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UNIQUENESS FOR RETARDED DELAY DIFFERENTIAL EQUATIONS WITHOUT LIPSCHITZ CONDITION

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Abstract

Consider the equation $\dot{x}(t) = f(t, x(t), x(t - r(t)))$ with the initial condition $x_0 = \phi$. Here f is a continuous real function, but it does not satisfy other regularity conditions. We prove that the initial value problem has a unique solution under the following monotonicity conditions:

$(x - y)f(t, x, y) \leq 0$ for all t, x, $y \in \mathbb{R}$,

$f(t, x_1, y) \geq f(t, x_2, y)$ for all t, $y \in \mathbb{R}$, and $x_1 < x_2$, and

if there is $t_0 \geq 0$ such that $r(t_0) = 0$, then the function $t_0 - t + r(t)$ does not change sign on an interval $[t_0, t_0 + \delta)$.

We show an example that the result cannot be applied in the state dependent case.

1. Introduction

It is well-known that the stability of the solution of the delay differential equation

$$\dot{x}(t) = G(t, x_t)$$

through a continuous function ϕ implies the uniqueness of this solution.

Consider the retarded differential equation

$$\dot{x}(t) = -g(x(t)) + g(x(t - r(t))), \tag{0}$$

where g and r are continuous real functions, $r(t) \geq 0$, and g is monotone increasing. According to a result of Razumikhin [4] the constant solution of Eq. (0) is stable, therefore uniqueness holds for this solution. For some interesting uniqueness results we refere the interested reader to [2], [3].

Our aim is to show uniqueness for every solution of Eq. (0) provided that $r(t)$ satisfies the following condition: if there is $t_0 \geq 0$ so that $r(t_0) = 0$, then there exists $\delta = \delta(t_0) > 0$ such that the function $t \mapsto t_0 - t + r(t)$ does not change sign on the interval $[t_0, t_0 + \delta)$. Note that the above assumption is

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common for several equations which arise in applications and it is satisfied, for example, when $r(t) > 0$ or the function $t - r(t)$ is monotone increasing.

We prove our uniqueness result for an equation more general than Eq.(0), that is

$$\dot{x}(t) = f(t, x(t), x(t - r(t)))$$

under certain monotonicity assumptions on f. Here f is continuous, but it does not satisfy other regularity conditions.

Our result cannot be applied when r depends on $x(t)$. We show that by an example.

2. Uniqueness result

Consider the initial value problem (IVP)

$$\dot{x}(t) = f(t, x(t), x(t - r(t))), \quad x_0 = \phi, \tag{1}$$

where $f : \mathbb{R}^3 \to \mathbb{R}$, $r : \mathbb{R} \to [0, \infty)$ and $\phi : (-\infty, 0] \to \mathbb{R}$ are continuous. The solution segment $x_t : (-\infty, 0] \to \mathbb{R}$ is given by $x_t(s) = x(t + s)$, $s \leq 0$.

Theorem 2.1. *Assume that*
(i) $(x - y)f(t, x, y) \leq 0$ *for all* x, $y \in \mathbb{R}$, $t \geq 0$,
(ii) $f(t, x_1, y) \geq f(t, x_2, y)$ *for all* $y \in \mathbb{R}$, x_1, $x_2 \in \mathbb{R}$, $x_1 < x_2$, *and* $t \geq 0$,
(iii) if there is $t_0 \geq 0$ *so that* $r(t_0) = 0$, *then there exists* $\delta = \delta(t_0) > 0$ *such that* $t_0 - t + r(t) \leq 0$ *or* $t_0 - t + r(t) \geq 0$ *for all* $t \in [t_0, t_0 + \delta)$.
Then IVP (1) has a unique solution.

Proof. Suppose by way of contradiction that there are two solutions $x_1(t)$ and $x_2(t)$ of IVP (1) on an interval $[0, A)$, $A \in \mathbb{R}$ such that $x_1(t) = x_2(t) = \phi(t)$ for all $t \leq 0$, and there is $\bar{t} > 0$ such that $x_1(\bar{t}) \neq x_2(\bar{t})$.

Set $H = \{s \in (0, A) : x_1(s) \neq x_2(s)\}$ and $t_0 = \inf H$. Since $t_0 \notin H$, it follows $x_1(t) = x_2(t)$ for all $t \leq t_0$.

Let $r(t_0) > 0$ or $r(t_0) = 0$ with $t_0 - t + r(t) \geq 0$ for all $t \in [t_0, t_0 + \delta)$, where δ is defined in assumption (iii). In both cases $t - r(t) \leq t_0$ for all $t \in [t_0, t_0 + \delta_1)$ with some $\delta_1 \in (0, \delta]$.

The definition of t_0 implies that there is a sequence (t_n) in H so that $t_n > t_0$, $t_n \to t_0$ and $x_1(t_n) \neq x_2(t_n)$ for all $n \in \mathbb{N}$. We can assume without loss of generality that $x_1(t_n) < x_2(t_n)$ for all $n \in \mathbb{N}$.

Define the functions

$$z(t) = x_2(t) - x_1(t) \quad \text{and} \quad u(t) = \max_{t_0 \leq s \leq t} z(s) \quad \text{for all} \quad t \in [t_0, A).$$

Clearly, we have $u(t_0) = 0$, $u(t)$ is monotone increasing on $[t_0, A)$ and $u(t) > 0$ for all (t_0, A). Further, define the function

$$D^+u(t) = \limsup_{h \to 0+} \frac{u(t + h) - u(t)}{h} \quad \text{for all} \quad t \in (t_0, A).$$

According to Theorem 2.3 (Appendix) [5] there is $\tau \in (t_0, t_0 + \delta_1)$ such that $D^+u(\tau) > 0$. The definition of $u(t)$ gives $z(\tau) \le u(\tau)$. We claim that $z(\tau) = u(\tau)$. Obviously, if $z(\tau) < u(\tau)$, then $u(t)$ is constant in a neighbourhood of τ, therefore $D^+u(\tau) = 0$, and this is a contradiction. Consequently, $z(\tau) = u(\tau)$.

Next we will show that $\dot{z}(\tau) > 0$ and $\dot{z}(\tau) \le 0$ at the same time, and this will prove the result in the studied case.

Since $D^+u(\tau) > 0$, there is a constant $K > 0$ such that $K < D^+u(\tau)$, and there exists a sequence (h_n), $h_n > 0$, $h_n \to 0$ so that

$$0 < K < \frac{u(\tau + h_n) - u(\tau)}{h_n}, \; n \in I\!N.$$

It is easy to see, that there is a sequence $(\overline{h}_n)$, $0 < \overline{h}_n \le h_n$ such that $u(\tau + h_n) = z(\tau + \overline{h}_n)$. Indeed, the definition of $u(t)$ yields $u(\tau + h_n) = \max(\max_{t_0 \le s \le \tau} z(s), \max_{\tau \le s \le \tau + h_n} z(s)) = \max(u(\tau), \max_{\tau \le s \le \tau + h_n} z(s))$. As $u(\tau) = z(\tau)$, we infer $u(\tau + h_n) = \max_{\tau \le s \le \tau + h_n} z(s)$. Th continuity of $z(s)$ on $[\tau, \tau + h_n]$ gives $\max_{\tau \le s \le \tau + h_n} z(s) = z(\tau + \overline{h}_n)$, where $0 < \overline{h}_n \le h_n$. Thus, $u(\tau + h_n) = z(\tau + \overline{h}_n)$. These facts lead to the following estimations:

$$0 < K < \frac{u(\tau + h_n) - u(\tau)}{h_n} \le \frac{z(\tau + \overline{h}_n) - z(\tau)}{\overline{h}_n}.$$

Letting $\overline{h}_n \to 0$, we conclude $\dot{z}(\tau) > 0$. Now, we show that $\dot{z}(\tau) \le 0$. Clearly, $\dot{z}(\tau) = \dot{x}_2(\tau) - \dot{x}_1(\tau)$. Being $x_1(t)$ and $x_2(t)$ solutions of IVP (1), we obtain $\dot{z}(\tau) = f(\tau, x_2(\tau), x_2(\tau - r(\tau))) - f(\tau, x_1(\tau), x_1(\tau - r(\tau)))$. Since $\tau - r(\tau) \le t_0$ and $x_1(t) = x_2(t)$ for $t \le t_0$, we infer $x_1(\tau - r(\tau)) = x_2(\tau - r(\tau))$. As $u(\tau) = z(\tau) = x_2(\tau) - x_1(\tau)$ and $u(\tau) > 0$, we find $x_1(\tau) < x_2(\tau)$. Finally, assumption (ii) implies $f(\tau, x_1(\tau), x_1(\tau - r(\tau))) \ge f(\tau, x_2(\tau), x_2(\tau - r(\tau)))$, that is $\dot{z}(\tau) \le 0$.

It remains to consider case $r(t_0) = 0$ and $t_0 - t + r(t) \le 0$ for $[t_0, t_0 + \delta)$.

The definition of t_0 implies the existence of a sequence (t_n) in H so that $t_n > t_0$, $t_n \to t_0$ and $x_1(t_n) \ne x_2(t_n)$ for all $n \in I\!N$. We have $x_1(t_n) \ne x_1(t_0)$ or $x_2(t_n) \ne x_2(t_0)$ for all $n \in I\!N$. We can assume without loss of generality that $x_2(t_n) \ne x_2(t_0)$ for all $n \in I\!N$. Define the functions

$$u(t) = \max_{t_0 \le s \le t} |x_2(s) - x_2(t_0)| \quad \text{and}$$

$$D^+u(t) = \limsup_{h \to 0+} \frac{u(t+h) - u(t)}{h} \quad \text{for all} \quad t \in (t_0, A).$$

Obviously, $u(t_0) = 0$, $u(t)$ is monotone increasing on $[t_0, A)$ and $u(t) > 0$ for all (t_0, A). According to Theorem 2.3 (Appendix)[5] there is $\tau \in (t_0, t_0 + \delta)$ such that $D^+u(\tau) > 0$. Arguing similarly as in the previous case, we obtain $u(\tau) = |x_2(\tau) - x_2(t_0)|$. $u(\tau) > 0$ yields $x_2(\tau) - x_2(t_0) \ne 0$.

Suppose $x_2(\tau) - x_2(t_0) > 0$. We can choose $\delta > 0$ in assumption (iii) so that $x_2(s) - x_2(t_0) > 0$ for all $s \in (\tau - \delta, \tau + \delta)$. We will show that $\dot{x}_2(\tau) > 0$ and $\dot{x}_2(\tau) \leq 0$ at the same time, and this contradiction will prove the result when $x_2(\tau) - x_2(t_0) > 0$. Since $D^+u(\tau) > 0$, it follows that there is a constant $K > 0$ such that $K < D^+u(\tau)$, and there is a sequence (h_n), $h_n > 0$, $h_n \to 0$ so that

$$0 < K < \frac{u(\tau + h_n) - u(\tau)}{h_n}, \; n \in I\!N.$$

It is easy to see, using the definition of $u(t)$ and the continuity of $x_2(s) - x_2(t_0)$ on $[\tau, \tau + h_n]$, that there is a sequence $(\overline{h}_n)$, $0 < \overline{h}_n \leq h_n$ such that $u(\tau + h_n) = x_2(\tau + \overline{h}_n) - x_2(t_0)$. These facts lead to the following estimations:

$$0 < K < \frac{u(\tau + h_n) - u(\tau)}{h_n} \leq \frac{x_2(\tau + \overline{h}_n) - x_2(\tau)}{\overline{h}_n}.$$

Letting $\overline{h}_n \to 0$, we conclude $\dot{x}_2(\tau) > 0$. Now, we prove $\dot{x}_2(\tau) \leq 0$. Since $t_0 \leq \tau - r(\tau) \leq \tau$, the monotone increasing property of u implies $u(\tau - r(\tau)) \leq u(\tau)$. Therefore $|x_2(\tau - r(\tau)) - x_2(t_0)| \leq x_2(\tau) - x_2(t_0)$. Hence $x_2(\tau - r(\tau)) \leq x_2(\tau)$. By assumption (i) we get $f(\tau, x_2(\tau), x_2(\tau - r(\tau))) \leq 0$, that is $\dot{x}_2(\tau) \leq 0$.

If $x_2(\tau) - x_2(t_0) < 0$, arguing similarly as above, we show that $\dot{x}_2(\tau) < 0$ and $\dot{x}_2(\tau) \geq 0$ at the same time using assumption (i). The proof of Theorem 2.1 is complete.

Remark. In case $r(t_0) = 0$ and $t_0 - t + r(t) \leq 0$ for all $t \geq t_0$, the unique solution of IVP (1) is the constant solution $x(t) = x(t_0)$ for all $t \geq t_0$.

Note that modifying slightly assumption (iii) of Theorem 2.1 and assuming condition (ii) of Theorem 2.1, we obtain the following result.

Theorem 2.2. *Suppose that*
a) $f(t, x_1, y) \geq f(t, x_2, y)$ *for all* $y \in I\!R$, x_1, $x_2 \in I\!R$, $x_1 < x_2$, *and* $t \geq 0$,
b) for all $t_0 \geq 0$ *there is* $\delta = \delta(t_0) > 0$ *such that* $t_0 - t + r(t) \geq 0$ *for all* $t \in [t_0, t_0 + \delta)$,
then IVP (1) has a unique solution.

We mention that Theorem 2.2 is a generalization of Ding's result [1] for scalar equations.

3. Example

Consider the functions

$$f(x, y) = -\sqrt[3]{x} + \sqrt[3]{y}, \quad (x, y) \in I\!R^2, \text{ and } \quad r(u) = N|u|^{\frac{1}{\alpha}} + r_0, \quad u \in I\!R,$$

where r_0, N and α are positive constants. Clearly, assumptions (i), (ii) and (iii) of Theorem 2.1 are satisfied.

Consider the IVP

$$\dot{x}(t) = -\sqrt[3]{x(t)} + \sqrt[3]{x(t-r(x(t)))}, \quad x_0 = \phi, \tag{2}$$

where

$$\phi(t) = \begin{cases} |t+r_0|^{\alpha}, & t \le -r_0 \\ 0, & -r_0 < t \le 0. \end{cases}$$

Our aim is to find two solutions of IVP (2) of form $x(t) = M\,t^{\alpha}$, namely we propose to choose two different sets of positive constants α, N, M and r_0 such that $x(t) = M\,t^{\alpha}$ is a solution, and hence IVP (2) is not uniquely solved.

The definition of r and the form of x imply

$$t - r(x(t)) = (1 - N\,M^{\frac{1}{\alpha}})t - r_0 \quad \text{for all} \quad t > 0.$$

We may assume that $1 - N\,M^{\frac{1}{\alpha}} < 0$. Then $(1 - N\,M^{\frac{1}{\alpha}})t - r_0 < -r_0 < 0$.

$$x(t - r(x(t))) = \phi(t - r(x(t))) = |t - r(x(t)) + r_0|^{\alpha}.$$
$$x(t - r(x(t))) = (N\,M^{\frac{1}{\alpha}} - 1)^{\alpha}\,t^{\alpha}.$$

Being $x(t) = M\,t^{\alpha}$ a solution of IVP (2), it follows

$$\alpha\,M\,t^{\alpha-1} = -M^{\frac{1}{3}}\,t^{\frac{\alpha}{3}} + (N\,M^{\frac{1}{\alpha}} - 1)^{\frac{\alpha}{3}}\,t^{\frac{\alpha}{3}} \text{ for all } t > 0. \tag{3}$$

Obviously, $\alpha - 1 = \frac{\alpha}{3}$, that is $\alpha = \frac{3}{2}$. We deduce from (3) that

$$3M + 2M^{\frac{1}{3}} = 2(N\,M^{\frac{2}{3}} - 1)^{\frac{1}{2}}.$$

Set $N = A^2$ and $M^{\frac{-2}{3}} = X$, where $A > 0$ and $X > 0$. Therefore

$$\frac{3}{X} + 2 = 2(A^2 - X)^{\frac{1}{2}}. \tag{4}$$

Comparing the graphs of the functions $(0, A^2] \ni X \mapsto \frac{3}{X} + 2$ and $(0, A^2] \ni X \mapsto 2(A^2 - X)^{\frac{1}{2}}$, it follows that, if $5 < 2(A^2 - 1)^{\frac{1}{2}}$, that is $A > \frac{\sqrt{29}}{2}$, then there are two solutions $X_1, X_2 \in (0, A^2)$ of (4). Then $1 - N\,M^{\frac{1}{\alpha}} < 0$, because $1 - N\,M^{\frac{1}{\alpha}} = 1 - A^2\,M^{\frac{2}{3}} = 1 - A^2\,X^{-1} < 0$. Let $A = 4 > \frac{\sqrt{29}}{2}$. From the definition of A and X, we obtain $N = 16$, $M_1 = X_1^{\frac{-3}{2}}$, $M_2 = X_2^{\frac{-3}{2}}$. Consider IVP (2), where

$$r(u) = 16|u|^{\frac{2}{3}} + r_0, \quad r_0 > 0,$$

and

$$\phi(t) = \begin{cases} |t+r_0|^{\frac{3}{2}}, & t \le -r_0 \\ 0, & -r_0 < t \le 0. \end{cases}$$

As we have shown, there are two positive constants $M_1 \neq M_2$ such that $x_1(t) = M_1\, t^{\frac{3}{2}}$ and $x_2(t) = M_2\, t^{\frac{3}{2}}$ are two different solutions of IVP (2).

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