathbb{R}^2$. Then the BVP (3.1),(3.2) has at least one positive solution.

Proof. Via the superlinearity at $x = 0$ assumption, for $\frac{1}{M} > 0$, there is $r_0 > 0$ such that for any $r \leq r_0$, it follows that $f\left(t, x, y, z\right)/x < 1/M$, for every $(t, x, y, z) \in \Delta_1$, where Δ_1 has been defined at the assumption (A_1) and R_0 therein will be defined below. Hence

$$f\left(t, x, y, z\right) < \frac{x}{M} \leq \frac{r}{M}, \qquad (t, x, y, z) \in \Delta_1.$$

Similarly by the superlinearity of f at infinity, for $1/\theta N > 0$ there exists $R_0 > r_0$, such that for every $R \geq R_0,\ \ f\left(t, x, y, z\right)/x > 1/\theta N,\ (t, x, y, z) \in \Delta_2$, that is

$$f\left(t, x, y, z\right) > \frac{x}{\theta N} \geq \frac{\theta R}{\theta N} \geq \frac{R}{N}, \qquad (t, x, y, z) \in \Delta_2.$$

Consequently assumptions $(A_1) - (A_2)$ are fulfilled and so Theorem 1 guarantee the result. ∎

4. Multiplicity Results

Theorem 2. *Suppose that assumptions (A_1) and (A_2) hold true. Then there exists a sequence $\{u_n\}$ of bounded and positive solutions to the BVP (E), such that*

$$\lim ||u_n|| = 0.$$

.

Proof. Since, by the nature of the vector field, for any $u \in \mathcal{X}(B)$ we have $u'(\eta) > 0$ and $u''(1) > 0$, in view of the continuity of solutions upon their initial values, we can find a sub-triangle

$$[K^*, A^*, B] \subseteq [K, A, B]$$

with the face $[K^*, A^*]$ parallel to $[K, A]$ such, that

$$u'(\eta) > 0 \quad \text{and} \quad u''(1) > 0, \quad u \in \mathcal{X}(P), \quad P \in [K^*, A^*, B]. \tag{4.1}$$

We set $K^* = (r_0, \hat{u}_0'')$ and consider a new simplex $[K_1, A_1, B_1]$ with

$$\begin{aligned} K_1 &= \left(r_1, -r_1 \frac{1+\eta^2}{\eta}\right), \quad B_1 = (0, r_1, 0) \quad \text{and} \\ A_1 &= \left(\frac{(1+\eta^2)\, r_1 + R_0}{\eta}, -r_1 \frac{1+\eta^2}{\eta}\right) \end{aligned}$$

(then $[K_1, A_1]$ is parallel to $[K, A]$) under the choice

$$r_1 \in (0, r_0) \quad \text{and} \quad -r_1 \frac{1+\eta^2}{\eta} > \hat{u}_0''. \tag{4.2}$$

Then in view of assumptions (A_1) and (A_2), we may apply once again the Krasnosel'skiĭ's theorem on the triangle $[K_1, A_1, B_1]$, to obtain another positive solution $u = u_2(t)$ of the BVP (3.1),(3.2). By the construction of $[K_1, A_1, B_1]$ and (4.1), it is obvious that $u = u_2(t)$ is different than the solution $u = u_1(t)$, $0 \le t \le 1$, obtained in the previous Theorem 1.

If we continue this procedure, choosing the sequence $\{r_n\}$ such that $\lim r_n = 0$, we may easily obtain a sequence $\{u_n\}$ of solutions to the BVP (E). Furthermore, by the boundary condition (3.2), we obtain

$$0 = u_n'(\eta) = u_n'(0) + \eta u_n''(0) + \eta^2 \int_0^1 (1-s)\, \alpha(s\eta)\, f[s\eta, u(s\eta), u'(s\eta), u''(s\eta)] ds$$

and so

$$0 = u_n'(\eta) \ge u_n'(0) + \eta u_n''(0)$$

that is

$$u_n'(0) \le -\eta u_n''(0), \ n = 1, 2, ...$$

By the above procedure (see (4.2)) and especially since $\lim r_n = 0$, we obtain

$$\lim u_n''(0) = 0$$

and given that $u_n'(t) \le u_n'(0)$, $0 \le t \le \eta$, we finally get

$$\lim u_n(\eta) = \lim[u_n'(0) + \int_0^\eta u_n'(t)\, dt] \le \lim (1+\eta)\, u_n'(0) = 0,$$

that is $\lim ||u_n|| = 0$. ∎

5. Discussion

If we assume that both functions $\alpha(t)$ and $f(t,x,y,z)$ are negative, we may easily demonstrate similar existence and multiplicity results. Indeed, considering the (x',x'') face semi-plane $(x' \le 0)$, we easily check that $x''' = \alpha(t) f(t,x,x',x'') < 0$. Thus, any trajectory $(x'(t),x''(t)),\ t \ge 0$, emanating from any point in the second quadrant

$$\{(x',x'') : x' < 0,\ x'' > 0\}$$

"evolutes" in a natural way, when $x'(t) < 0$, toward the positive $x''-$semi-axis and then, when $x'(t) \ge 0$ toward the positive $x'-$semi-axis. As a result, under a certain growth rate on f , we can control the vector field in a way that assures the existence of a trajectory satisfying the given boundary conditions. Let's notice that in present situation, the obtaining solution $(x'(t), x''(t))$ is convex, in contrast to the previous case, where it is concave (see Fig. 1).

Furthermore we could easily get analogous results, for the case when the nonlinearity is sublinear.

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