Electronic Journal of Qualitative Theory of Differential Equations
2014, No. **62**, 1–11; http://www.math.u-szeged.hu/ejqtde/

The best constant of Sobolev inequality corresponding to anti-periodic boundary value problem

Jozef Kiseľák✉

Faculty of Science, P. J. Šafárik University, Jesenná 5, Košice, 040 01, Slovakia

Received 19 June 2014, appeared 17 December 2014
Communicated by Ivan Kiguradze

Abstract. In this paper we establish the best constant of $\mathcal{L}^p$ Sobolev inequality for a function with anti-periodic boundary conditions. The best constant is expressed by $\mathcal{L}^q$ norm of $(M-1)$-th order Euler polynomial. Lyapunov-type inequality for certain higher order differential equation including 1-dim p-Laplacian is obtained by the usage of this constant.

Keywords: L^p Sobolev inequality, best constant, Green function, Euler polynomial, Lyapunov-type inequality.

2010 Mathematics Subject Classification: 46E35, 41A44, 26D10, 34B27.

1 Introduction

It is well-known that sharp Sobolev inequalities are very important in the study of partial and ordinary differential equations, especially in the study of problems arising from geometry and physics. They are relevant for the study of boundary value problems. In this paper, we treat the anti-periodic case. Throughout the paper, we assume $p > 1$, $a < b$ and $\frac{1}{p} + \frac{1}{q} = 1$, i.e. $q = \frac{p}{p-1} > 1$ is the conjugate exponent of p. For $M = 1, 2, 3, \ldots$ let us consider a sequence of Sobolev spaces

$$W_M = \left\{ u \;\middle|\; u^{(M)} \in \mathcal{L}^p(a,b),\;\; u^{(i)}(a) + u^{(i)}(b) = 0,\;\; 0 \le i \le M-1 \right\}$$

and the following one-dimensional Sobolev inequality:

$$\|u(x)\|_\infty \le C \|u^{(M)}(x)\|_p, \tag{1.1}$$

where $u \in W_M$ and $\|\cdot\|_\infty$ and $\|\cdot\|_p$ are the usual $\mathcal{L}^\infty$ and $\mathcal{L}^p$ norms.

When $p = 2$, the engineering meaning of this inequality is that the square of the maximum bending of a string $(M = 1)$ or a beam $(M = 2)$ is estimated from above by the constant multiple of the potential energy due to internal forces. Notice that anti-periodic boundary value problems appear in physics also in other situations, see, for example [4, 8]. The purpose

✉ Email: jozef.kiselak@upjs.sk

of this paper is to derive a Sobolev inequality corresponding to an anti-periodic boundary value problem, and to obtain the best constant by using the property as the reproducing kernel of Green function. As an application, we give Lyapunov-type inequalities for certain half-linear higher order differential equations with anti-periodic boundary conditions.

1.1 Polynomials

To state the conclusion, we need to introduce the Appell polynomials (or sequences). The sequence $P_n(x)$ is Appel for $g(t)$ if and only if

$$\frac{\mathrm{e}^{xt}}{g(t)} = \sum_{k=0}^{\infty} P_k(x)\,\frac{t^k}{k!}, \quad |t| < \sigma,\ 0 < \sigma < \infty \tag{1.2}$$

for all x in the field C of field characteristic 0 and $\frac{1}{g(t)}$ is analytic function, see [9], where the author summarizes properties of more general Sheffer sequences and gives a number of specific examples.

The Bernoulli polynomials have been studied since the 18th century. There are many applications in mathematics and physics. Many functions are used to obtain the generating function of them, but also Euler and Genocchi polynomials. We first define Bernoulli polynomials $B_n(x)$ using the generating function (1.2) with $g(t) = \frac{\mathrm{e}^t-1}{t}$, i.e.

$$\frac{te^{xt}}{e^t-1} = \sum_{k=0}^{\infty} B_k(x)\frac{t^k}{k!}, \quad |t| < 2\pi. \tag{1.3}$$

Similarly, we denote the Euler polynomials and Genocchi polynomials, which are defined by means of the generating function (1.2) with $g(t) = \frac{1}{2}(\mathrm{e}^t + 1)$, $g(t) = \frac{1}{2t}(\mathrm{e}^t + 1)$, as $E_n(x)$ and $G_n(x)$ respectively, i.e.

$$\frac{2\,e^{tx}}{e^t+1} = \sum_{k=0}^{\infty} E_k(x)\,\frac{t^k}{k!}, \quad |t| < \pi \tag{1.4}$$

$$\frac{2t\,e^{tx}}{e^t+1} = \sum_{k=0}^{\infty} G_k(x)\,\frac{t^k}{k!}, \quad |t| < \pi \tag{1.5}$$

Although it does not immediately yield their explicit form, the manipulation of (1.3), (1.4), (1.5), along with the uniqueness theorem for power series expansions, leads to many properties of these polynomials. For example symmetry is easily obtained in this way. For $k \in \mathbb{N}_0$ and all $x \in \mathbb{R}$ we have

$$P_k(1-x) = (-1)^k P_k(x), \tag{1.6}$$

$$G_k(1-x) = (-1)^{k+1} G_k(x), \tag{1.7}$$

$$(-1)^k B_k(-x) = B_k(x) + kx^{k-1}, \tag{1.8}$$

$$(-1)^k E_k(-x) = -E_k(x) + 2x^k, \tag{1.9}$$

$$(-1)^{k-1} G_k(-x) = -G_k(x) + 2kx^{k-1}, \tag{1.10}$$

where P_k can be replaced by B_k or E_k. Since all these polynomial sequences are Appell sequences, also property concerning derivation must hold (it is sometimes used as an equivalent definition). We summarize this:

$$\frac{\mathrm{d}}{\mathrm{d}x} P_k(x) = kP_{k-1}(x), \qquad k \geq 1, \tag{1.11}$$

where P_k can be replaced by B_k, E_k or G_k. Notice that another convention followed by some authors (see [1, p. 169]) defines this concept in a different way, conflicting with Appell's original definition, by using the identity

$$\frac{\mathrm{d}}{\mathrm{d}x}P_k(x) = P_{k-1}(x)$$

instead. This is reflected in the fact that $B_k(E_k, G_k)$ is $\frac{1}{k!}$ multiple of the "original" one. Various interesting and potentially useful properties and relationships involving the Bernoulli, Euler and Genocchi polynomials have been studied. We need only few of them, which can be summarized in the following lemma, see e.g. [2].

Lemma 1.1. *We have*

$$G_n(x) = 2^n \left[B_n\left(\frac{x+1}{2}\right) - B_n\left(\frac{x}{2}\right) \right], \quad \forall n \in \mathbb{N}_0 \ \textit{and} \ \forall x \in \mathbb{R}, \tag{1.12}$$

$$G_n(x) = n\, E_{n-1}(x), \quad \forall n \in \mathbb{N} \ \textit{and} \ \forall x \in \mathbb{R}. \tag{1.13}$$

In the Table 1.1 we give explicit forms of the first four Bernoulli, Euler and Genocchi polynomials.

n	$B_n(x)$	$E_n(x)$	$G_n(x)$
0	1	1	0
1	$x-\frac{1}{2}$	$x-\frac{1}{2}$	1
2	$x^2-x+\frac{1}{6}$	x^2-x	$2x-1$
3	$x^3-\frac{3}{2}x^2+\frac{1}{2}x$	$x^3-\frac{3}{2}x^2+\frac{1}{4}$	$3x^2-3x$
4	$x^4-2x^3+x^2-\frac{1}{30}$	x^4-2x^3+x	$4x^3-6x^2+1$

Table 1.1: Explicit forms of the first four polynomials.

2 Boundary value problem

In this section, we present the main theorems of this paper. For the case $p = 2$, the problem of finding the best constants of (1.1) is solved completely, whereas the method of maximizing the diagonal value of reproducing kernels was used, see references in Table 2.1. For the general case the difficulty of obtaining the best constants increases and cases of clamped and Dirichlet boundary conditions remain unsolved, again see Table 2.1.

We consider the boundary value problem

$$\begin{cases} (-1)^{\left\lfloor \frac{M+1}{2} \right\rfloor} u^{(M)} = f(x), & a < x < b \\ u^{(k)}(a) + u^{(k)}(b) = 0, & 0 \le k \le M-1. \end{cases} \tag{2.1}$$

In [14] the authors obtained a Green function for even M and $a = 0,\ b = 1$. As it is pointed out in [13], using the method of reflection and some algebra one can show that for the problem (2.1) the expression of the Green function has the form

$$G(2M; x, y) = (-1)^{M+1}(2(b-a))^{2M-1} \left[B_{2M}\left(\frac{|x-y|}{2(b-a)}\right) - B_{2M}\left(\frac{1}{2} - \frac{|x-y|}{2(b-a)}\right) \right].$$

Once the Green's function is obtained, one can write down the solution of the problem (2.1) very easily using an integral.

Theorem 2.1. *For any $f \in BC(a,b)$ boundary value problem (2.1) has one and only one classical solution $u(x)$ given by*

$$u(x) = \int_a^b G(M;x,y)f(y)\,\mathrm{d}y, \quad a \le x \le b,$$

where the Green function has the form

$$G(M;x,y) = \frac{(-1)^{\lfloor \frac{M-1}{2} \rfloor}(b-a)^{M-1}}{2(M-1)!} E_{M-1}\left(\frac{|x-y|}{b-a}\right)(\mathrm{sgn}(x-y))^M \tag{2.2}$$

Proof. Differentiating $u(x)$, the properties listed in Lemma 2.2 and the following fact yields the existence and uniqueness of the solution. It is not difficult to show that for general M the result from [13] should be modified to

$$G(M;x,y) = \Omega\left[B_M\left(\frac{|x-y|}{2(b-a)}\right) - (-1)^M B_M\left(\frac{1}{2} - \frac{|x-y|}{2(b-a)}\right)\right](\mathrm{sgn}(x-y))^M, \tag{2.3}$$

where

$$\Omega = \frac{(-1)^{\lfloor \frac{M+1}{2} \rfloor}}{M!(2(b-a))^{1-M}}.$$

Due to the symmetry of Bernoulli polynomials it is true that $B_n\left(\frac{1}{2} - y\right) = (-1)^n B_n\left(\frac{1}{2} + y\right)$, $\forall n \in \mathbb{N}_0$, $\forall y \in \mathbb{R}$. Using that fact together with (1.12) in Lemma 1.1 we have $G(M;x,y) = -2^{-M}\Omega\, G_M\left(\frac{|x-y|}{b-a}\right)(\mathrm{sgn}(x-y))^M$. Now it is sufficient to use (1.13) in Lemma 1.1. □

Lemma 2.2. *The Green function $G(M;x,y)$ defined by (2.2) satisfies the following properties.*

(a) $G(M;x,y) = (-1)^M G(M;y,x) \quad (a < x,y < b)$.

(b) $\partial_x^M G(M;x,y) = 0 \quad (a < x,y < b,\ x \neq y)$.

(c) *For* $0 \le k \le M-1$, *we have*
$\partial_x^k G(M;x,y)|_{x=b} + \partial_x^k G(M;x,y)|_{x=a} = 0, \quad (a < y < b)$,

(d) $\partial_x^k G(M;x,y)|_{y=x^+} - \partial_x^k G(M;x,y)|_{y=x^-} = \begin{cases} 0, & 0 \le k \le M-2 \\ (-1)^{\lfloor \frac{M+1}{2} \rfloor}, & k = M-1, \quad (a < x < b), \end{cases}$

(e) $\partial_x^k G(M;x,y)|_{x=y^-} - \partial_x^k G(M;x,y)|_{x=y^+} = \begin{cases} 0, & 0 \le k \le M-2 \\ (-1)^{\lfloor \frac{M+1}{2} \rfloor}, & k = M-1, \quad (a < y < b). \end{cases}$

Proof. Obviously (a) and (b) holds. Differentiating (2.2) k times with respect to x, we obtain

$$\partial_x^k G(M;x,y) = \tilde{\Omega}\, E_{M-1-k}\left(\frac{|x-y|}{b-a}\right)(\mathrm{sgn}(x-y))^{M+k} \tag{2.4}$$

for $a < x,y < b$, $x \neq y$ and $0 \le k \le M-1$, where $\tilde{\Omega} = \frac{(-1)^{\lfloor \frac{M-1}{2} \rfloor}(b-a)^{M-1-k}}{2(M-1-k)!}$. From (2.4) and symmetry property (1.6), we have

$$\begin{aligned}
&\partial_x^k G(M;x,y)|_{x=b} + \partial_x^k G(M;x,y)|_{x=a} \\
&= \tilde{\Omega}\left[E_{M-1-k}\left(\frac{b-y}{b-a}\right) + E_{M-1-k}\left(\frac{y-a}{b-a}\right)(-1)^{M+k}\right] \\
&= \tilde{\Omega}\left[E_{M-1-k}\left(\frac{b-y}{b-a}\right) + E_{M-1-k}\left(1 - \frac{y-a}{b-a}\right)(-1)\right] \\
&= 0.
\end{aligned}$$

So we have (c). Again from (2.4), it is true that

$$\partial_x^k G(M;x,y)|_{y=x^+} - \partial_x^k G(M;x,y)|_{y=x^-} = \tilde{\Omega}\, E_{M-1-k}(0) \left[(-1)^{M-1-k} + 1\right],$$

for $a < x < b$, $0 \le k \le M-1$. Item (d) follows from symmetry property (1.9) and (e) is equivalent to (d). This completes the proof. □

2.1 Reproducing kernel

Thanks to the solution of the boundary value problem (2.1) we have found reproducing kernel for some specific Hilbert space, which helps us to solve the problem of finding the best constant for Sobolev inequality (1.1). We denote by W_M^* the dual space of W_M, i.e.

$$W_M^* = \left\{u \;\middle|\; u^{(M)} \in \mathcal{L}^q(a,b),\;\; u^{(i)}(a) + u^{(i)}(b) = 0,\;\; 0 \le i \le M-1\right\}.$$

We show that the Green function $G(2M;x,y)$ is a reproducing kernel for function spaces W_M, W_M^* and duality pairing $\langle\cdot,\cdot\rangle_M$ defined as $\langle u,v\rangle_M = \int_a^b u^{(M)}(x) v^{(M)}(x)\,\mathrm{d}x$ on $W_M \times W_M^*$.

Lemma 2.3. *(a) If $u^{(M)} \in \mathcal{L}^p(a,b)$, then*

$$\begin{aligned}
&\int_a^b u^{(M)}(x)\, \partial_x^M G(2M;x,y)\,\mathrm{d}x \\
&\quad = \Theta \int_a^b u^{(M)}(x)\, E_{M-1}\left(\frac{|x-y|}{b-a}\right) (\mathrm{sgn}(x-y))^M\,\mathrm{d}x \\
&\quad = -u(y) + \sum_{j=0}^{M-1} \frac{(a-b)^j}{2j!} E_j\left(\frac{b-y}{b-a}\right)\left[u^{(j)}(b) + u^{(j)}(a)\right], \qquad \Theta = \frac{(a-b)^{M-1}}{2(M-1)!}
\end{aligned}$$

(b) For any $u \in W_M$, we have the following reproducing relation

$$\begin{aligned}
u(y) &= \langle u(x), G(2M;x,y)\rangle_M = \int_a^b u^{(M)}(x)\, \partial_x^M G(2M;x,y)\mathrm{d}x \\
&= -\Theta \int_a^b u^{(M)}(x)\, E_{M-1}\left(\frac{|x-y|}{b-a}\right) (\mathrm{sgn}(x-y))^M\,\mathrm{d}x, \qquad a \le y \le b.
\end{aligned} \tag{2.5}$$

Proof. We first prove the first part. For any two smooth functions u and v, we have

$$u^{(M)} v^{(M)} = (-1)^M u\, v^{(2M)} + \left(\sum_{j=0}^{M-1} (-1)^{M-1-j} u^{(j)}\, v^{(2M-1-j)}\right)'. \tag{2.6}$$

Now, we put $v(x) = G(2M;x,y)$, $y \in (a,b)$ and integrate this identity with respect to x over intervals $a < x < y$ and $y < x < b$, we obtain

$$\int_a^b u^{(M)}(x)\, \partial_x^M G(2M;x,y)\,\mathrm{d}x = \Phi(y^-) - \Phi(y^+) + \Phi(b) - \Phi(a),$$

where

$$\Phi(z) := \sum_{j=0}^{M-1} (-1)^{M-1-j} u^{(j)}(x)\, \partial_x^{2M-1-j} G(2M;x,y)\bigg|_{x=z}.$$

Using Lemma 2.2 we have

$$\int_a^b u^{(M)}\,\partial_x^M G(2M;x,y)\,\mathrm{d}x$$
$$= -u(y) + \sum_{j=0}^{M-1} \frac{(a-b)^j}{2j!}\left[u^{(j)}(b)E_j\left(\frac{b-y}{b-a}\right) - u^{(j)}(a)E_j\left(1-\frac{b-y}{b-a}\right)(-1)^{j+1}\right].$$

It suffices now to use symmetry property (1.6). The second part results directly from the fact that $u \in W_M$. □

BVP	$p=2$	$p>1$ (general case)	$C(M)$
Periodic	[14]	[3]	$\begin{cases} \\|B_M(x)\\|_{\mathcal{L}^q(0,1)}, & M \text{ is odd} \\ \\|B_M(\alpha_0;x)\\|_{\mathcal{L}^q(0,1)}, & M \text{ is even}\end{cases}$
Anti-periodic	[14]	this paper	$\frac{(b-a)^{M-\frac{1}{p}}}{2(M-1)!}\\|E_{M-1}(x)\\|_{\mathcal{L}^q(0,1)}$
Clamped	[12]	$M=1,2,3$ [11]	see [11]
Dirichlet	[14]	$M=2m$ [5], $M=1,3,5$ [6]	$2^{2M-2}\\|\delta_M(x)\\|_{\mathcal{L}^q(-1,1)}$ and see [6]
Neumann	[14]	[7]	$\begin{cases} 2^M\\|B_M(x)\\|_{\mathcal{L}^q(0,1)}, & M \text{ is odd} \\ 2^M\\|B_M(\alpha_0;x)\\|_{\mathcal{L}^q(0,1)}, & M \text{ is even}\end{cases}$
Dirichlet–Neumann	[14]	[15]	$2^{2M-1}\\|\gamma_M(x)\\|_{\mathcal{L}^q(0,1)}$, M is odd

Table 2.1: Various boundary conditions and best constants.
Note that most of the authors solved specific problem on the interval $[0,1]$, but it can be simply extended to $[a,b]$. We recall that α_0 is the unique solution to the equation

$$\int_0^\alpha \left((-1)^{\lfloor\frac{M-1}{2}\rfloor}[B_M(x)-B_M(\alpha)]\right)^{q-1}\mathrm{d}x = \int_\alpha^{\frac{1}{2}}\left((-1)^{\lfloor\frac{M}{2}\rfloor}[B_M(x)-B_M(\alpha)]\right)^{q-1}\mathrm{d}x$$

in the interval $(0,\frac{1}{2})$ and notation
$\gamma_M(x) = (-1)^{M+1}B_M\left(\frac{1-x}{4}\right) + B_M\left(\frac{1+x}{4}\right)$, $\delta_M(x) = B_M\left(\frac{|x|}{4}\right) - B_M\left(\frac{2-x}{4}\right)$ is used.

We now present the main result of the paper. The technique used in the proof of the following theorem is much like that employed in [3].

Theorem 2.4. *The best constant of the Sobolev inequality or the supremum of the Sobolev functional*

$$C(M) = \sup_{\substack{u\in W_M\\ u\not\equiv 0}} \frac{\|u(x)\|_\infty}{\|u^{(M)}(x)\|_p}$$

is given by the formula

$$C(M) = \frac{(b-a)^{M-\frac{1}{p}}}{2(M-1)!}\|E_{M-1}(x)\|_{\mathcal{L}^q(0,1)}, \tag{2.7}$$

where the supremum is attained for

$$u(x) = \frac{(-1)^{\lfloor\frac{M+1}{2}\rfloor}(b-a)^{M-1}}{2(M-1)!}\int_a^b E_{M-1}\left(\frac{|x-y|}{b-a}\right)(\operatorname{sgn}(x-y))^M f(y)\,\mathrm{d}y \quad (a \le x \le b),$$

where $f(x) = (-1)^{\lceil\frac{M-1}{2}\rceil}\left|E_{M-1}\left(\frac{x-a}{b-a}\right)\right|^{q-1}\operatorname{sgn}\left(E_{M-1}\left(\frac{x-a}{b-a}\right)\right).$

Proof. Applying Hölder's inequality to the identity (2.5) in Lemma 2.3, we have

$$|u(y)| \leq \frac{(b-a)^{M-1}}{2(M-1)!} \left\| E_{M-1}\left(\frac{|x-y|}{b-a}\right) \right\|_{\mathcal{L}^q(a,b)} \left\| u^{(M)}(x) \right\|_{\mathcal{L}^p(a,b)}, \qquad a \leq y \leq b.$$

Since $\int_a^b \left|E_{M-1}(\frac{|x-y|}{b-a})\right|^q \mathrm{d}x = (b-a)\int_0^1 |E_{M-1}(x)|^q \,\mathrm{d}x$ holds for all $y \in [a,b]$, previous inequality can be rewritten as follows

$$\sup_{a \leq y \leq b} |u(y)| \leq \frac{(b-a)^{M-1+\frac{1}{q}}}{2(M-1)!} ||E_{M-1}(x)||_{\mathcal{L}^q(0,1)} \left\| u^{(M)}(x) \right\|_{\mathcal{L}^p(a,b)}. \tag{2.8}$$

This shows that the best constant is not greater than right-hand side of (2.7). Now, we prove second part of the theorem. Let f be defined as above. Then u is the solution to the boundary value problem (2.1). Note that $u \in W_M$. Interchanging x and y, we obtain

$$u(y) = \frac{(-1)^{M+\lfloor\frac{M+1}{2}\rfloor}(b-a)^{M-1}}{2(M-1)!} \int_a^b E_{M-1}\left(\frac{|x-y|}{b-a}\right) (\mathrm{sgn}(x-y))^M f(x)\mathrm{d}x \quad (a \leq y \leq b).$$

Moreover, we have

$$u(a) = \frac{(-1)^{M+\lfloor\frac{M+1}{2}\rfloor}(b-a)^{M-1}}{2(M-1)!} \int_a^b E_{M-1}\left(\frac{x-a}{b-a}\right) f(x)\,\mathrm{d}x$$

and this yields

$$\begin{aligned}
u(a) &= \frac{(b-a)^{M-1}}{2(M-1)!} \int_a^b \left| E_{M-1}\left(\frac{x-a}{b-a}\right) \right|^q \mathrm{d}x \\
&= \frac{(b-a)^{M-1}}{2(M-1)!} \left\| E_{M-1}\left(\frac{x-a}{b-a}\right) \right\|^q_{\mathcal{L}^q(a,b)} \\
&= \frac{(b-a)^{M-1}}{2(M-1)!} \left\| E_{M-1}\left(\frac{x-a}{b-a}\right) \right\|_{\mathcal{L}^q(a,b)} \left\| E_{M-1}\left(\frac{x-a}{b-a}\right) \right\|^{\frac{q}{p}}_{\mathcal{L}^q(a,b)} \\
&= \frac{(b-a)^{M-\frac{1}{p}}}{2(M-1)!} \|E_{M-1}(x)\|_{\mathcal{L}^q(0,1)} \|f(x)\|_{\mathcal{L}^p(a,b)} \\
&= \frac{(b-a)^{M-\frac{1}{p}}}{2(M-1)!} \|E_{M-1}(x)\|_{\mathcal{L}^q(0,1)} \left\| u^{(M)}(x) \right\|_{\mathcal{L}^p(a,b)}.
\end{aligned}$$

Since $u(a) \leq \sup_{a \leq y \leq b} |u(y)|$, this together with (2.8) shows that we have constructed u in which the supremum of Sobolev inequality is attained. □

3 Application

The well-known Lyapunov inequality states that if $r\colon [a,b] \to \mathbb{R}$ is a continuous function, then a necessary condition for the Dirichlet boundary value problem

$$\begin{cases} u'' + r(t)u = 0, & a < t < b \\ u(a) = u(b) = 0, \end{cases} \tag{3.1}$$

M	$C(M)^p$	$u(x)$ on $[a,b]$
1	$\frac{(b-a)^{p-1}}{2^p}$	$\frac{a+b}{2} - x$
2	$\frac{(b-a)^{2p-1}}{2^{2p}(q+1)^{p-1}}$	$\frac{(b-a)(x-\frac{a+b}{2})}{2^q(q+1)}\left[\frac{1}{2^{q+1}} - \frac{\lvert x-\frac{a+b}{2}\rvert^{q+1}}{(b-a)^{q+1}(q+2)}\right]$
3	$\frac{(b-a)^{3p-1}\Gamma(1+q)^{2(p-1)}}{2^{2p}\Gamma(2q+2)^{p-1}}$	$\frac{\int_a^b \operatorname{sgn}(x-y)[(b-y)(y-a)]^q\,\mathrm{d}y}{4(b-a)^{2(q-1)}}$

Table 2.2: Special cases of best constants and best functions.

to have nontrivial solutions is that

$$\int_a^b |r(t)|\,\mathrm{d}t > \frac{4}{b-a} \tag{3.2}$$

and the constant 4 cannot be replaced by a larger number. Such result has found many practical uses in problems as oscillation theory or eigenvalue problems (spectral properties of differential equations). Several proofs and generalizations or improvements for various boundary conditions have appeared in the literature. Recently, the author in [10] obtained a new Lyapunov-type inequality, a generalization of (3.2), for a certain anti-periodic problem.

Theorem 3.1. *Consider the following $(m+1)$-order half-linear boundary value problem*

$$\begin{cases} \left(|u^{(m)}|^{p-2}u^{(m)}\right)' + r(t)|u|^{p-2}u = 0, & a < t < b \\ u^{(k)}(a) + u^{(k)}(b) = 0, & k = 0,1,2,\dots,m. \end{cases} \tag{3.3}$$

If u is its nonzero solution, then

$$\int_a^b |r(t)|\mathrm{d}t > 2\left(\frac{2}{b-a}\right)^{m(p-1)} =: \frac{1}{\tilde{C}(m)^p}. \tag{3.4}$$

We now introduce an assertion, which sharpen the result of [10]. In the Table 3.1 we show how the inequality (3.6) improves (3.4).

Theorem 3.2. *If u is a nonzero solution of*

$$\begin{cases} \left(|u^{(m)}|^{p-2}u^{(m)}\right)' + r(t)|u|^{p-2}u = 0, & a < t < b \\ u^{(k)}(a) + u^{(k)}(b) = 0, & k = 0,1,2,\dots,m, \end{cases} \tag{3.5}$$

then the inequality

$$\int_a^b |r(t)|\,\mathrm{d}t > \frac{1}{C(m)^{p-1}C(1)} \tag{3.6}$$

holds.

Proof. Multiplying the first equation in (3.5) by $u^{(m-1)}(t)$ and integrating over $[a,b]$, we have

$$\int_a^b |u^{(m)}(t)|^p\,\mathrm{d}t = \int_a^b r(t)|u(t)|^{p-2}u(t)u^{(m-1)}(t)\,\mathrm{d}t.$$

Applying Theorem 2.4 to u and $u^{(m-1)}$ respectively, yields

$$\int_a^b |u^{(m)}(t)|^p \,\mathrm{d}t \le \int_a^b |r(t)| \,\mathrm{d}t \; C(1) \left(\int_a^b |u^{(m)}(t)|^p \,\mathrm{d}t \right)^{\frac{1}{p}} \; C^{p-1}(m) \left(\int_a^b |u^{(m)}(t)|^p \,\mathrm{d}t \right)^{\frac{p-1}{p}}.$$

Since u is nonzero solution, by dividing both sides by $\int_a^b |u^{(m)}(t)|^p \,\mathrm{d}t$, we obtain inequality (3.6). Moreover, this inequality is strict, since $u(t)$ is not a constant. □

m	1	2	3	4	5
$\tilde{C}(m)^2$	$\frac{b-a}{4}$	$\frac{(b-a)^2}{8}$	$\frac{(b-a)^3}{16}$	$\frac{(b-a)^4}{32}$	$\frac{(b-a)^5}{64}$
$C(m)C(1)$	$\frac{b-a}{4}$	$\frac{\sqrt{3}(b-a)^2}{24}$	$\frac{\sqrt{30}(b-a)^3}{240}$	$\frac{\sqrt{595}(b-a)^4}{3360}$	$\frac{\sqrt{2170}(b-a)^5}{20160}$

m	1	2	3
$\tilde{C}(m)^p$	$\frac{(b-a)^{p-1}}{2^p}$	$\frac{(b-a)^{2p-2}}{2^{2p-1}}$	$\frac{(b-a)^{3p-3}}{2^{3p-2}}$
$C(m)^{p-1}C(1)$	$\frac{(b-a)^{p-1}}{2^p}$	$\frac{(b-a)^{2p-2}}{2^{2p-1}(q+1)^{\frac{p-1}{q}}}$	$\frac{(b-a)^{3p-1}\Gamma(q+1)^{\frac{2(p-1)}{q}}}{2^{2p-1}\Gamma(2q+2)^{\frac{p-1}{q}}}$

Table 3.1: Comparison of constants of Lyapunov inequalities in Theorems 3.1, 3.2.

Now we establish a Lyapunov-type inequality for the half-linear equation of higher order with anti-periodic boundary value conditions. Notice that for $m = 1$ problems (3.5) and (3.7) coincide.

Theorem 3.3. *If u is a nonzero solution of*

$$\begin{cases} \left(|u^{(m)}|^{p-2} u^{(m)} \right)^{(m)} + r(t)|u|^{p-2}u = 0, & a < t < b \\ u^{(k)}(a) + u^{(k)}(b) = 0, & k = 0, 1, 2, \dots, 2m-1, \end{cases} \tag{3.7}$$

then the following inequality holds:

$$\int_a^b |r(t)| \,\mathrm{d}t > \frac{1}{C(m)^p}. \tag{3.8}$$

Proof. Let u be a solution of (3.7). Multiplying both sides of the first equation in (3.7) by $(-1)^m u$ and integrating from a to b we have

$$(-1)^m \int_a^b u(t) v^{(m)}(t) \,\mathrm{d}t = (-1)^{m+1} \int_a^b r(t)|u(t)|^p \,\mathrm{d}t,$$

where $v(t) = |u^{(m)}|^{p-2} u^{(m)}$. Integrating by parts the left-hand side of the former equation we obtain (notice that by a solution we mean a classical one, thus u, v belong to $C^m([a, b])$)

$$\sum_{i=0}^{m-1} \left[(-1)^{m+i} u^{(i)}(t) v^{(m-i-1)}(t) \right]_a^b + \int_a^b |u^{(m)}|^p \,\mathrm{d}t = (-1)^{m+1} \int_a^b r(t)|u(t)|^p \,\mathrm{d}t.$$

Moreover using the anti-periodic boundary conditions from (3.7) and the fact that $v^{(k)}(a) + v^{(k)}(b) = 0$, $k = 1, 2, \dots, m-1$, we have

$$\int_a^b |u^{(m)}|^p \,\mathrm{d}t = (-1)^{m+1} \int_a^b r(t)|u(t)|^p \,\mathrm{d}t \le \int_a^b |r(t)| \,\mathrm{d}t \left(\sup_{a \le t \le b} |u(t)| \right)^p.$$

Now we use Sobolev inequality from Theorem 2.4:

$$\frac{\left(\sup_{a\le t\le b}|u(t)|\right)^p}{C(m)^p} \le \int_a^b |r(t)|\,\mathrm{d}t\left(\sup_{a\le t\le b}|u(t)|\right)^p.$$

Dividing both sides by $\left(\sup_{a\le t\le b}|u(t)|\right)^p$, we obtain desired inequality. Again this inequality is strict. □

Acknowledgements

The present work was supported by grant VEGA MŠ SR 1/0344/14 and PF UPJŠ internal grant system (grant No. VVGS-2013-105). The author would like to thank the reviewer for careful reading and valuable comments that help improve the manuscript.

References

[1] T. S. Chihara, *An introduction to orthogonal polynomials*, Mathematics and its Applications, Vol. 13., Gordon and Breach Science Publishers, New York, 1978. MR0481884

[2] A. F. Horadam, Genocchi polynomials, *Applications of Fibonacci Numbers (Proc. of the 4th Internat. Conf., Bergum G. E., Philippou A. N., Horadam A. F., 1990)*, Kluwer Acad. Publ., Dordrecht, 1991, 145–166. MR1193711; url

[3] Y. Kametaka, Y. Oshime, K. Watanabe, H. Yamagishi, A. Nagai, K. Takemura, The best constant of L^p Sobolev inequality corresponding to the periodic boundary value problem for $(-1)^M(d/dx)^{2M}$, *Sci. Math. Jpn.* **66**(2007), 169–181. MR2348461

[4] H. Kleinert, A. Chervyakov, Functional determinants from Wronski Green functions, *J. Math. Phys.* **40**(1999), 6044–6051. MR1722352; url

[5] Y. Oshime, Y. Kametaka, H. Yamagishi, The best constant of L^p Sobolev inequality corresponding to Dirichlet boundary value problem for $(d/dx)^{4m}$, *Sci. Math. Jpn.*, **68**(2008), No. 3, 313–321. MR2468619

[6] Y. Oshime, K. Watanabe, The best constant of L^p Sobolev inequality corresponding to Dirichlet boundary value problem. II, *Tokyo J. Math.* **34**(2011), 115–133. MR2866638; url

[7] Y. Oshime, H. Yamagishi, K. Watanabe, The best constant of L^p Sobolev inequality corresponding to Neumann boundary value problem for $(-1)^M(d/dx)^{2M}$, *Hiroshima Math. J.* **42**(2012), 293–299. MR3050123

[8] S. Pinsky, U. Trittmann, Antiperiodic boundary conditions in supersymmetric discrete light cone quantization, *Phys. Rev. D* **62**(2000), 87701-87704. MR1804109; url

[9] S. Roman, *The umbral calculus*, Pure and Applied Mathematics, Academic Press, Inc., New York, 1984. MR0741185

[10] Y. Wang, Lyapunov-type inequalities for certain higher order differential equations with anti-periodic boundary conditions, *Appl. Math. Lett.* **25**(2012), 2375–2380. MR2967847; url

[11] K. Watanabe, Y. Kametaka, A. Nagai, H. Yamagishi, K. Takemura, Symmetrization of functions and the best constant of 1-dim L^p Sobolev inequality, *J. Inequal. Appl.* **12**(2009) MR2566064; url.

[12] K. Watanabe, Y. Kametaka, H. Yamagishi, A. Nagai, K. Takemura, The best constant of Sobolev inequality corresponding to clamped boundary value problem, *Bound. Value Probl.* **2011**, Art. ID 875057, 17 pp. MR2783107; url

[13] K. Watanabe, H. Yamagishi, Y. Kametaka, Riemann zeta function and Lyapunov-type inequalities for certain higher order differential equations, *Appl. Math. Comput.* **218**(2011), 3950–3953. MR2851492; url

[14] H. Yamagishi, Y. Kametaka, A. Nagai, K. Watanabe, K. Takemura, Riemann zeta function and the best constants of five series of Sobolev inequalities, in: *Expansion of integrable systems*, RIMS Kôkyûroku Bessatsu, B13, Res. Inst. Math. Sci. Kôkyûroku Bessatsu, 2009, 125–139. MR2642632

[15] H. Yamagishi, K. Watanabe, Y. Kametaka, The best constant of L^p Sobolev inequality corresponding to Dirichlet–Neumann boundary value problem, *Math. J. Okayama Univ.* **56**(2014), 145–155. MR3155087