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ON THE DIFFERENTIAL AND DIFFERENCE EQUATION WITH A POWER DELAYED ARGUMENT

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Abstract. We study the qualitative behaviour of solutions of the differential equation

$$\dot{y}(t) = a\,y(t^\alpha) + b\,y(t), \qquad t \in [1, \infty),$$

where $0 < \alpha < 1$, $a \neq 0$, $b \leq 0$ are real scalars. Our aim is to present (and using the analysis of the numerical discretization also partly explain) some specific properties of solutions on the unbounded as well as the compact domain.

1. Introduction

This paper is concerned with the delay differential equation

$$\dot{y}(t) = a\,y(t^\alpha) + b\,y(t), \qquad t \in [1, \infty), \tag{1.1}$$

where $0 < \alpha < 1$, $a \neq 0$ and $b \leq 0$ are real scalars. This equation belongs to the wide and natural class of equations with unbounded lags (i.e. such that the difference between the present time and the delayed time becomes unbounded as $t \to \infty$).

The prototype of such equations is the equation

$$\dot{y}(t) = a\,y(\lambda t) + b\,y(t), \qquad t \in [0, \infty), \tag{1.2}$$

where $0 < \lambda < 1$, arising as a mathematical idealization of industrial problem involving wave motion in the overhead supply line to an electrified railway system (see [13]). The qualitative properties of the equation (1.2) as well as its modifications and numerical discretizations have been the subject of enormous interest of many authors (we mention at least papers [5-6] or [9-10]). As it may be expected, the behaviour of solutions of (1.2) as well as (1.1) substantially depends on the sign of the coefficient b. In this paper, we are going to discuss these equations especially with non-positive values of b. It is perhaps curious that the case $b = 0$ is in many ways the most difficult one to analyze.

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The qualitative investigation of the equation (1.1) goes back to [4], where this equation with $\alpha > 1$ (the advanced case) has been considered. The derived results indicated some related qualitative properties of the equations (1.1) and (1.2) with $\alpha > 1$ and $\lambda > 1$, respectively. More generally, they seemed to indicate a common general property for equations with unbounded (positive) deviations. Later on, this conjecture has been at least partly confirm not only for advanced equations (see [1]), but even also for delay equations (see [2] or [12]).

All the resemblances have been described for two-term equations involving besides the delayed term also the non-delayed term. Therefore we take into our considerations also one-term equations (the case $b = 0$) and demonstrate another related (and sometimes very surprising) properties of solutions of equations (1.1) and (1.2). The analysis of the simple numerical discretization turns out to be the key tool in these investigations. Moreover, we pose a hypothesis that such a behaviour is a general property for linear autonomous differential equations with unbounded lags.

2. Preliminary results

First we mention some results and calculations concerning the equation

$$\dot{y}(t) = a\,y(\lambda t), \qquad 0 < \lambda < 1, \qquad t \in [0, \infty). \tag{2.1}$$

The solution y_λ of (2.1) subject to the initial condition $y(0) = 1$ can be expanded into the Taylor series form

$$y_\lambda(t) = \sum_{k=0}^{\infty} \frac{a^k \lambda^{\binom{k}{2}}}{k!} t^k, \qquad t \geq 0\,.$$

Now consider the equation (2.1) with $a < 0$. Letting $\lambda \to 1^-$ it seems to be "natural" to expect that the solution y_λ approaches the function $\exp\{at\}$ solving the corresponding "limit" problem. Indeed, if $a = -1$, $\lambda = 0.99$ and $t \in (100, 300)$, then the values $y_{0.99}(t)$ are extremely small in modulus ($|y_{0.99}(t)| \approx 10^{-40}$) and we can believe that the solution tends to zero as $t \to \infty$. However, this intuitive conclusion is wrong.

To describe the correct asymptotic behaviour of the equation (2.1) we utilize the relevant theoretical result. First we introduce auxiliary functions

$$\varphi_\lambda(t) = t^k (\log t)^l \exp(-\frac{1}{2c}(\log t - \log\log t)^2), \tag{2.2}$$

where $c = \log\lambda$, $k = 1/2 - (1 + \log|ac|)/c$, $l = -1 + \frac{1}{c}\log|ac|$. Let us emphasize at least the following properties of φ_λ ($0 < \lambda < 1$) which are relevant to our next investigations:

- $\varphi_\lambda(t)$ is unbounded as $t \to \infty$ for any $0 < \lambda < 1$
- $\lim\limits_{t\to\infty} \frac{\varphi_r(t)}{\varphi_s(t)} = 0, \qquad 0 < r < s < 1\,.$

Now we can formulate the asymptotic result describing the asymptotics of (2.1) in terms of functions (2.2). In particular, this result implies that y_λ is not bounded as $t \to \infty$ for any $0 < \lambda < 1$.

Theorem 1 ([7, Theorem 7])**.** *Let y_λ be a solution of (2.1), where $a \neq 0$. Then*

$$y_\lambda(t) = O(\varphi_\lambda(t)) \qquad \text{as } t \to \infty\,.$$

Moreover, if $y_\lambda(t) = o(\varphi_\lambda(t))$ as $t \to \infty$, then y_λ is the zero solution.

We note that the above described phenomenon can be observed also for two-term differential equation (1.2) under the additional condition $a < b$ (we continue to assume $b \leq 0$). Expanding the solution y_λ of (1.2) into the Taylor series form and considering values of λ close to 1 we can again find that y_λ is almost zero for a long time interval. Nevertheless, the following result shows that the behaviour of y_λ at infinity is not approaching zero.

Theorem 2 ([7, Theorem 3])**.** *Let y_λ be a solution of (1.2), where $a \neq 0$, $b < 0$ and $0 < \lambda < 1$. Then there exists a continuous periodic function g of period $\log \lambda^{-1}$ such that*

$$y_\lambda(t) = t^\gamma g(\log t) + O(t^{\gamma_r - 1}) \qquad \text{as } t \to \infty\,,$$

where γ is a root of $a\lambda^\gamma + b = 0$ and $\gamma_r = \mathrm{Re}\ \gamma$.

Notice that if $a < b$ then $\mathrm{Re}\ \gamma > 0$, hence the solution y_λ of (1.2) cannot be bounded as $t \to \infty$ for any $0 < \lambda < 1$.

The discussion concerning the discrepancy between our naive expectations and the correct qualitative behaviour was the subject matter of the paper [11]. In particular, a simple numerical discretization of the equation (1.2) was analyzed with respect to its asymptotic stability properties. It follows from these considerations that there exists a critical time t^* such that the solution y_λ of (1.2) displays a tendency to decrease (in modulus) to zero before $t = t^*$, but it is blowing up after $t = t^*$. Note that this behaviour of solutions is usually reffered to as the "numerical nightmare".

The next section discusses the occurrence of this phenomenon provided the equation (1.1) with a power delayed argument is considered. We show that some qualitative properties of solutions of (1.1) are closely related to those described for pantograph equations (1.2) and (2.1) in the previous parts.

3. Main results

We start with the discussion of the two-term equation (1.1) and its simplest (convergent) discretization

$$z(n+1) = (1 + bh)z(n) + ahz(r_n), \qquad n = 0, 1, \ldots, \tag{3.1}$$

where r_n is the integer part of $[(1+nh)^\alpha - 1]/h$. As it is customary, $z(n)$ means the approximation of $y(t)$ at $t = t_n := 1+nh$ (with $t_0 = 1$ as the initial point) and $h > 0$ is the stepsize. Considering the condition $a < b \leq 0$ and the solution z_α of (3.1) with the parameter α satisfying $0 < 1 - \alpha << 1$, we can again observe very small absolute values of $z_\alpha(n)$ for h sufficiently small and many n large enough. However, the qualitative analysis of the corresponding exact equation (1.1) contradicts the "natural" hypothesis on the asymptotic stability of this equation.

To precise this, we first present the asymptotic result concerning the two-term equation (1.1), i.e. with $b < 0$. The formulation of the following statement is a consequence of Theorem 3.2 of [2].

Theorem 3. *Let y_α be a solution of (1.1), where $a \neq 0$, $b < 0$ and $0 < \alpha < 1$. Then there exists a continuous periodic function g of period $\log \alpha^{-1}$ such that*

$$y_\alpha(t) = (\log t)^\gamma g(\log\log t) + O((\log t)^{\gamma_r - 1}) \qquad \text{as } t \to \infty\,,$$

where γ is a root of $a\alpha^\gamma + b = 0$ and $\gamma_r = \mathrm{Re}\,\gamma$.

Obviously, if $a < b$ then the solution y_α of (1.1) cannot be bounded as $t \to \infty$ for any $0 < \alpha < 1$.

As we have already mentioned earlier, the asymptotic investigation of one-term delay equations can be more difficult task than the analysis of the corresponding two-term equations. However, in the particular case of the equation

$$\dot{y}(t) = a\,y(t^\alpha), \qquad 0 < \alpha < 1, \qquad t \in [1, \infty) \tag{3.2}$$

it is possible to transform the equation (3.2) into a very convenient form, namely (1.2) (of course, with parameters a, b having a different meaning). Indeed, let us introduce the change of variables

$$u = \log t, \qquad x(u) = t^\eta y(t)$$

converting (3.2) into the equation

$$x'(u) = \eta x(u) + a\exp\{u(1 + \eta - \alpha\eta)\}x(\alpha u)\,.$$

Setting $\eta = 1/(\alpha - 1) < 0$ we arrive at the equation

$$x'(u) = \frac{1}{\alpha - 1}x(u) + ax(\alpha u)\,. \tag{3.3}$$

By Theorem 2, the asymptotics of any solution x of (3.3) can be related to the function u^β, $\beta = \frac{\log(|a|(1-\alpha))}{\log \alpha^{-1}}$. To apply conclusions of Theorem 2 we introduce the functions

$$\psi_\alpha(t) = t^k(\log t)^l, \qquad k = \frac{1}{1 - \alpha}, \qquad l = \frac{\log|a(1-\alpha)|}{\log \alpha^{-1}}. \tag{3.4}$$

Similarly as in the case of the system of functions φ_λ introduced and discussed above it holds:

- $|\psi_\alpha(t)|$ is unbounded as $t \to \infty$ for any $0 < \alpha < 1$
- $\lim\limits_{t\to\infty} \frac{|\psi_r(t)|}{|\psi_s(t)|} = 0, \qquad 0 < r < s < 1\,.$

Summarizing the previous calculations we arrive at the following description of the asymptotics of (3.2) in terms of functions (3.4).

Theorem 4. *Consider the equation (3.2), where $a \neq 0$. Then for any solution y_α of (3.2) there exists a continuous periodic function g of period $\log \alpha^{-1}$ such that*

$$y_\alpha(t) = \psi_\alpha(t)g(\log\log t) + O\left(t^{\frac{1}{1-\alpha}}(\log t)^{\frac{\log(|a|(1-\alpha))}{\log \alpha^{-1}} - 1}\right) \quad \text{as } t \to \infty.$$

In particular, both previous results imply that the equation (1.1) is not stable provided $a < b$ regardless of $b = 0$ or $b < 0$. To give (at least partial) explanation of the discrepancy between this theoretical conclusion and the expectations based on our numerical calculations we analyze the discretization (3.1) with respect to its stability properties. Since the next ideas and calculations are modifications of those presented in [11] for the discretization of the pantograph equations (1.2) and (2.1), we only sketch them.

The formula (3.1) is a linear difference equation with constant coefficients and of non-constant order. If we are given with a fixed $m \in \mathbb{Z}^+$, then considering all $n \in \mathbb{N}$ such that

$$n - m \le \frac{(t_n)^\alpha - 1}{h} < n - m + 1 \tag{3.5}$$

we can rewrite (3.1) as the difference equation of the constant order $m + 1$

$$z(n+1) = (1 + bh)z(n) + ahz(n-m)\,. \tag{3.6}$$

Now applying the well known criterion on the asymptotic stability of linear autonomous difference equations of constant order (see, e.g., [3]) we get the following characterization of this property.

Corollary 1. *Any solution of the difference equation (3.6) tends to zero as $n \to \infty$ if and only if the corresponding characteristic polynomial*

$$P_m(\lambda) := \lambda^{m+1} - (1 + bh)\lambda^m - ah$$

has all its zeros inside the open unit disk.

The detailed discussion of the effectiveness of this criterion has been done in [8] and [11]. We mention here the following result which is of great importance in our effort to explain the curious and unexpected change of the behaviour of (1.1) and (3.2) for values α close to 1.

Proposition 1. *Consider the equation (3.6) with $a < b \le 0$, $0 < h < \frac{1}{|a|+|b|}$. Then there exists $m^* := m^*(h)$ such that the corresponding characteristic polynomial $P_m(\lambda)$ has all its zeros inside the open unit disk if and only if $m \le m^*$. Moreover, $P_m(\lambda)$ has at least one root which is outside the closed unit disk provided $m \ge m^* + 2$ and it holds*

$$\rho(a,b) := \lim_{h\to 0} m^* h = \begin{cases} \frac{\pi}{2|a|}, & b = 0, \\ \frac{1}{\sqrt{a^2-b^2}}\left(\pi + \arctan\frac{\sqrt{a^2-b^2}}{b}\right), & b < 0\,. \end{cases} \tag{3.7}$$

In other words, there exists the critical index $n^* := n^*(h)$ such that any solution z of (3.1) displays a tendency to have a zero limit (as $n \to \infty$) before $n = n^*$, but it is blowing up after $n = n^*$. Then the natural question arises, namely what does happen with the value of n^* when $h \to 0$? To answer this we rewrite the inequality (3.5) as

$$t_n - hm \le (t_n)^\alpha < t_n - hm + h\,.$$

Then letting $h \to 0$ and substituting the corresponding $m = m^*(h)$ we can conclude that the value t^* representing the qualitative change of the behaviour of the exact solution of (1.1) can be estimated as the (unique) root of the nonlinear equation

$$t^* - (t^*)^\alpha = \rho(a, b), \qquad t^* \geq 1\,, \tag{3.8}$$

where ρ is given by (3.7).

To illustrate this result, we consider the equation (1.1) with the parameters $a = -1$, $b = -0.2$ and $\alpha = 0.99$. Then (3.8) becomes

$$t^* - (t^*)^{0.99} = 1.81$$

and admits the root $t^* \approx 47.8$. Calculating the corresponding numerical solution of (3.1) with h sufficiently small we can confirm that the exact solution has actually a tendency to rapidly decrease (in modulus) before $t = t^*$ and increase after $t = t^*$. Note also that, by (3.8), if $\alpha \to 1^-$ then $t^* \to \infty$. Furthermore, Theorem 3 and Theorem 4 (with the above stated properties of ψ_α) imply that the more is parameter α close to 1, the more is the corresponding solution y_α of (1.1) "exploding" for $t > t^*$.

The previous results and ideas discussed the problem of the qualitative change of the behaviour of the differential equation (1.1). We considered its convergent numerical discretization (3.1) and analyzed its behaviour on the compact domain. Then letting $h \to 0$ and using Theorem 3 and Theorem 4 we arrived at the description of the correct behaviour of solutions of (1.1).

To complete this discussion, we mention our last problem, namely whether the numerical solution based on the formula (3.1) can correctly mimic the behaviour of the exact solution of (1.1) also for $n \to \infty$. In particular, we show that under a restriction on the stepsize h the asymptotic bound of all solutions of (3.1) coincides with the asymptotics of the exact equation (1.1) described by Theorem 3. For the sake of simplicity we consider the case $|a| + b \geq 0$ which is consistent with the related assumption involved in our previous considerations. Nevertheless, let us emphasize that this assumption can be omitted in the formulation of the following assertion (the corresponding proof is rather longer and it requires some additional operations).

Theorem 5. *Consider the equation (3.1), where $b < 0$, $|a| + b \geq 0$, $h > 0$, $1 + bh > 0$, $0 < \alpha < 1$ and let $t_n := 1 + nh$. Then*

$$z(n) = O((\log t_n)^{\gamma_r}), \qquad \gamma_r = \frac{\log \frac{|a|}{(-b)}}{\log \alpha^{-1}} \tag{3.9}$$

for any solution z of (3.1).

Proof. Let z be a solution of (3.1). We introduce the change of the dependent variable $w(n) = z(n)/\rho(n)$, where

$$\rho(n) = (\log t_n)^{\gamma_r}\,. \tag{3.10}$$

Then w defines a solution of

$$\rho(n+1)w(n+1) = (1 + bh)\rho(n)w(n) + ah\rho(r_n)w(r_n)$$

which is equivalent to

$$\Delta\left(\frac{\rho(n)w(n)}{(1+bh)^n}\right)=\frac{ah}{(1+bh)^{n+1}}\rho(r_n)w(r_n)\,. \tag{3.11}$$

Now let $n_0>1$ be such that $(1+n_0h+h)^\alpha\le 1+n_0h$ and put $I_0:=[1,n_0]$, $n_{p+1}:=\sup\{m\in\mathbb{N}:r_m\le n_p\}$ and $I_{p+1}:=[n_p,n_{p+1}]$, $p=0,1,\dots$. Considering any $\bar n\in I_{p+1}$, $\bar n>n_p$ and summating (3.11) from n_p to $\bar n-1$ we get

$$w(\bar n)=\frac{\rho(n_p)(1+bh)^{\bar n}}{\rho(\bar n)(1+bh)^{n_p}}w(n_p)+\frac{(1+bh)^{\bar n}}{\rho(\bar n)}\sum_{s=n_p}^{\bar n-1}\frac{ah}{(1+bh)^{s+1}}\rho(r_s)w(r_s)\,.$$

To estimate $w(\bar n)$ we introduce the notation $B_p:=\sup\{|w(n)|,\ n\in\cup_{j=0}^{p}I_j\}$. Then

$$|w(\bar n)|\le\frac{\rho(n_p)(1+bh)^{\bar n}}{\rho(\bar n)(1+bh)^{n_p}}B_p+\frac{(1+bh)^{\bar n}}{\rho(\bar n)}\sum_{s=n_p}^{\bar n-1}\frac{-bh\rho(s)}{(1+bh)^{s+1}}B_p$$

because of the inequality

$$|a|\rho(r_s)\le -b\rho(s)$$

following from (3.10). Further, we can write $(-bh)/(1+bh)^{s+1}=\Delta(1/(1+bh))^s$, hence

$$\begin{aligned}|w(\bar n)|\le&B_p\left(\frac{\rho(n_p)(1+bh)^{\bar n}}{\rho(\bar n)(1+bh)^{n_p}}+\frac{(1+bh)^{\bar n}}{\rho(\bar n)}\sum_{s=n_p}^{\bar n-1}\rho(s)\Delta(1/(1+bh))^s\right)\\ =&B_p\left(\frac{\rho(n_p)(1+bh)^{\bar n}}{\rho(\bar n)(1+bh)^{n_p}}+\frac{(1+bh)^{\bar n}}{\rho(\bar n)}\right.\\ &\left.\times\left[\frac{\rho(\bar n)}{(1+bh)^{\bar n}}-\frac{\rho(n_p)}{(1+bh)^{n_p}}-\sum_{s=n_p}^{\bar n-1}(1/(1+bh))^{s+1}\Delta\rho(s)\right]\right)\\ \le&B_p\left(1-\frac{(1+bh)^{\bar n}}{\rho(\bar n)}\sum_{s=n_p}^{\bar n-1}(1/(1+bh))^{s+1}\Delta\rho(s)\right)\end{aligned}$$

by use of the summation by parts.

Since $|a|+b\ge 0$, the difference $\Delta\rho$ is non-negative, hence $|w(\bar n)|\le B_p$. This implies $B_{p+1}\le B_p$ for $p=0,1,\dots$ and the sequence (B_p) is bounded as $p\to\infty$. The estimate (3.9) is proved. □

4. Concluding remarks

In previous sections we described some common specific properties of linear autonomous differential equations with a proportional and power delayed argument. We conjecture that these properties are typical for a wide class of differential equations with a (general) unbounded lag. Indeed, there are some results generalizing certain qualitative properties of differential equations with the above mentioned particular delays. E.g., Theorem 3.2 of [2] extends the validity of Theorem 2 and Theorem 3 to a general class of linear differential equations with a delayed argument

τ satisfying $0 < \dot{\tau} \le \kappa < 1$. Similarly, it is not difficult to see that the discussion concerning the estimate of the critical time t^* (characterizing the change of the qualitative behaviour of solutions) can be extended to the case involving a general delayed argument τ and leads to the nonlinear equation

$$t^* - \tau(t^*) = \rho(a, b)\,.$$

On the other hand, the investigations of the asymptotic behaviour of one-term linear delay equations involving a general unbounded lag as well the investigations of other numerical discretizations remain still open.

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