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On Nonlinear Spectra for some Nonlocal Boundary Value Problems

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Abstract

We consider some second order nonlinear differential equation with nonlocal (integral) condition. This spectrum differs essentially from the known ones.

1 Introduction

Investigations of Fučik spectra have started in sixties of XX century. Let us mention the work [5] and the bibliography therein. Of the recent works let us mention [8], [9]. The Fučík spectra have been investigated for the second order equation with different two-point boundary conditions. There are fewer works about the higher order problems.

Our goal is to get formulas for the spectra (λ, μ) of the second order BVP

$$x'' = \begin{cases} -(\alpha+1)\mu^{2\alpha+2}|x|^{2\alpha}x, & \text{if} \quad x \geq 0, \\ -(\beta+1)\lambda^{2\beta+2}|x|^{2\beta}x, & \text{if} \quad x < 0, \end{cases} \tag{1}$$

($\alpha \geq 0$, $\beta \geq 0$, $\lambda > 0$, $\mu > 0$) with the boundary conditions

$$x(0) = 0, \quad \int\limits_0^1 x(s)ds = 0. \tag{2}$$

The spectra were obtained under the normalization condition $|x'(0)| = 1$, because without this condition problems may have continuous spectra.

This paper is organized as follows.

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We try to extend investigation of Fučík type spectra in two directions. The first one considers the classical equation with integral boundary condition ([10]). The second direction deals with equation of the type $x'' = -\mu f(x^+) + \lambda g(x^-)$ and the good reference is the work [9].

In Section 2 we present results on the Fučík spectrum for the problem $x'' = -\mu^2 x^+ + \lambda^2 x^-$ with the boundary conditions (2).

In Section 3 we consider the equation (1) with Dirichlet conditions.

In Section 4 we provide formulas for Fučík spectrum of the problem (1), (2). This is the main result of our work.

Our formulas for the spectra are given in terms of the functions $S_\alpha(t)$ and $S_\beta(t)$, which are generalizations of the lemniscatic functions [11] ($\mathrm{sl}\, t$ and $\mathrm{cl}\, t$), and their primitives $I_\alpha(t)$ and $I_\beta(t)$. Some properties of these functions are given in Subsection 4.1. The formulas for relations between lemniscatic functions and their derivatives are known from [3].The specific case of $\alpha = 0$, $\beta = 1$ is considered in details as in Example.

2 About some nonlinear problem with integral condition

Consider the second order BVP

$$x'' = -\mu^2 x^+ + \lambda^2 x^-, \quad \mu, \lambda > 0, \tag{3}$$

where $x^+ = \max\{x, 0\}$, $x^- = \max\{-x, 0\}$, with the boundary conditions

$$x(0) = 0; \quad \int_0^1 x(s)ds = 0. \tag{4}$$

Definition 2.1 *The Fučík spectrum is a set of points (λ, μ) such that the problem (3), (4) has nontrivial solutions.*

The first result describes decomposition of the spectrum into branches F_i^+ and F_i^- $(i = 0, 1, 2, \dots)$ for the problem (3), (4).

Proposition 2.1 *The Fučík spectrum $\sum = \bigcup\limits_{i=0}^{+\infty} F_i^\pm$ consists of a set of branches*
$F_i^+ = \{(\lambda, \mu) |\ x'(0) > 0,$ *the nontrivial solution $x(t)$ of the problem has exactly i zeroes in $(0, 1)\}$;*
$F_i^- = \{(\lambda, \mu) |\ x'(0) < 0,$ *the nontrivial solution $x(t)$ of the problem has exactly i zeroes in $(0, 1)\}$.*

Theorem 2.1 ([10], section 2) *The Fučík spectrum* $\sum = \bigcup_{i=1}^{+\infty} F_i^{\pm}$ *for the problem* (3), (4) *consists of the branches given by*

$$F_{2i-1}^{+} = \Big\{(\lambda,\mu)\Big|\ \frac{2i\lambda}{\mu} - \frac{(2i-1)\mu}{\lambda} - \frac{\mu\cos(\lambda - \frac{\lambda\pi i}{\mu} + \pi i)}{\lambda} = 0,\ \frac{i\pi}{\mu} + \frac{(i-1)\pi}{\lambda} \le 1,\ \frac{i\pi}{\mu} + \frac{i\pi}{\lambda} > 1\Big\},$$

$$F_{2i}^{+} = \Big\{(\lambda,\mu)\Big|\ \frac{(2i+1)\lambda}{\mu} - \frac{2i\mu}{\lambda} - \frac{\lambda\cos(\mu - \frac{\mu\pi i}{\lambda} + \pi i)}{\mu} = 0,\ \frac{i\pi}{\mu} + \frac{i\pi}{\lambda} \le 1,\ \frac{(i+1)\pi}{\mu} + \frac{i\pi}{\lambda} > 1\Big\},$$

$$F_{2i-1}^{-} = \Big\{(\lambda,\mu)\Big|\ \frac{2i\mu}{\lambda} - \frac{(2i-1)\lambda}{\mu} - \frac{\lambda\cos(\mu - \frac{\mu\pi i}{\lambda} + \pi i)}{\mu} = 0,\ \frac{(i-1)\pi}{\mu} + \frac{i\pi}{\lambda} \le 1,\ \frac{i\pi}{\mu} + \frac{i\pi}{\lambda} > 1\Big\},$$

$$F_{2i}^{-} = \Big\{(\lambda,\mu)\Big|\ \frac{(2i+1)\mu}{\lambda} - \frac{2i\lambda}{\mu} - \frac{\mu\cos(\lambda - \frac{\lambda\pi i}{\mu} + \pi i)}{\lambda} = 0,\ \frac{i\pi}{\mu} + \frac{i\pi}{\lambda} \le 1,\ \frac{i\pi}{\mu} + \frac{(i+1)\pi}{\lambda} > 1\Big\},$$

where $i = 1, 2, \ldots$.

Visualization of the spectrum to this problem is given in Figure 1.

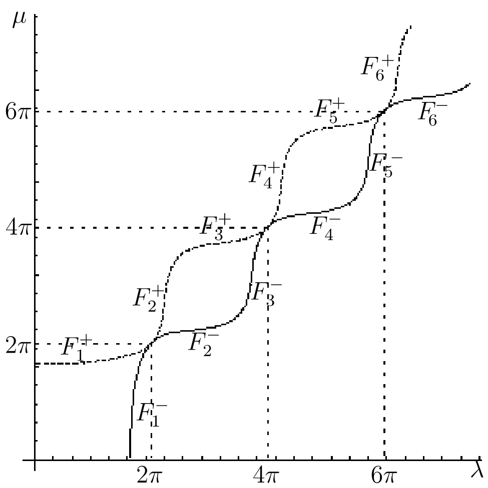

Fig. 1. The Fučík spectrum for the problem (3), (4).

3 Spectrum for the Fučík type problem with Dirichlet conditions

Consider the equation

$$x'' = \begin{cases} -(\alpha+1)\mu^{2\alpha+2}|x|^{2\alpha}x, & \text{if} \quad x \geq 0, \\ -(\beta+1)\lambda^{2\beta+2}|x|^{2\beta}x, & \text{if} \quad x < 0, \end{cases} \tag{5}$$

with the boundary conditions

$$x(0) = x(1) = 0, \quad |x'(0)| = 1, \tag{6}$$

where $\alpha, \beta \geq 0$, $\lambda, \mu > 0$.

Theorem 3.1 ([9], subsection 3.2.1) *The Fučík spectrum* $\sum = \bigcup\limits_{i=0}^{+\infty} F_i^{\pm}$ *for the problem* (5), (6) *consists of the branches given by*

$$\begin{aligned} F_0^+ &= \Big\{(\lambda, 2A_\alpha)\Big\}, \\ F_0^- &= \Big\{(2A_\beta, \mu)\Big\}, \\ F_{2i-1}^+ &= \Big\{(\lambda,\mu)\Big|\ i\frac{2A_\alpha}{\mu} + i\frac{2A_\beta}{\lambda} = 1\Big\}, \\ F_{2i}^+ &= \Big\{(\lambda,\mu)\Big|\ (i+1)\frac{2A_\alpha}{\mu} + i\frac{2A_\beta}{\lambda} = 1\Big\}, \\ F_{2i-1}^- &= \Big\{(\lambda,\mu)\Big|\ i\frac{2A_\alpha}{\mu} + i\frac{2A_\beta}{\lambda} = 1\Big\}, \\ F_{2i}^- &= \Big\{(\lambda,\mu)\Big|\ i\frac{2A_\alpha}{\mu} + (i+1)\frac{2A_\beta}{\lambda} = 1\Big\}, \end{aligned}$$

where $A_\alpha = \int_0^1 \frac{ds}{\sqrt{1-s^{2\alpha+2}}}$, $A_\beta = \int_0^1 \frac{ds}{\sqrt{1-s^{2\beta+2}}}$ $i = 1, 2, \ldots$.

The respective Fučík spectrum in the case of semilinear problem (for $\alpha = 0$, $\beta = 1$)

$$x'' = -\mu^2 x^+ + 2\lambda^4 x^{3-}, \quad \lambda, \mu > 0, \quad x(0) = x(1) = 0, \quad |x'(0)| = 1 \tag{7}$$

is depicted in Figure 2.

Remark 3.1 *To simplify our formulas we consider equation in the form* (5), *but in the work [9] the authors consider the eguation* $x'' = -\mu x^+ + \lambda x^-$.

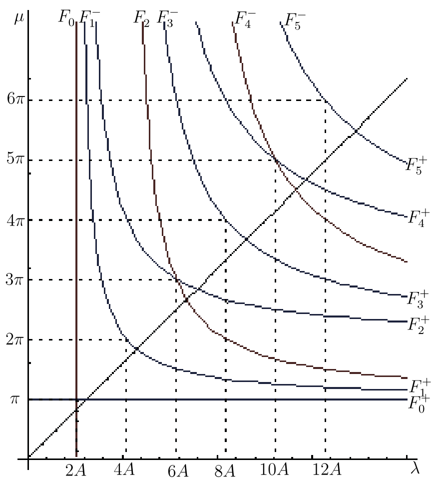

Fig. 2. The Fučík spectrum for the problem (5), (6), the first six pairs of branches.

4 Spectrum for the Fučík type problem with integral condition

4.1 Some auxiliary results

The function $S_n(t)$ is defined as a solution of the initial value problem

$$x'' = -(n+1)x^{2n+1}, \qquad x(0) = 0, \quad x'(0) = 1. \tag{8}$$

where n is a positive integer.

Functions $S_n(t)$ possess some properties of the usual $\sin t$ functions (notice that $S_0(t) = \sin t$). We mention several properties of these functions which are needed in our investigations. The functions $S_n(t)$:

1. are continuous and differentiable;

2. are periodic with the minimal period $4A_n$, where $A_n = \int\limits_0^1 \frac{ds}{\sqrt{1-s^{2n+2}}}$;

3. take maximal value $+1$ at the points $(4i+1)A_n$ and minimal value -1 at the points $(4i-1)A_n$ $(i = 0, \pm 1, \pm 2, \ldots)$;

4. take zeroes at the points $2iA_n$.

For boundary value problems with the integral condition the following remark may be of value.

Consider

$$I_n(t) := \int_0^t S_n(\xi)\, d\xi. \tag{9}$$

This function is periodic with minimal period $4A_n$ and can be expressed in terms of the so called hypergeometric functions.

Remark 4.1 *A solution of the problem*

$$x'' = -(n+1)\gamma^{2n+2}|x|^{2n}x, \qquad x(0) = 0, \;\; x'(0) = 1. \tag{10}$$

can be written in terms of $S_n(t)$ *as* $x(t) = \frac{1}{\gamma}S_n(\gamma t)$.

4.2 The spectrum

Consider the problem

$$x'' = \begin{cases} -(\alpha+1)\mu^{2\alpha+2}|x|^{2\alpha}x, & \text{if} \quad x \geq 0, \\ -(\beta+1)\lambda^{2\beta+2}|x|^{2\beta}x, & \text{if} \quad x < 0, \end{cases} \tag{11}$$

$$x(0) = 0, \;\; \int_0^1 x(s)ds = 0, \;\; |x'(0)| = 1, \tag{12}$$

where $\alpha, \beta \geq 0$, $\lambda, \mu > 0$.

Theorem 4.1 *The Fučík spectrum* $\sum = \bigcup_{i=1}^{+\infty} F_i^{\pm}$ *for the problem* (11), (12) *consists of the branches given by*

$$F_{2i-1}^{+} = \Big\{(\lambda,\mu)\Big|\; i\frac{\lambda}{\mu}I_\alpha(2A_\alpha) - (i-1)\frac{\mu}{\lambda}I_\beta(2A_\beta) + \frac{\mu}{\lambda}I_\beta(\lambda - \frac{2i\lambda A_\alpha}{\mu} - 2iA_\beta) = 0,$$
$$\frac{2A_\alpha}{\mu}i + \frac{2A_\beta}{\lambda}(i-1) \leq 1, \;\; \frac{2A_\alpha}{\mu}i + \frac{2A_\beta}{\lambda}i > 1\Big\},$$

$$F_{2i}^{+} = \Big\{(\lambda,\mu)\Big|\; i\frac{\lambda}{\mu}I_\alpha(2A_\alpha) - i\frac{\mu}{\lambda}I_\beta(2A_\beta) + \frac{\lambda}{\mu}I_\alpha(\mu - \frac{2i\mu A_\beta}{\lambda} - 2iA_\alpha) = 0,$$
$$\frac{2A_\alpha}{\mu}i + \frac{2A_\beta}{\lambda}i \leq 1, \;\; \frac{2A_\alpha}{\mu}(i+1) + \frac{2A_\beta}{\lambda}i > 1\Big\},$$

$$F_{2i-1}^{-} = \Big\{(\lambda,\mu)\Big|\; i\frac{\mu}{\lambda}I_\beta(2A_\beta) - (i-1)\frac{\lambda}{\mu}I_\alpha(2A_\alpha) + \frac{\lambda}{\mu}I_\alpha(\mu - \frac{2i\mu A_\beta}{\lambda} - 2iA_\alpha) = 0,$$
$$\frac{2A_\alpha}{\mu}(i-1) + \frac{2A_\beta}{\lambda}i \leq 1, \;\; \frac{2A_\alpha}{\mu}i + \frac{2A_\beta}{\lambda}i > 1\Big\},$$

$$F_{2i}^{-} = \Big\{(\lambda,\mu)\Big|\; i\frac{\mu}{\lambda}I_\beta(2A_\beta) - i\frac{\lambda}{\mu}I_\alpha(2A_\alpha) + \frac{\mu}{\lambda}I_\beta(\lambda - \frac{2i\lambda A_\alpha}{\mu} - 2iA_\beta) = 0,$$
$$\frac{2A_\alpha}{\mu}i + \frac{2A_\beta}{\lambda}i \leq 1, \;\; \frac{2A_\alpha}{\mu}i + \frac{2A_\beta}{\lambda}(i+1) > 1\Big\},$$

where $i = 1, 2, \ldots$.

Proof. Consider the problem (11), (12).

It is clear that $x(t)$ must have zeroes in $(0,1)$. That is why $F_0^{\pm} = \emptyset$.

We will prove the theorem for the case of F_{2i-1}^{+}. Suppose that $(\lambda,\mu) \in F_{2i-1}^{+}$ and let $x(t)$ be a respective nontrivial solution of the problem (11), (12). The solution has $2i-1$ zeroes in $(0,1)$ and $x'(0) = 1$. Let these zeroes be denoted by τ_1, τ_2 and so on.

Consider a solution of the problem (11), (12) in the intervals $(0,\tau_1)$, (τ_1,τ_2), $\dots$, $(\tau_{2i-1},1)$. Notice that $|x'(\tau_j)| = 1$ $(j = 1,\dots,2i-1)$. We obtain that the problem (11), (12) in these intervals reduces to the eigenvalue problems. So in the odd intervals we have the problem $x'' = -(\alpha+1)\mu^{2\alpha+2}x^{2\alpha+1}$ with boundary conditions $x(0) = x(\tau_1) = 0$ in the first such interval and with boundary conditions $x(\tau_{2i-2}) = x(\tau_{2i-1}) = 0$ in the remaining ones, but in the even intervals we have the problem $x'' = -(\beta+1)\lambda^{2\beta+2}x^{2\beta+1}$ with boundary condition $x(\tau_{2i-3}) = x(\tau_{2i-2}) = 0$ in each such interval, but for the last one the only condition is $x(\tau_{2i-1}) = 0$. In view of (12) a solution $x(t)$ must satisfy the condition

$$\int_0^{\tau_1} x(s)ds + \int_{\tau_2}^{\tau_3} x(s)ds + \ldots + \int_{\tau_{2i-2}}^{\tau_{2i-1}} x(s)ds =$$

$$= \Big| \int_{\tau_1}^{\tau_2} x(s)ds + \int_{\tau_3}^{\tau_4} x(s)ds + \ldots + \int_{\tau_{2i-1}}^{1} x(s)ds \Big|. \quad (13)$$

Since $x(t) = S_\alpha(\mu t)$ in the interval $(0,\tau_1)$ and $x(\tau_1) = 0$ we obtain $\tau_1 = \frac{2A_\alpha}{\mu}$. Similarly we obtain for the other zeroes

$$\tau_2 = \frac{2A_\alpha}{\mu} + \frac{2A_\beta}{\lambda},$$

$$\tau_3 = 2\frac{2A_\alpha}{\mu} + \frac{2A_\beta}{\lambda},$$

$$\dots,$$

$$\tau_{2i-2} = (i-1)\frac{2A_\alpha}{\mu} + (i-1)\frac{2A_\beta}{\lambda},$$

$$\tau_{2i-1} = i\frac{2A_\alpha}{\mu} + (i-1)\frac{2A_\beta}{\lambda}.$$

In view of these facts it is easy to get that $\int_0^{\tau_1} x(s)ds = \frac{1}{\mu^2} I_\alpha(2A_\alpha)$.

Analogously

$$\int_{\tau_2}^{\tau_3} x(s)ds = \int_{\tau_4}^{\tau_5} x(s)ds = \ldots = \int_{\tau_{2i-2}}^{\tau_{2i-1}} x(s)ds = \frac{1}{\mu^2} I_\alpha(2A_\alpha).$$

Therefore

$$\int_{0}^{\tau_1} x(s)ds + \int_{\tau_2}^{\tau_3} x(s)ds + \ldots + \int_{\tau_{2i-2}}^{\tau_{2i-1}} x(s)ds = i\frac{1}{\mu^2} I_\alpha(2A_\alpha).$$

Now we consider a solution of the problem (11), (12) in the remaining intervals. Since $x(t) = -S_\beta(\lambda t - \lambda\tau_1)$ in (τ_1, τ_2) we obtain $\int_{\tau_1}^{\tau_2} x(s)ds = -\frac{1}{\lambda^2} I_\beta(2A_\beta)$.

Analogously

$$\int_{\tau_3}^{\tau_4} x(s)ds = \int_{\tau_5}^{\tau_6} x(s)ds = \ldots = \int_{\tau_{2i-3}}^{\tau_{2i-2}} x(s)ds = -\frac{1}{\lambda^2} I_\beta(2A_\beta).$$

But in the last interval $(\tau_{2i-1}, 1)$ we obtain

$$\int_{\tau_{2i-1}}^{1} x(s)ds = \frac{1}{\lambda^2} I_\beta\big(\lambda - \lambda\frac{2A_\alpha}{\mu} i - 2A_\beta i\big).$$

It follows from the last two lines that

$$\Big|\int_{\tau_1}^{\tau_2} x(s)ds + \int_{\tau_3}^{\tau_4} x(s)ds + \ldots + \int_{\tau_{2i-3}}^{\tau_{2i-2}} x(s)ds + \int_{\tau_{2i-1}}^{1} x(s)ds\Big| =$$
$$= (i-1)\frac{1}{\lambda^2} I_\beta(2A_\beta) - \frac{1}{\lambda^2} I_\beta\big(\lambda - \lambda\frac{2A_\alpha}{\mu} i - 2A_\beta i\big).$$

In view of the last equality and (13) we obtain

$$i\frac{1}{\mu^2} I_\alpha(2A_\alpha) = (i-1)\frac{1}{\lambda^2} I_\beta(2A_\beta) - \frac{1}{\lambda^2} I_\beta\big(\lambda - \lambda\frac{2A_\alpha}{\mu} i - 2A_\beta i\big).$$

Multiplying it by $\mu\lambda$, we obtain

$$i\frac{\lambda}{\mu} I_\alpha(2A_\alpha) - (i-1)\frac{\mu}{\lambda} I_\beta(2A_\beta) + \frac{\mu}{\lambda} I_\beta(\lambda - \frac{2i\lambda A_\alpha}{\mu} - 2iA_\beta) = 0. \qquad (14)$$

Considering the solution of the problem (11), (12) it is easy to prove that $\tau_{2i-1} \leq 1 < \tau_{2i}$ or $\frac{2A_\alpha}{\mu} i + \frac{2A_\beta}{\lambda}(i-1) \leq 1 < \frac{2A_\alpha}{\mu} i + \frac{2A_\beta}{\lambda} i$.

This result and (14) prove the theorem for the case of F^+_{2i-1}. The proof for other branches is analogous. ■

Remark 4.2 *If* $\alpha = \beta = 0$ *we obtain the problem* (3), (4). *The spectrum of this problem is given in Figure 1.*

4.3 Some properties of the spectrum

Now we would like to point out some properties of the spectrum for the problem (11), (12).

Theorem 4.2 *The following properties of the spectrum for the problem* (11), (12) *hold:*

1. *the union of the positive branches and the negative ones form the continuous curves;*
2. *each branch is finite;*
3. *the positive part of the spectrum and the negative one intersect at the points* $\Big(i(\frac{2A_\alpha}{\Delta}+2A_\beta), i(2A_\beta\Delta+2A_\alpha)\Big)$. *These points belong to the straight line* $\mu=\lambda\Delta$. *The odd-numbered and the even-numbered branches are separated by these points.*
4. *the even-numbered and the odd-numbered branches are separated by the points*

$$F^+_{2i}\cap F^+_{2i+1}=\Big(\sqrt{i(i+1)}\frac{2A_\alpha}{\Delta}+i2A_\beta;2A_\beta\sqrt{i(i+1)}\Delta+(i+1)2A_\alpha)\Big),$$

$$F^-_{2i}\cap F^-_{2i+1}=\Big(\sqrt{i(i+1)}\frac{2A_\alpha}{\Delta}+(i+1)2A_\beta;2A_\beta\sqrt{i(i+1)}\Delta+i2A_\alpha)\Big),$$

where $\Delta=\sqrt{\frac{I_\alpha(2A_\alpha)}{I_\beta(2A_\beta)}}$, $i=1,2,\ldots$.

Proof. The properties 1 and 2 are direct consequence of the proof of Theorem 4.1.

The proof of the properties 3 and 4.

It is clear that the positive part of the spectrum and the negative one intersect at the point in which the odd branches of the Fučík spectrum for the problem (11), (12) intersect with the respective branches of the problem (5), (6).

Thus there are such (λ,μ) values that

$$i\frac{2A_\alpha}{\mu}+i\frac{2A_\beta}{\lambda}=1 \tag{15}$$

and at the same time

$$i\frac{1}{\mu^2}I_\alpha(2A_\alpha)=i\frac{1}{\lambda^2}I_\beta(2A_\beta). \tag{16}$$

In view of (16) we obtain that $\mu=\lambda\sqrt{\frac{I_\alpha(2A_\alpha)}{I_\beta(2A_\beta)}}$. It follows from this expression and (15) that $\lambda=i\Big(2A_\alpha\sqrt{\frac{I_\beta(2A_\beta)}{I_\alpha(2A_\alpha)}}+2A_\beta\Big)$, but $\mu=i\Big(2A_\alpha+2A_\beta\sqrt{\frac{I_\alpha(2A_\alpha)}{I_\beta(2A_\beta)}}\Big)$.

This result proves the property 3. The proof for the property 4 is analogous. ■

4.4 The example for $\alpha = 0$, $\beta = 1$

Now we consider the problem (11), (12) for the case of $\alpha = 0$, $\beta = 1$.

The problem can be written also as

$$x'' = -\mu^2 x^+ + 2\lambda^4 (x^-)^3, \quad \mu, \lambda \geq 0, \tag{17}$$

$$x^+ = \max\{x, 0\}, \quad x^- = \max\{-x, 0\}$$

with the boundary conditions

$$x(0) = 0, \quad \int_0^1 x(s)ds = 0, \quad |x'(0)| = 1. \tag{18}$$

Theorem 4.3 *The Fučík spectrum for the problem* (17), (18) *consists of the branches given by*

$$F^+_{2i-1} = \Big\{(\lambda,\mu)\Big| \; \frac{2i\lambda}{\mu} - \frac{(2i-1)\mu}{\lambda}\frac{\pi}{4} - \frac{\mu \arctan \mathrm{cl}\,(\lambda - \lambda\frac{\pi}{\mu}i - 2Ai)}{\lambda} = 0,$$
$$i\frac{\pi}{\mu} + (i-1)\frac{2A}{\lambda} \leq 1, \; i\frac{\pi}{\mu} + i\frac{2A}{\lambda} > 1\Big\},$$

$$F^+_{2i} = \Big\{(\lambda,\mu)\Big| \; \frac{(2i+1)\lambda}{\mu} - \frac{2i\mu}{\lambda}\frac{\pi}{4} - \frac{\lambda\cos(\mu - \mu\frac{2A}{\lambda}i - \pi i)}{\mu} = 0,$$
$$i\frac{\pi}{\mu} + i\frac{2A}{\lambda} \leq 1, \; (i+1)\frac{\pi}{\mu} + i\frac{2A}{\lambda} > 1\Big\},$$

$$F^-_{2i-1} = \Big\{(\lambda,\mu)\Big| \; \frac{2i\mu}{\lambda}\frac{\pi}{4} - \frac{(2i-1)\lambda}{\mu} - \frac{\lambda\cos(\mu - \mu\frac{2A}{\lambda}i - \pi i)}{\mu} = 0,$$
$$(i-1)\frac{\pi}{\mu} + i\frac{2A}{\lambda} \leq 1, \; i\frac{\pi}{\mu} + i\frac{2A}{\lambda} > 1\Big\},$$

$$F^-_{2i} = \Big\{(\lambda,\mu)\Big| \; \frac{(2i+1)\mu}{\lambda}\frac{\pi}{4} - \frac{2i\lambda}{\mu} - \frac{\mu \arctan \mathrm{cl}\,(\lambda - \lambda\frac{\pi}{\mu}i - 2Ai)}{\lambda} = 0,$$
$$i\frac{\pi}{\mu} + i\frac{2A}{\lambda} \leq 1, \; i\frac{\pi}{\mu} + (i+1)\frac{2A}{\lambda} > 1\Big\},$$

where $\mathrm{cl}\,(t)$ *is the lemniscatic cosine function,* $A = \int_0^1 \frac{ds}{\sqrt{1-s^4}}$, $i = 1, 2, \ldots$.

Proof. We will prove this theorem only for F^+_{2i-1}. The proof for other branches is analogues.

It is well-known that $S_0(t) = \sin(t)$, $S_1(t) = \mathrm{sl}\,(t)$, where $\mathrm{sl}\, t$ is the lemniscatic sine function.

It is known ([1]) that $\int_0^t \mathrm{sl}\, s ds = \frac{\pi}{4} - \arctan \mathrm{cl}\, t$.

Direct computation shows that $A_0 = \int_0^1 \frac{ds}{\sqrt{1-s^2}} = \frac{\pi}{2}$.

Thus we obtain

$$I_0(2A_0) = I_0(\pi) = \int_0^\pi \sin s\, ds = 2,$$

$$I_1(2A) = \frac{\pi}{4} - \arctan \operatorname{cl} 2A = \frac{\pi}{4} - \arctan(-1) = 2\frac{\pi}{4},$$

$$I_1(\lambda - \frac{2i\lambda\pi}{\mu} - 2iA) = \frac{\pi}{4} - \arctan \operatorname{cl}(\lambda - \frac{2i\lambda\pi}{\mu} - 2iA).$$

Equation from Theorem 4.1 using these expressions can be written as

$$i\frac{\lambda}{\mu}I_0(\pi) - (i-1)\frac{\mu}{\lambda}I_1(2A) + \frac{\mu}{\lambda}I_1(\lambda - \frac{2i\lambda\pi}{\mu} - 2iA) =$$
$$= 2i\frac{\lambda}{\mu} - 2(i-1)\frac{\mu}{\lambda}\frac{\pi}{4} + \frac{\mu}{\lambda}(\frac{\pi}{4} - \arctan \operatorname{cl}(\lambda - \frac{2i\lambda\pi}{\mu} - 2iA)) =$$
$$= \frac{2i\lambda}{\mu} - \frac{(2i-1)\mu}{\lambda}\frac{\pi}{4} - \frac{\mu}{\lambda}\arctan \operatorname{cl}(\lambda - \frac{2i\lambda\pi}{\mu} - 2iA)) = 0.$$

■

Visualization of the spectrum to this problem is given in Figure 3.

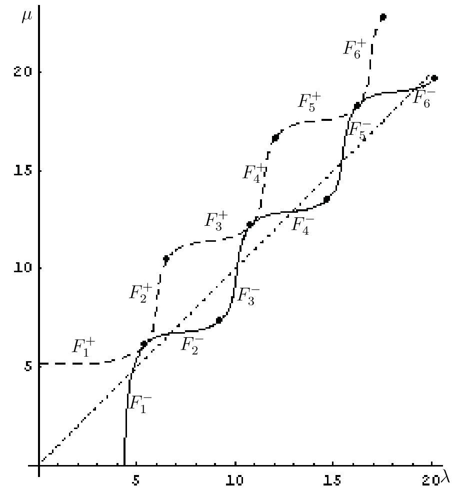

Fig. 3. The Fučík spectrum for the problem (17), (18).

It is easy to see that all properties mentioned in Theorem 4.2 hold.

Remark 4.3 *Let us mention also that the proof of Theorem 4.3 can be conducted as that for the Theorem 4.1.*

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