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Oscillation Criteria for Second Order Nonlinear Retarded Differential Equations

Dáša Lacková

Abstract

The aim of this paper is to deduce oscillatory and asymptotic behavior of the solutions of the second order nonlinear retarded differential equation

$$\Big[r(t)\big|\big[x(t)-p(t)x\left[\tau(t)\right]\big]'\big|^{\alpha-1}\big[x(t)-p(t)x\left[\tau(t)\right]\big]'\Big]'+$$

$$+\,q(t)\big|x\left[\sigma(t)\right]\big|^{\alpha-1}x\left[\sigma(t)\right]=0,$$

where α is a positive constant and $\tau(t)$ and $\sigma(t)$ are delayed arguments.

1 Introduction

In this paper we are concerned with the problem of oscillatory properties of the retarded differential equation of the form

$$\Big[r(t)\big|\big[x(t)-p(t)x\left[\tau(t)\right]\big]'\big|^{\alpha-1}\big[x(t)-p(t)x\left[\tau(t)\right]\big]'\Big]'+$$

$$+\,q(t)\big|x\left[\sigma(t)\right]\big|^{\alpha-1}x\left[\sigma(t)\right]=0. \tag{E^-}$$

For convenience and further references, we introduce the notation

$$R(t)=\int_{t_0}^{t}\frac{1}{r^{1/\alpha}(s)}\,ds,\quad t\geq t_0.$$

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We suppose throughout the paper that the following hypotheses hold:

$(H1)$ α is a positive constant;

$(H2)$ $\tau(t), \sigma(t) \in C^1[t_0, \infty)$, $\tau(t) \le t$, $\sigma(t) \le t$, $\lim_{t\to\infty} \tau(t) = \infty$, $\lim_{t\to\infty} \sigma(t) = \infty$, $\sigma'(t) > 0$;

$(H3)$ $r(t) \in C^1[t_0, \infty)$, $r(t) > 0$, $\lim_{t\to\infty} R(t) = \infty$;

$(H4)$ $q(t), p(t) \in C[t_0, \infty)$, $q(t) > 0$, $0 \le p(t) \le p < 1$.

We put $z(t) = x(t) - p(t)x[\tau(t)]$. By a solution of Eq. (E^-) we mean a function $x(t) \in C^1[T_x, \infty)$, $T_x \ge t_0$, which has the property $r(t)\left|z'(t)\right|^{\alpha-1} z'(t) \in C^1[T_x, \infty)$ and satisfies Eq. (E^-) on $[T_x, \infty)$. We consider only those solutions $x(t)$ of Eq. (E^-) which satisfy $\sup\{|x(t)| : t \ge T\} > 0$ for all $T \ge T_x$. We assume that (E^-) possesses such a solution.

A solution of (E^-) is called oscillatory if it has arbitrarily large zeros on $[T_x, \infty)$ and otherwise it is said to be nonoscillatory. Eq. (E^-) is said to be oscillatory if every its solution is oscillatory.

This paper is motivated by the papers [4, 7] where the oscillation of differential equations of the form

$$\left[r(t)\left|x'(t)\right|^{\alpha-1} x'(t)\right]' + q(t)\left|x\left[\sigma(t)\right]\right|^{\alpha-1} x\left[\sigma(t)\right] = 0 \qquad (E_1)$$

is studied and by the papers [1, 8] where the oscillation criteria for differential equations of the form

$$\left[r(t)\left|\left[x(t) + p(t)x\left[\tau(t)\right]\right]'\right|^{\alpha-1}\left[x(t) + p(t)x\left[\tau(t)\right]\right]'\right]' + $$
$$+ q(t)\left|x\left[\sigma(t)\right]\right|^{\alpha-1} x\left[\sigma(t)\right] = 0, \qquad (E_2)$$

respectively

$$\left[r(t)\left|\left[x(t) + p(t)x\left(t-\tau\right)\right]'\right|^{\alpha-1}\left[x(t) + p(t)x\left(t-\tau\right)\right]'\right]' + $$
$$+ q(t)f\left(x\left[\sigma(t)\right]\right) = 0 \qquad (E_3)$$

with $\dfrac{f(u)}{|u|^{\alpha-1}u} \ge \beta > 0$ for $u \ne 0$, β is a constant, were presented.

2 Main results

We need the following lemma.

Lemma 2.1 (See [5]) *If A and B are nonnegative constants, then*

$$A^{\lambda} - \lambda A B^{\lambda-1} + (\lambda - 1) B^{\lambda} \geq 0, \quad \lambda > 1$$

and the equality holds if and only if $A = B$.

Proof. The case $A = 0$ holds evidently, so we can assume that $A \neq 0$. Then the left side of the inequality can be written in the form

$$1 - \lambda C^{\lambda-1} + (\lambda - 1) C^{\lambda}, \tag{1}$$

where $C = \frac{B}{A}$. Denote (1) by $f(C)$. Clearly (1) is satisfied for $C = 0$. On the other hand, if $C \neq 0$ then function $f(C)$ is decreasing for $C \in (0, 1)$ and increasing for $C \in (1, \infty)$. Furthermore $f(1) = 0$. Hence the inequality holds too. The proof is complete. □

The following theorem presents the oscillatory criterion for Eq. (E^{-}).

Theorem 2.1 *Let*

$$\int^{\infty} \left[R^{\alpha}[\sigma(t)]\, q(t) - \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} \frac{\sigma'(t)}{R[\sigma(t)]\, r^{\frac{1}{\alpha}}[\sigma(t)]} \right] dt = \infty, \tag{2}$$

$$\int^{\infty} \frac{1}{r^{\frac{1}{\alpha}}(u)} \left[\int_{u}^{\infty} q(s)\, ds \right]^{\frac{1}{\alpha}} du = \infty. \tag{3}$$

Then every nonoscillatory solution of Eq. (E^{-}) tends to zero as $t \to \infty$.

Proof. Assume to the contrary that $x(t)$ is a nonoscillatory solution of Eq. (E^{-}). We may assume that $x(t) > 0$. The case of $x(t) < 0$ can be proved by the same arguments.
Set

$$z(t) = x(t) - p(t) x[\tau(t)]. \tag{4}$$

Then $z(t) < x(t)$ and Eq. (E^{-}) can be written in the following form

$$\left[r(t) \left| z'(t) \right|^{\alpha-1} z'(t) \right]' + q(t)\, x^{\alpha}[\sigma(t)] = 0. \tag{5}$$

We claim that $x(t)$ is bounded. To prove it we assume, on the contrary, that $x(t)$ is unbounded. Hence there exists a sequence $\{t_m\}$ such that $\lim\limits_{m\to\infty} t_m = \infty$ moreover $\lim\limits_{m\to\infty} x(t_m) = \infty$ and $x(t_m) = \max\{x(s); t_0 \le s \le t_m\}$. Since $\tau(t) \to \infty$ as $t \to \infty$, we can choose sufficiently large m such that $\tau(t_m) > t_0$. As $\tau(t) \le t$, we have

$$\begin{aligned} x(\tau(t_m)) &\le \max\{x(s);\, t_0 \le s \le \tau(t_m)\} \\ &\le \max\{x(s);\, t_0 \le s \le t_m\} = x(t_m). \end{aligned}$$

Therefore for all large m

$$z(t_m) = x(t_m) - p(t_m)x[\tau(t_m)] \ge (1 - p(t_m))x(t_m).$$

Thus $z(t_m) \to \infty$ as $m \to \infty$.

Eq. (5) implies, that function $r(t)\,|z'(t)|^{\alpha-1}\,z'(t)$ is nonincreasing and we get two possibilities for $z'(t)$:

(i) $z'(t) > 0$,

(ii) $z'(t) < 0$ for $t \ge t_1 \ge t_0$.

The condition (ii) implies that for some positive constant M and $\forall\, t \ge t_1 \ge t_0$

$$r(t)\,|z'(t)|^{\alpha-1}\,z'(t) \le -M < 0.$$

Thus

$$-z'(t) \ge \left(\frac{M}{r(t)}\right)^{\frac{1}{\alpha}}.$$

Integrating the above inequality from t_1 to t, we obtain

$$z(t) \le z(t_1) - M^{\frac{1}{\alpha}}\big(R(t) - R(t_1)\big).$$

Letting $t \to \infty$ in the above inequality and using $(H3)$, we get $z(t) \to -\infty$. This contradiction proves that (i) holds.

For the case (i) we obtain that $z(t) > 0$ and $r(t)\,|z'(t)|^{\alpha-1}\,z'(t) = r(t)\,[z'(t)]^{\alpha}$. Combining these facts together with $z^{\alpha}(t) < x^{\alpha}(t)$, we are led to

$$\left[r(t)\,[z'(t)]^{\alpha}\right]' + q(t)\,z^{\alpha}\,[\sigma(t)] \le 0 \tag{6}$$

and

$$\left[r(t)\,[z'(t)]^{\alpha}\right]' \le 0.$$

Therefore

$$r(t)\left[z'(t)\right]^{\alpha} \leq r\left[\sigma(t)\right]\left[z'\left[\sigma(t)\right]\right]^{\alpha},$$

which implies that

$$\frac{z'\left[\sigma(t)\right]}{z'(t)} \geq \left(\frac{r(t)}{r\left[\sigma(t)\right]}\right)^{\frac{1}{\alpha}}. \tag{7}$$

Define

$$w(t) = R^{\alpha}\left[\sigma(t)\right]\frac{r(t)\left[z'(t)\right]^{\alpha}}{z^{\alpha}\left[\sigma(t)\right]} > 0 \tag{8}$$

for $t \geq t_1$.
Differentiating $w(t)$, we have

$$\begin{aligned} w'(t) &= \frac{\alpha R^{\alpha-1}\left[\sigma(t)\right]\sigma'(t)}{r^{\frac{1}{\alpha}}\left[\sigma(t)\right]} \cdot \frac{r(t)\left[z'(t)\right]^{\alpha}}{z^{\alpha}\left[\sigma(t)\right]} + R^{\alpha}\left[\sigma(t)\right]\frac{\left[r(t)[z'(t)]^{\alpha}\right]'}{z^{\alpha}\left[\sigma(t)\right]} \\ &\quad - \alpha R^{\alpha}\left[\sigma(t)\right]\frac{r(t)[z'(t)]^{\alpha} z'\left[\sigma(t)\right]\sigma'(t)}{z^{\alpha+1}\left[\sigma(t)\right]}. \end{aligned} \tag{9}$$

Using (6), (7) and (8), we have

$$\begin{aligned} w'(t) &\leq \frac{\alpha\sigma'(t)}{R\left[\sigma(t)\right]r^{\frac{1}{\alpha}}\left[\sigma(t)\right]} w(t) - R^{\alpha}\left[\sigma(t)\right]q(t) \\ &\quad - \frac{\alpha\sigma'(t)}{R\left[\sigma(t)\right]r^{\frac{1}{\alpha}}\left[\sigma(t)\right]} \cdot \frac{R^{\alpha+1}\left[\sigma(t)\right]r^{\frac{\alpha+1}{\alpha}}(t)[z'(t)]^{\alpha+1}}{z^{\alpha+1}\left[\sigma(t)\right]} \\ w'(t) &\leq \frac{\alpha\sigma'(t)}{R\left[\sigma(t)\right]r^{\frac{1}{\alpha}}\left[\sigma(t)\right]}\left[w(t) - w^{\frac{\alpha+1}{\alpha}}(t)\right] - R^{\alpha}\left[\sigma(t)\right]q(t). \end{aligned} \tag{10}$$

Set $A = w(t)$ and $B = \lambda^{\frac{1}{1-\lambda}}$, where $\lambda = \frac{\alpha+1}{\alpha} > 1$. Applying the Lemma 2.1 to (10), we obtain

$$w'(t) \leq \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} \cdot \frac{\sigma'(t)}{R\left[\sigma(t)\right]r^{\frac{1}{\alpha}}\left[\sigma(t)\right]} - R^{\alpha}\left[\sigma(t)\right]q(t).$$

Integrating the above inequality from t_1 to t, we get

$$w(t) \leq w(t_1) - \int_{t_1}^{t}\left[R^{\alpha}\left[\sigma(s)\right]q(s) - \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\frac{\sigma'(s)}{R\left[\sigma(s)\right]r^{\frac{1}{\alpha}}\left[\sigma(s)\right]}\right]ds. \tag{11}$$

Letting $t \to \infty$ in (11), we get $w(t) \to -\infty$ in view of (2). This contradicts to positivity of $w(t)$ and we conclude that $x(t)$ is bounded. Consequently, in view of (4) $z(t)$ is bounded too.

Eq. (5) implies, that function $r(t)\,|z'(t)|^{\alpha-1}\,z'(t)$ is nonincreasing and we get two possibilities for $z'(t)$:

(i) $z'(t) > 0$,

(ii) $z'(t) < 0$ for $t \geq t_2 \geq t_1$.

The condition (ii) implies that for some positive constant N and $\forall\, t \geq t_2$

$$r(t)\,|z'(t)|^{\alpha-1}\,z'(t) \leq -N < 0.$$

Proceeding similarly as in the previous we obtain

$$-z'(t) \geq \left(\frac{N}{r(t)}\right)^{\frac{1}{\alpha}}.$$

Integrating the above inequality from t_2 to t, we obtain

$$z(t) \leq z(t_2) - N^{\frac{1}{\alpha}}\big(R(t) - R(t_2)\big).$$

Letting $t \to \infty$ in the above inequality and using $(H3)$, we get $z(t) \to -\infty$. This contradicts that $z(t)$ is bounded, e.g. (i) holds.

Now we shall discuss the following two cases:

1. $z(t) > 0$,
2. $z(t) < 0$.

Case 1. Let $z(t) > 0$.

Since $z(t)$ is bounded and $z'(t) > 0$, there exists

$$\lim_{t\to\infty} z(t) = 2c, \quad 0 < c < \infty. \tag{12}$$

Integrating (6) from t to ∞ and taking into account monocity of $z^{\alpha}[\sigma(t)]$ and (12) one gets

$$[z'(t)]^{\alpha} \geq c^{\alpha} \cdot \frac{1}{r(t)} \int_t^{\infty} q(s)\,ds.$$

Raising to $\frac{1}{\alpha}$ power and integrating from t_3 to t we acquire

$$z(t) \geq z(t_3) + c \int_{t_3}^{t} \frac{1}{r^{\frac{1}{\alpha}}(u)} \left[\int_u^{\infty} q(s)\,ds\right]^{\frac{1}{\alpha}} du. \tag{13}$$

Letting $t \to \infty$ in the previous inequality, we get $z(t) \to \infty$ in view of (3) and this contradicts the boundedness of the function $z(t)$.

Case 2. Let $z(t) < 0$.

Since $z(t)$ is bounded and $z'(t) > 0$, there exists

$$\lim_{t\to\infty} z(t) = c, \quad -\infty < c \le 0. \tag{14}$$

The boundedness of $x(t)$ yields $\limsup_{t\to\infty} x(t) = a$, $0 \le a < \infty$. Then there exists a sequence $\{t_k\}$ such that $\lim_{k\to\infty} t_k = \infty$, $\lim_{k\to\infty} x(t_k) = a$. If $a > 0$, choosing $\epsilon = \frac{a(1-p)}{2p}$ we see that $x[\tau(t)] < a + \epsilon$, eventually. Moreover

$$0 \ge \lim_{k\to\infty} z(t_k) \ge \lim_{k\to\infty} \big(x(t_k) - p(a+\epsilon)\big) = \frac{a}{2}(1-p) > 0.$$

Thus $a = 0$ and that is $\lim_{t\to\infty} x(t) = 0$. □

Now we provide easily verifiable oscillatory criterion for Eq. (E^-).

Corollary 2.1 *Let* (3) *holds and*

$$\liminf_{t\to\infty} \frac{R^{\alpha+1}[\sigma(t)]\, r^{\frac{1}{\alpha}}[\sigma(t)]\, q(t)}{\sigma'(t)} > \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}, \tag{15}$$

Then every nonoscillatory solution of Eq. (E^-) *tends to zero as* $t \to \infty$.

Proof. Let (15) holds. Then there exists $\epsilon > 0$ such that for all large t, say $t \ge t_1$

$$\frac{R^{\alpha+1}[\sigma(t)]\, r^{\frac{1}{\alpha}}[\sigma(t)]\, q(t)}{\sigma'(t)} \ge \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} + \epsilon,$$

which follows that

$$R^{\alpha}[\sigma(t)]\, q(t) - \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} \frac{\sigma'(t)}{R[\sigma(t)]\, r^{\frac{1}{\alpha}}[\sigma(t)]} \ge \epsilon \frac{\sigma'(t)}{R[\sigma(t)]\, r^{\frac{1}{\alpha}}[\sigma(t)]}.$$

Integrating the above inequality from t_1 to t, we obtain

$$\int_{t_1}^{t} \left[R^{\alpha}[\sigma(s)]\, q(s) - \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} \frac{\sigma'(s)}{R[\sigma(s)]\, r^{\frac{1}{\alpha}}[\sigma(s)]} \right] ds \ge$$

$$\ge \epsilon \big[\ln R[\sigma(t)] - \ln R[\sigma(t_1)] \big] \to \infty \quad \text{as} \quad t \to \infty.$$

Now the assertion of Corollary 2.1 follows from Theorem 2.1. □

Corollary 2.2 *If*

$$\int^{\infty}\left[[\sigma(s)]^{\alpha}q(s)-\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\frac{\sigma'(s)}{\sigma(s)}\right]ds=\infty, \tag{16}$$

$$\int^{\infty}\left[\int_u^{\infty}q(s)\,ds\right]^{\frac{1}{\alpha}}du=\infty, \tag{17}$$

then every nonoscillatory solution of Eq. (18)

$$\Big[\big|\left[x(t)-p(t)x\left[\tau(t)\right]\right]'\big|^{\alpha-1}\left[x(t)-p(t)x\left[\tau(t)\right]\right]'\Big]'+$$

$$+\,q(t)\big|x\left[\sigma(t)\right]\big|^{\alpha-1}x\left[\sigma(t)\right]=0 \tag{18}$$

tends to zero as $t\to\infty$.

Proof. It is easy to see that the conditions (2) and (3) reduce to (16) and (17) for $r(t)\equiv 1$. □

Corollary 2.3 *If*

$$\int^{\infty}\left[R\left[\sigma(s)\right]q(s)-\frac{\sigma'(s)}{4R\left[\sigma(s)\right]r\left[\sigma(s)\right]}\right]ds=\infty, \tag{19}$$

$$\int^{\infty}\frac{1}{r(u)}\int_u^{\infty}q(s)\,ds\,du=\infty, \tag{20}$$

then every nonoscillatory solution of Eq. (21)

$$\Big[r(t)\left[x(t)-p(t)x\left[\tau(t)\right]\right]'\Big]'+q(t)x\left[\sigma(t)\right]=0 \tag{21}$$

tends to zero as $t\to\infty$.

Proof. It is easy to see that (2) and (3) reduce to (19) and (20) for $\alpha=1$.
□

Corollary 2.4 *Let* (2) *and* (3) *hold. If* $p(t)$ *oscilates, then Eq.* (E^{-}) *is oscillatory.*

Proof. Let $x(t)$ is a positive solution of (E^-). Arguing exactly as in the proof of Theorem 2.1 we can show that $z(t) < 0$. If $\{t_k\}$ is a sequence of zeros of $p(t)$, then

$$0 > z(t_k) = x(t_k) - p(t_k)x(\tau(t_k)) > 0.$$

That is a contradiction. □

Now we will use so-called the integral averaging technique. Let us consider a function $H(t,s)$ satisfying the following properties
(i) $H(t,s) > 0$ for $t > s \geq t_0$,
(ii) $H(t,t) = 0$ and $\frac{\partial H(t,s)}{\partial s} < 0$.

Denote

$$\begin{aligned} h(t,s) &= \frac{-\frac{\partial H(t,s)}{\partial s}}{\sqrt{H(t,s)}}, \\ Q(t,s) &= \sqrt{H(t,s)} \cdot \frac{\alpha\sigma'(s)}{R[\sigma(s)]r^{\frac{1}{\alpha}}[\sigma(s)]} - h(t,s), \quad \text{for } t > s\,. \end{aligned}$$

Theorem 2.2 *Let $\alpha \geq 1$ and (3) holds. Assume that for some $k \in (0,1)$*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_1)} \int_{t_1}^{t} \Bigg[H(t,s)R^{\alpha}\left[\sigma(s)\right] q(s) - \frac{R\left[\sigma(s)\right] r^{\frac{1}{\alpha}}\left[\sigma(s)\right]}{4\alpha k\sigma'(s)} Q^2(t,s) \Bigg] ds = \infty. \tag{22}$$

Then every nonoscillatory solution of Eq. (E^-) tends to zero as $t \to \infty$.

Proof. Assume to the contrary that $x(t)$ is a nonoscillatory solution of Eq. (E^-). Without loss of generality we may assume that $x(t) > 0$. Proceeding similarly as in the proof of Theorem 2.1 we have $z(t) > 0$, $z'(t) > 0$ and using the fact that $[r(t)\,(z'(t))^{\alpha}]^{\frac{1}{\alpha}}$ is nonincreasing, we see that for any $k_1 \in (0,1)$ and for all large t $(t \geq t_1)$

$$\begin{aligned} z\left[\sigma(t)\right] &\geq \int_{t_1}^{\sigma(t)} z'(s)ds = \int_{t_1}^{\sigma(t)} \frac{1}{r^{\frac{1}{\alpha}}(s)} \left(r^{\frac{1}{\alpha}}(s)z'(s)\right) ds \\ &\geq r^{\frac{1}{\alpha}}\left[\sigma(t)\right] z'\left[\sigma(t)\right] \big(R\left[\sigma(t)\right] - R(t_1)\big) \\ &> k_1 R\left[\sigma(t)\right] r^{\frac{1}{\alpha}}\left[\sigma(t)\right] z'\left[\sigma(t)\right]. \end{aligned} \tag{23}$$

Taking into account (23) and the monotonicity of $r(t)\left[z'(t)\right]^{\alpha}$, we conclude that

$$\begin{aligned}\frac{z'\left[\sigma(t)\right]}{z\left[\sigma(t)\right]} &= \frac{1}{r\left[\sigma(t)\right]}\cdot\frac{r\left[\sigma(t)\right]\left[z'\left[\sigma(t)\right]\right]^{\alpha}}{z^{\alpha}\left[\sigma(t)\right]}\cdot\left(\frac{z\left[\sigma(t)\right]}{z'\left[\sigma(t)\right]}\right)^{\alpha-1}\\ &\geq \frac{r(t)\left[z'(t)\right]^{\alpha}}{z^{\alpha}\left[\sigma(t)\right]}\cdot\frac{kR^{\alpha-1}\left[\sigma(t)\right]}{r^{\frac{1}{\alpha}}\left[\sigma(t)\right]}\geq\frac{k}{R\left[\sigma(t)\right]r^{\frac{1}{\alpha}}\left[\sigma(t)\right]}w(t) \qquad (24)\end{aligned}$$

where $k = k_1^{\alpha-1}\in(0,1)$.

Using the function $w(t)$ defined in (8), $w'(t)$ in (9) and the inequality (24) we obtain

$$\begin{aligned} w'(t) &\leq \frac{\alpha\sigma'(t)}{R\left[\sigma(t)\right]r^{\frac{1}{\alpha}}\left[\sigma(t)\right]}w(t) - R^{\alpha}\left[\sigma(t)\right]q(t)\\ &\quad -\alpha R^{\alpha}\left[\sigma(t)\right]\sigma'(t)\frac{r(t)\left[z'(t)\right]^{\alpha}}{z^{\alpha}\left[\sigma(t)\right]}\cdot\frac{z'\left[\sigma(t)\right]}{z\left[\sigma(t)\right]}\\ &\leq \frac{\alpha\sigma'(t)}{R\left[\sigma(t)\right]r^{\frac{1}{\alpha}}\left[\sigma(t)\right]}w(t) - R^{\alpha}\left[\sigma(t)\right]q(t) - \frac{\alpha k\sigma'(t)}{R\left[\sigma(t)\right]r^{\frac{1}{\alpha}}\left[\sigma(t)\right]}w^2(t).\end{aligned}$$

Multiplying this inequality with $H(t,s)>0$ and following integrating from t_1 to t we have

$$\begin{aligned}&\int_{t_1}^{t}H(t,s)R^{\alpha}\left[\sigma(s)\right]q(s)ds\leq\\ &\leq \int_{t_1}^{t}H(t,s)\frac{\alpha\sigma'(s)}{R\left[\sigma(s)\right]r^{\frac{1}{\alpha}}\left[\sigma(s)\right]}w(s)ds\\ &\quad-\int_{t_1}^{t}H(t,s)\frac{\alpha k\sigma'(s)}{R\left[\sigma(s)\right]r^{\frac{1}{\alpha}}\left[\sigma(s)\right]}w^2(s)ds-\int_{t_1}^{t}H(t,s)w'(s)ds.\end{aligned}$$

Now integrating (per partes) from t_1 to t and using definition of the functions $h(t,s)$ and $Q(t,s)$ we are led to

$$\begin{aligned}&\int_{t_1}^{t}H(t,s)R^{\alpha}\left[\sigma(s)\right]q(s)ds\leq\\ &\leq H(t,t_1)w(t_1)-\int_{t_1}^{t}H(t,s)\frac{\alpha k\sigma'(s)}{R\left[\sigma(s)\right]r^{\frac{1}{\alpha}}\left[\sigma(s)\right]}w^2(s)ds\\ &\quad+\int_{t_1}^{t}\sqrt{H(t,s)}\left[\sqrt{H(t,s)}\cdot\frac{\alpha\sigma'(s)}{R\left[\sigma(s)\right]r^{\frac{1}{\alpha}}\left[\sigma(s)\right]}-h(t,s)\right]w(s)ds\leq\end{aligned}$$

$$
\begin{aligned}
&\leq H(t,t_1)w(t_1) \\
&\quad - \int_{t_1}^{t} H(t,s)\frac{\alpha k\sigma'(s)}{R\left[\sigma(s)\right] r^{\frac{1}{\alpha}}\left[\sigma(s)\right]} w^2(s)ds + \int_{t_1}^{t} \sqrt{H(t,s)}Q(t,s)w(s)ds.
\end{aligned}
$$

Consequently

$$
\begin{aligned}
&\int_{t_1}^{t} H(t,s)R^{\alpha}\left[\sigma(s)\right]q(s)ds \leq \\
&\leq H(t,t_1)w(t_1) + \int_{t_1}^{t} \frac{R\left[\sigma(s)\right] r^{\frac{1}{\alpha}}\left[\sigma(s)\right]}{4\alpha k\sigma'(s)} Q^2(t,s)ds \\
&\quad - \int_{t_1}^{t} \left[\sqrt{H(t,s)\frac{\alpha k\sigma'(s)}{R\left[\sigma(s)\right] r^{\frac{1}{\alpha}}\left[\sigma(s)\right]}} w(s) - \frac{1}{2}\sqrt{\frac{R\left[\sigma(s)\right] r^{\frac{1}{\alpha}}\left[\sigma(s)\right]}{\alpha k\sigma'(s)}} Q(t,s)\right]^2 ds.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\frac{1}{H(t,t_1)}\int_{t_1}^{t}\left[H(t,s)R^{\alpha}\left[\sigma(s)\right]q(s)\right. \\
&\quad \left.- \frac{R\left[\sigma(s)\right] r^{\frac{1}{\alpha}}\left[\sigma(s)\right]}{4\alpha k\sigma'(s)} Q^2(t,s)\right] ds \leq w(t_1).
\end{aligned}
$$

Letting $t \to \infty$ we get the contradiction with (22). The rest of proof is similar to the proof of Theorem 2.2. □

Let us have $H(t,s)$ defined by (25).

$$
H(t,s) = (t-s)^n, \qquad n \text{ is a positive integer.} \tag{25}
$$

Then Theorem 2.2 provides the following criterion:

Theorem 2.3 *Let $\alpha \geq 1$ and (3) holds. Assume that for some $k \in (0,1)$*

$$
\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{(t-t_1)^n}\int_{t_1}^{t}\left[(t-s)^n R^{\alpha}\left[\sigma(s)\right]q(s)\right. \\
&\quad \left.- \frac{R\left[\sigma(s)\right] r^{\frac{1}{\alpha}}\left[\sigma(s)\right]}{4\alpha k\sigma'(s)} Q^2(t,s)\right] ds = \infty,
\end{aligned} \tag{26}
$$

where

$$Q(t,s) = (t-s)^{\frac{n}{2}} \left(\frac{\alpha\sigma'(s)}{R\big[\sigma(s)\big] r^{\frac{1}{\alpha}}[\sigma(s)]} - \frac{n}{t-s} \right).$$

Then every nonoscillatory solution of Eq. (E^-) *tends to zero as* $t \to \infty$.

Remark 1 *Theorem 2.1 extends results presented for neutral differential equations of the forms*

$$\big(x(t) - px(t-\tau)\big)'' + q(t)x\left[\sigma(t)\right] = 0,$$

$$\big(x(t) \pm p(t)x\left[\tau(t)\right]\big)^{(n)} + q(t)x\left[\sigma(t)\right] = 0,$$

$$\big(x(t) - p(t)x\left[\tau(t)\right]\big)^{(n)} + q(t)f\big(x\left[\sigma(t)\right]\big) = 0$$

presented in [3], [2] *and* [6].

Remark 2 *Putting* $p(t) \equiv 0$, *Theorem 2.1 generalizes results presented in* [4] *and* [7], *where the differential equations of the form* (E_1) *are studied.*

Remark 3 *Theorems 2.1, 2.2 and 2.3 complement results presented in* [1, 8], *where authors deal with the neutral differential equations of the form* (E_2), *respectively* (E_3).

Example 1 *We consider differential equation*

$$\left[\left| \left(x(t) - px\left(\frac{t}{2}\right) \right)' \right|^{\alpha-1} \left(x(t) - px\left(\frac{t}{2}\right) \right)' \right]' +$$

$$+ \frac{2\alpha\beta^{\alpha}(2p-1)^{\alpha}}{t^{\alpha+1}} \big|x(\beta t)\big|^{\alpha-1} x(\beta t) = 0, \tag{27}$$

with $t > 0$, $r(t) = 1$, $\tau(t) = \frac{t}{2}$, $p(t) = p$, $\frac{1}{2} < p < 1$, $\sigma(t) = \beta t$, $0 < \beta < 1$, $q(t) = \frac{2\alpha\beta^{\alpha}(2p-1)^{\alpha}}{t^{\alpha+1}}$. *If*

$$2\alpha\beta^{2\alpha}(2p-1)^{\alpha} > \left(\frac{\alpha}{\alpha+1} \right)^{\alpha+1},$$

then by Theorem 2.1 every nonoscillatory solution of Eq. (27) *tends to zero as* $t \to \infty$. *One of the solutions of Eq.* (27) *is for example* $x(t) = \frac{1}{t}$.

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Department of Applied Mathematics and Business Informatics, Faculty of Economics, Technical University of Košice, Němcovej 32, 040 01 Košice, Slovakia.
E-mail address: dasa.lackova@tuke.sk