Disconjugate operators and related differential equations

Mariella Cecchi, Zuzana Došlá and Mauro Marini

Dedicated to J. Vosmanský on occasion of his 65^{th} *birthday*

Abstract: There are studied asymptotic properties of solutions of the nonlinear differential equation $L_n x + p(t)f(x) = 0$, where L_n is disconjugate operator, p is of one sign and $f(u)u > 0$ for $u \neq 0$. Some comparison theorems in terms of property A and property B between linear and nonlinear equations are also given which generalize known results for $n = 3$.

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1 Introduction

Consider the $n-$th order differential equations

$$\left(\frac{1}{a_{n-1}(t)}\left(\cdots\left(\frac{1}{a_1(t)}x'\right)'\cdots\right)'\right)' + p(t)f(x(t)) = 0 \tag{N}$$

$$\left(\frac{1}{a_1(t)}\left(\cdots\left(\frac{1}{a_{n-1}(t)}u'\right)'\cdots\right)'\right)' + (-1)^n p(t)f(u(t)) = 0, \tag{N$^{\mathcal{A}}$}$$

where

$$\begin{gathered} a_i,\, p \in C^0([0,\infty),\mathbb{R}),\ a_i(t) > 0,\ i = 1,\ldots,n-1,\ p(t) \neq 0, \\ f \in C^0(\mathbb{R},\mathbb{R}),\quad f(u)u > 0 \ \text{ for } u \neq 0. \end{gathered} \tag{H}$$

[1] This paper is in the final form and no version of it will be submitted for publication elsewhere.

The notation ($\mathrm{N}^{\mathcal{A}}$) is suggested by the fact that in the linear case, i.e. for equation

$$\left(\frac{1}{a_{n-1}(t)}\left(\cdots\left(\frac{1}{a_1(t)}x'\right)'\cdots\right)'\right)' + p(t)x = 0 \tag{L}$$

the *adjoint equation* to (L) is

$$\left(\frac{1}{a_1(t)}\left(\cdots\left(\frac{1}{a_{n-1}(t)}u'\right)'\cdots\right)'\right)' + (-1)^n p(t)u = 0. \tag{$\mathrm{L}^{\mathcal{A}}$}$$

When, for some $i \in \{1, 2, .., n-1\}$, the function a_i is not suitably continuously differentiable, then (N) is to be interpreted as a first order differential system for the vector $(x^{[0]}, x^{[1]}, ..., x^{[n-1]})$ given by

$$x^{[0]}(t) = x(t), \quad x^{[1]}(t) = \frac{1}{a_1(t)}x'(t), \ldots, \; x^{[n-1]}(t) = \frac{1}{a_{n-1}(t)}\left(x^{[n-2]}(t)\right)'.$$

The functions $x^{[i]}, i = 0, 1, ..., n-1$ are called the *quasiderivatives* of x. Similarly $x^{[n]}$ denote the function given by $x^{[n]}(t) = (x^{[n-1]}(t))'$. For ($\mathrm{N}^{\mathcal{A}}$) we may proceed in a similar way.

As usual, a function u defined on (t_0, ∞) $(t_0 \geq 0)$ is said to be *a proper solution* of (N) if for any $t \in (t_0, \infty)$ it satisfies (N) and for any $\tau \in (t_0, \infty)$

$$\sup \{|u(t)| : \; t \in [\tau, \infty)\} > 0.$$

A proper solution of (N) is called *oscillatory* if it has a sequence of zeros tending to ∞; otherwise it is called *nonoscillatory*.

The $n-$th order differential operator

$$L_n x \equiv \frac{d}{dt}\frac{1}{a_{n-1}(t)}\cdots\frac{d}{dt}\frac{1}{a_1(t)}\frac{d}{dt}x \tag{1}$$

associated to (N) is disconjugate on the interval $[0, \infty)$, i.e. any solution of $L_n x = 0$ has at most $n-1$ zeros on $[0, \infty)$. Differential equations associated to disconjugate operators have been deeply studied: see, e.g., the monographs [7, 10, 13, 14] and references contained therein. In particular, a classification of solutions has been given in [2, 8] and the case $n = 3$ has been considered in [3, 4, 5]. For other references and related results we refer the reader to these papers.

To study the asymptotic behavior of higher order differential equations associated to disconjugate operators, in sixties, A. Kondratev and I. Kiguradze introduced the following definitions

Definition 1. A proper solution x of the equation (N) is said to be a *Kneser solution* if it satisfies for all large t

$$x^{[i]}(t)x^{[i+1]}(t) < 0, \qquad i = 0, 1, \ldots, n-1. \tag{2}$$

A proper solution x of the equation (N) is said to be *strongly monotone solution* if it satisfies for all large t

$$x^{[i]}(t)x^{[i+1]}(t) > 0, \qquad i = 0, 1, \ldots, n-1. \tag{3}$$

Definition 2. Assume p positive. Equation (N) is said to have *property A* if, for n even, any proper solution x of (N) is oscillatory, and, for n odd, it is either oscillatory or Kneser solution satisfying

$$\lim_{t\to\infty} x^{[i]}(t) = 0, \quad i = 0, 1, \ldots, n-1. \tag{4}$$

Assume p negative. Equation (N) is said to have *property B* if, for n even, any proper solution x of (N) is either oscillatory or Kneser solution satisfying (4) or strongly monotone solution satisfying

$$\lim_{t\to\infty} |x^{[i]}(t)| = \infty, \quad i = 0, 1, \ldots, n-1, \tag{5}$$

and, for n odd, any solution is either oscillatory or strongly monotone solution satisfying (5).

When $n = 3$, in [4, 5] some comparison results for properties A and B have been given jointly with some relationships between the linear and nonlinear case. The aim of this paper is to extend such a study to nonlinear higher order differential equations. More precisely, by using a linearization device and a recent result [6] on equivalence between property A or B for the linear equation (L) and its adjoint ($\mathrm{L}^{\mathcal{A}}$), we will give some comparison results concerning properties A or B. Our assumptions on nonlinearity are related with the behavior of f only in a neighbourhood of zero and/or of infinity. No monotonicity conditions are required as well as no assumptions involving the behavior of f in the whole $\mathbb{R}$ are supposed.

The paper is organized as follows. Section 2 summarizes the main results on properties A and B for linear equations proved in [6] and section 3 is devoted to nonlinear case: some comparison theorems are established jointly with some results on the asymptotic behavior of nonoscillatory solutions.

We close the introduction with some notation. Let u_i, $1 \le i \le n$, be positive and continuous functions on I. As in [8, 10, 17], put for $r > s \ge 0$ and $k = 1, \dots, n$

$$I_0 \equiv 1,$$
$$I_k(r, s; u_1, \dots, u_k) = \int_s^r u_1(\tau) I_{k-1}(\tau, s; u_2, \dots u_k)\, d\tau$$

that is,

$$I_k(r, s; u_1, \dots, u_k) = \int_s^r u_1(\tau_1) \int_s^{\tau_1} \cdots \int_s^{\tau_{k-1}} u_k(\tau_k)\, d\tau_k \dots\, d\tau_1.$$

The class of Kneser and strongly monotone solutions will be denoted by $\mathcal{N}_0$ and $\mathcal{N}_n$, respectively. Remark that, taking into account the sign of $x^{[n]}$, Kneser solutions of (N) and strongly monotone solutions of ($\mathrm{N}^{\mathcal{A}}$) can exist only in the cases

$$I)\ p \text{ positive},\ n \text{ odd}; \qquad II)\ p \text{ negative},\ n \text{ even}. \tag{6}$$

Similarly Kneser solutions of ($\mathrm{N}^{\mathcal{A}}$) and strongly monotone solutions of (N) can exist only when p is negative.

Finally, in addition to (H), some of the following conditions in sections 3-4 will be assumed:

$$\limsup_{u \to 0} \frac{f(u)}{u} < \infty, \tag{H1}$$

$$\liminf_{|u| \to \infty} \frac{f(u)}{u} > 0, \tag{H2}$$

$$\liminf_{u \to 0} \frac{f(u)}{u} > 0, \tag{H3}$$

$$\limsup_{|u|\to\infty} \frac{f(u)}{u} < \infty, \tag{H4}$$

and

$$\int_0^\infty a_i(t)\, dt = \infty \qquad \text{for } i = 1, \ldots, n-1. \tag{C}$$

We recall that, when (C) holds, the disconjugate operator L_n is called *in the canonical form* , see, e.g., [21].

Assumptions (H1)-(H4) are motivated by the Emden–Fowler equation

$$\left(\frac{1}{a_{n-1}(t)}\left(\cdots\left(\frac{1}{a_1(t)}x'(t)\right)'\cdots\right)'\right)' + p(t)|x(t)|^\lambda \operatorname{sgn} x(t) = 0, \tag{EF}$$

where $\lambda > 0$. Clearly, nonlinearity in (EF) satisfies either (H1), (H2) (if $\lambda \geq 1$) or (H3), (H4) (if $\lambda \in (0, 1]$).

2 The linear case

Concerning the existence of Kneser solutions or strongly monotone solutions, in the linear case the following holds:

Lemma 1. *For the linear equation (L) it holds:*
(a) Let n be odd and $p(t) > 0$. Then $\mathcal{N}_0 \neq \emptyset$.
(b) Let n be even and $p(t) < 0$. Then $\mathcal{N}_0 \neq \emptyset$, $\mathcal{N}_n \neq \emptyset$.
(c) Let n be odd and $p(t) < 0$. Then $\mathcal{N}_n \neq \emptyset$.

Proof. It follows from [12, 16]. □

In the linear case the properties of solutions of (L) are strictly related with those of its adjoint ($\mathrm{L}^{\mathcal{A}}$) (see, for instance, Th. 8.24, Th.8.33 in [10] and, for $n = 3$, Ths.1.3–1.5 in [11]). In particular in [6], we have given an equivalence result concerning properties A or B for (L) and its adjoint ($\mathrm{L}^{\mathcal{A}}$). The main result is the following:

Theorem 1. ***(Equivalence Theorem)***

(a) Let n be even and $p(t) > 0$. Equation (L) has property A if and only if equation ($L^{\mathcal{A}}$) has property A.

(b) Let n be odd and $p(t) > 0$. Equation (L) has property A if and only if equation ($L^{\mathcal{A}}$) has property B.

(c) Let n be even and $p(t) < 0$. Equation (L) has property B if and only if equation ($L^{\mathcal{A}}$) has property B.

(d) Let n be odd and $p(t) < 0$. Equation (L) has property B if and only if equation ($L^{\mathcal{A}}$) has property A.

Some applications of this result has been given in [6]. For instance, when n is odd, it enables to obtain new criteria on property A from existing ones on property B and vice versa. When n is even it permits to produce new criteria on property A [B] from existing ones by interchanging roles of coefficients a_i and a_{n-i}.

A relationship between Kneser and strongly monotone solutions of (L) and ($L^{\mathcal{A}}$) is given by the following:

Proposition 1. *Let* (6) *be satisfied. The following statements are equivalent:*

(a) there exists a Kneser solution x of (L) satisfying $\lim_{t\to\infty} x^{[i]}(t) = 0$ *for* $i = 0, 1, \dots, n-1$*;*

(b) every Kneser solution x of (L) satisfies $\lim_{t\to\infty} x^{[i]}(t) = 0$ *for* $i = 0, 1, \dots, n-1$*;*

(c) it holds for $t \ge T \ge 0$

$$\lim_{t\to\infty} I_n(t, T; a_i, \dots, a_1, p, a_{n-1}, \dots, a_{i+1}) = \infty, \quad i = 1, \dots, n-1; \tag{7}$$

(d) there exists a strongly monotone solution u of ($L^{\mathcal{A}}$) satisfying $\lim_{t\to\infty} |u^{[i]}(t)| = \infty$ *for* $i = 0, 1, \dots, n-1$.

(e) every strongly monotone solution u of ($L^{\mathcal{A}}$) satisfies $\lim_{t\to\infty} |u^{[i]}(t)| = \infty$ *for* $i = 0, 1, \dots, n-1$.

Proof. The assertion is an easy consequence of Lemma 1 and Lemmas 3-5 in [6]. More precisely, from Lemma 1, (L) and ($L^{\mathcal{A}}$) have both Kneser and strongly monotone solutions. The statement (a) $\Longrightarrow$ (c) follows from Lemma 3-(a), (c) $\Longrightarrow$ (b) from Lemma 4-(a). From Lemma 5-(a), we get (d) $\Longrightarrow$ (c) and (c) $\Longrightarrow$ (e). Obviously, (b) $\Longrightarrow$ (a) and (e) $\Longrightarrow$ (d). Therefore, all statements are equivalent. $\square$

3 The nonlinear case

This section is devoted to nonlinear equations (N) and ($N^{\mathcal{A}}$). The first result, similar to Proposition 1, establishes some asymptotic properties for Kneser and strongly monotone solutions in the nonlinear case and generalizes some results for Emden–Fowler equation (EF), where $a_i = 1$, $i = 1, \ldots, n-1$, see [13, Th.s.16.2, 16.3].

Theorem 2. *Let* (6) *be satisfied.*

(a) Assume (H1). If there exists a Kneser solution x of (N) such that $\lim_{t\to\infty} x^{[i]}(t) = 0$ *for* $i = 0, 1, \ldots, n-1$*, then* (7) *holds.*

(b) Assume (H3). If (7) *holds, then every Kneser solution x of (N) (if it exists) satisfies* $\lim_{t\to\infty} x^{[i]}(t) = 0$ *for* $i = 0, 1, \ldots, n-1$.

(c) Assume (H4). If there exists a strongly monotone solution u of ($N^{\mathcal{A}}$) *such that* $\lim_{t\to\infty} |u^{[i]}(t)| = \infty$ *for* $i = 0, 1, \ldots, n-1$*, then* (7) *holds.*

(d) Assume (H2). If (7) *holds, then every strongly monotone solution u of ($N^{\mathcal{A}}$) (if it exists) satisfies* $\lim_{t\to\infty} |u^{[i]}(t)| = \infty$ *for* $i = 0, 1, \ldots, n-1$.

Proof. Claim (a). Let F the function given by

$$F(t) = \frac{f(x(t))}{x(t)}$$

and consider, for t sufficiently large, the linearized equation

$$\left(\frac{1}{a_{n-1}(t)}\left(\cdots\left(\frac{1}{a_1(t)}w'\right)'\cdots\right)'\right)' + p(t)F(t)w = 0 \ . \qquad (\mathrm{L}_F)$$

Because $w \equiv x$ is an its solution, (L_F) has a Kneser solution such that $\lim_{t\to\infty} w^{[i]}(\infty) = 0$ for $i = 0, 1, \ldots, n-1$. By Proposition 1, we obtain

$$\lim_{t\to\infty} I_n(t, T; a_i, \ldots, a_1, Fp, a_{n-1}, \ldots, a_{i+1}) = \infty, \quad i = 0, 1, \ldots, n-1. \qquad (8)$$

Because (H1) holds, there exists a constant m such that

$$0 < F(t) < m \quad \text{for all large } t$$

and so

$$I_n(t, T; a_i, \ldots, a_1, Fp, a_{n-1}, \ldots, a_{i+1}) \leq \\ \leq m I_n(t, T; a_i, \ldots, a_1, p, a_{n-1}, \ldots, a_{i+1}), \quad i = 0, 1, \ldots, n-1. \qquad (9)$$

Hence (8) gives the assertion.

Claim (b). Suppose that there exists an eventually positive Kneser solution x of (N) such that for some $k \in \{0, 1, \dots, n-1\}$ $\lim_{t\to\infty} (-1)^k x^{[k]}(t) > 0$ and consider the linearized equation (L_F): by Proposition 1 there exists $i \in \{0, 1, \dots, n-1\}$ such that

$$\lim_{t\to\infty} I_n(t, T; a_i, a_{i-1}, \dots, a_1, Fp, a_{n-1}, \dots, a_{i+1}) < \infty. \tag{10}$$

Two cases are possible: I) $x(\infty) > 0$, II) $x(\infty) = 0$. If I) holds, because x is an eventually positive decreasing function, there exists a positive constant h such that

$$F(t) > h \quad \text{for all large } t. \tag{11}$$

If II) holds, in virtue of (H3), (11) holds too. Hence, taking into account (10), we obtain

$$\lim_{t\to\infty} I_n(t, T; a_i, a_{i-1}, \dots, a_1, p, a_{n-1}, \dots, a_{i+1}) < \infty, \tag{12}$$

which contradicts (7).

Claim (c). Let u be a strongly monotone solution u of ($\mathrm{N}^{\mathcal{A}}$) satisfying $\lim_{t\to\infty} |u^{[i]}(t)| = \infty$ for $i = 0, 1, \dots, n-1$. Consider the function G given by

$$G(t) = \frac{f(u(t))}{u(t)}.$$

In virtue of (H4), there exists a positive constant H such that

$$G(t) < H \quad \text{for all large } t.$$

Consider, for t sufficiently large, the linearized equation equation

$$\left(\frac{1}{a_1(t)}\left(\cdots\left(\frac{1}{a_{n-1}(t)}w'\right)'\cdots\right)'\right)' + (-1)^n p(t) G(t) w(t) = 0. \tag{$\mathrm{L}_G^{\mathcal{A}}$}$$

Using an argument similar to this given in claim (a), we get the assertion.

Claim (d). Suppose there exists an eventually positive strongly monotone solution u of ($\mathrm{N}^{\mathcal{A}}$) such that for some $k \in \{0, 1, \dots, n-1\}$ $\lim_{t\to\infty} |u^{[k]}(t)| <$

∞ and consider the linearized equation ($\mathrm{L}_G^{\mathcal{A}}$). By Proposition 1 there exists $i \in \{0, 1, \ldots, n-1\}$ such that

$$\lim_{t\to\infty} I_n(t, T; a_i, a_{i-1}, \ldots, a_1, Gp, a_{n-1}, \ldots, a_{i+1}) < \infty. \tag{13}$$

In virtue of (H2), there exists a positive constant M such that

$$G(t) > M \quad \text{for all large } t.$$

Hence from (13) we obtain (12), which contradicts (7). $\square$

From now we will suppose that the disconjugate operator L_n is in the canonical form, i.e. (C) holds. Then the set $\mathcal{N}$ of all nonoscillatory solutions of (N) can be divided into the following classes (see, e.g., [8, Lemma 1]):

$$\begin{aligned}
&\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_3 \cup \cdots \cup \mathcal{N}_{n-1} && \text{for } n \text{ even, } p(t) > 0,\\
&\mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_2 \cup \cdots \cup \mathcal{N}_{n-1} && \text{for } n \text{ odd, } p(t) > 0,\\
&\mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_2 \cup \cdots \cup \mathcal{N}_{n} && \text{for } n \text{ even, } p(t) < 0,\\
&\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_3 \cup \cdots \cup \mathcal{N}_{n} && \text{for } n \text{ odd, } p(t) < 0,
\end{aligned} \tag{14}$$

where $x \in \mathcal{N}_l$ if and only if for all large t

$$\begin{aligned}
x(t)x^{[i]}(t) &> 0, & 0 \le i \le l,\\
(-1)^{i+l}x(t)x^{[i]}(t) &> 0, & l \le i \le n-1.
\end{aligned}$$

We recall that solutions in class $\mathcal{N}_0$ are Kneser solutions and solutions in class $\mathcal{N}_n$ are strongly monotone solutions. The asymptotic properties of these solutions are described in the following lemma, which proof is based on a similar argument as in the proof of [19, 20]. For the completeness, we present it with the proof.

Lemma 2. *Assume (C) and let x be a nonoscillatory solution of (N).*
(a) If $x \in \mathcal{N}_0$, then $x^{[i]}(\infty) = 0$ for $i = 1, 2, .., n-1$.
(b) If $u \in \mathcal{N}_n$, then $\left|u^{[i]}(\infty)\right| = \infty$ for $i = 0, 1, .., n-2$.

Proof. Claim (a). Without loss of generality, assume x eventually positive. Then there exists t_x such that the functions $(-1)^i x^{[i]}, i = 0, 1, ..n-1,$

are positive decreasing. Assume there exists $k \in \{1, 2, .., n-1\}$ such that $x^{[k]}(\infty) \neq 0$, i.e.

$$(-1)^k x^{[k]}(\infty) = c_k > 0.$$

From $x^{[k]}(t) = \frac{1}{a_k(t)} \left(x^{[k-1]}(t)\right)'$we obtain $(t \geq t_x)$

$$x^{[k-1]}(t) - x^{[k-1]}(t_x) = \int_{t_x}^{t} a_k(s) x^{[k]}(s) ds.$$

Because $(-1)^k x^{[k]}$ is positive decreasing, we obtain

$$(-1)^k x^{[k-1]}(t) - (-1)^k x^{[k-1]}(t_x) = \int_{t_x}^{t} a_k(s)(-1)^k x^{[k]}(s)\, ds > c_k \int_{t_x}^{t} a_k(s)\, ds.$$

Taking into account (C), as $t \to \infty$ we obtain $(-1)^{k-1} x^{[k-1]}(\infty) < 0$, that is a contradiction.

Claim (b). Without loss of generality, assume u eventually positive. Then there exists t_x such that the functions $u^{[\ell]}, \ell = 0, 1, ..n-1$, are positive increasing. From $u^{[i+1]}(t) = \frac{1}{a_{i+1}(t)} \left(u^{[i]}(t)\right)'$ we obtain $(t \geq t_x)$

$$u^{[i]}(t) - u^{[i]}(t_x) = \int_{t_x}^{t} a_{i+1}(s) u^{[i+1]}(s) ds.$$

Because $u^{[i+1]}, i = 0, 1, .., n-2$ are positive increasing, we obtain

$$u^{[i]}(t) - u^{[i]}(t_x) > u^{[i+1]}(t_x) \int_{t_x}^{t} a_i(s) ds, \quad i = 0, 1, .., n-2.$$

As $t \to \infty$, we get the assertion. $\square$

If (C) holds, in view of Lemma 2, properties A and B can be formulated by the following way:

Property A for (N)

$\mathbf{n}$ even, $\mathbf{p} > \mathbf{0}$: $\qquad$ Property A $\iff \mathcal{N} = \emptyset.$

$\mathbf{n}$ odd, $\mathbf{p} > \mathbf{0}$: $\qquad$ Property A $\iff \begin{cases} \mathcal{N} = \mathcal{N}_0, \\ x \in \mathcal{N}_0 \Longrightarrow x(\infty) = 0. \end{cases}$

Property B for (N)

n even, $\mathbf{p} < \mathbf{0}$: Property B $\Longleftrightarrow \begin{cases} \mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_n, \\ x \in \mathcal{N}_0 \Longrightarrow x(\infty) = 0, \\ u \in \mathcal{N}_n \Longrightarrow \left|u^{[n-1]}(\infty)\right| = \infty. \end{cases}$

n odd, $\mathbf{p} < \mathbf{0}$: Property B $\Longleftrightarrow \begin{cases} \mathcal{N} = \mathcal{N}_n, \\ u \in \mathcal{N}_n \Longrightarrow \left|u^{[n-1]}(\infty)\right| = \infty. \end{cases}$

Remark 1. In the nonlinear case property A [B] does not ensure the existence of all type of solutions occuring in its definition, as Emden–Fowler equation shows [13, Th.s. 16.15, 16.8, Cor. 16.1].

Concerning the existence of Kneser solutions in the nonlinear case, the following holds:

Proposition 2. *Assume $n \geq 3$, (H1) and $(-1)^{n-1}p(t) \geq 0$ (i.e.* (6) *holds). Then there exists a Kneser solution of* (N)*.*

Proof. It is shown in [1] (Corollary, p.332) that, if

$$\int_0^1 \frac{du}{f(u)} = \infty,$$

then there exists a solution of (N) such that $x(t) > 0$, $(-1)^i x^{[i]}(t) \geq 0$ for $k = 1, \ldots, n-1$, $t \geq 0$. It remains to show that quasiderivatives of x are eventually different from zero. Assume there exists an integer k and a sequence $\{T_j\}$, $T_j \to \infty$, such that $x^{[k]}(T_j) = 0$. Hence $x^{[k+1]}$ has infinitely zeros approaching ∞. Repeating the same argument, we get that $x^{[n]}$ has infinitely large zeros, which contradicts the positiveness of x. □

When the disconjugate operator is in the canonical form, then the claim (c) of Theorem 2 can be proved without assumptions involving the behavior of the nonlinearity in a neighbourhood of zero and infinity. Indeed the following holds:

Proposition 3. *Assume (C) and* (6)*. If*

$$\lim_{t\to\infty} I_n(t, T; p, a_{n-1}, \ldots, a_1) = \infty, \tag{15}$$

then every Kneser solution x of (N) (if it exists) satisfies $x^{[i]}(\infty) = 0$ for $i = 0, 1, \ldots, n-1$.

Proof. By Lemma 2 every Kneser solution x of (N) satisfies $x^{[i]}(\infty) = 0$ for $i = 1, 2, ..., n-1$. Suppose that there exists an eventually positive Kneser solution x of (N) such that

$$\lim_{t\to\infty} x(t) = c > 0. \tag{16}$$

Again, using a linearization device, we get that $w \equiv x$ is a solution of (L_F) with $F(t) = f(x(t))/x(t)$. Taking into account (C) and Proposition 1, we obtain

$$\lim_{t\to\infty} I_n(t, T; Fp, a_{n-1}, \ldots, a_1) < \infty. \tag{17}$$

Because x is an eventually positive decreasing solution, taking into account (16), there exists a positive constant μ such that

$$F(t) > \mu > 0 \quad \text{for all large } t.$$

Hence, by (17), we obtain

$$\mu \lim_{t\to\infty} I_n(t, T; p, a_{n-1}, \ldots, a_1) \leq$$
$$\leq \lim_{t\to\infty} I_n(t, T; Fp, a_{n-1}, \ldots, a_1) < \infty,$$

which is a contradiction. □

By Theorem 2 and Proposition 3, we obtain also the following

Corollary 1. *Assume (C), (H1) and* (6). *If there exists a Kneser solution* x_1 *of (N) such that* $\lim_{t\to\infty} x_1^{[i]}(\infty) = 0$ *for* $i = 0, 1, \ldots, n-1$, *then every Kneser solution* x *of (N) satisfies* $x^{[i]}(\infty) = 0$ *for* $i = 0, 1, \ldots, n-1$.

Proof. By Theorem 2-(a), (7) holds. Particularly, (15) is verified and Proposition 3 gives the assertion. □

Now we give a comparison theorem, which generalizes of Theorem 4 in [4], stated for $n = 3$. It compares property A/B between the linear equation

$$\left(\frac{1}{a_{n-1}(t)}\left(\cdots\left(\frac{1}{a_1(t)}x(t)'\right)'\cdots\right)'\right)' + kp(t)x(t) = 0, \qquad k > 0 \tag{L_k}$$

and nonlinear equations (N), ($\mathrm{N}^{\mathcal{A}}$).

Theorem 3. *Assume (C) and (H2).*

(a) Let n be even and $p(t) > 0$. If (L_k) has property A for every $k > 0$, then (N) and $(N^{\mathcal{A}})$ have property A.

(b) Let n be odd and $p(t) > 0$. If (L_k) has property A for every $k > 0$, then (N) has property A and $(N^{\mathcal{A}})$ has property B.

(c) Let n be even and $p(t) < 0$. If (L_k) has property B for every $k > 0$, then (N) and $(N^{\mathcal{A}})$ have property B.

(d) Let n be odd and $p(t) < 0$. If (L_k) has property B for every $k > 0$, then (N) has property B and $(N^{\mathcal{A}})$ has property A.

Proof. (a) If all the proper solutions of (N) are oscillatory, then (N) has property A. Assume there exists a proper nonoscillatory solution x of (N), and, without loss of generality, suppose there exists $T_x \geq 0$ such that $x(t) > 0$, for $t \geq T_x$. In view of (14) we have

$$x \in \mathcal{N}_1 \cup \mathcal{N}_3 \cup \cdots \cup \mathcal{N}_{n-1}, \tag{18}$$

which implies that there exists a positive constant c_x such that

$$x(t) \geq c_x > 0 \quad \text{for } t \geq T_x. \tag{19}$$

Consider, for $t \geq T_x$, the linearized equation (L_F): because the function $v = x$ is an its nonoscillatory solution, (L_F) does not have property A. In view of (19), there exists k such that for $t \geq T_x$

$$F(t) = \frac{f(x(t))}{x(t)} \geq k. \tag{20}$$

Thus, by a classical comparison result (see, e.g. [8, Th.1]), (L_k) does not have property A, which is a contradiction. Now we prove that $(\mathrm{N}^{\mathcal{A}})$ has property A. Assume that there exists a proper nonoscillatory solution x of $(\mathrm{N}^{\mathcal{A}})$ and, without loss of generality, suppose x eventually positive. Then (18) holds and hence (19) and (20) hold, too. By the same argument as above, the linearized equation

$$\left(\frac{1}{a_1(t)}\left(\cdots\left(\frac{1}{a_{n-1}(t)}w'\right)'\cdots\right)'\right)' + (-1)^n p(t)F(t)w(t) = 0. \tag{$\mathrm{L}_F^{\mathcal{A}}$}$$

does not have property A because $w \equiv x$ is an its nonoscillatory solution. Then, by using again the quoted comparison result [8, Theorem 1], the linear equation

$$\left(\frac{1}{a_1(t)}\left(\cdots\left(\frac{1}{a_{n-1}(t)}u'\right)'\cdots\right)'\right)' + (-1)^n kp(t)u(t) = 0 \qquad (\mathrm{L}_k^{\mathcal{A}})$$

does not have property A, that contradicts Theorem 1-(a).

Claim (b). If all the proper solutions of (N) are oscillatory, then (N) has property A. Assume that (N) has nonoscillatory solutions. Thus, in view of (C), we have

$$\mathcal{N}_0 \cup \mathcal{N}_2 \cdots \cup \mathcal{N}_{n-1} \neq \emptyset.$$

Suppose that (N) does not have property A. Taking into account Lemma 2, there exists a proper nonoscillatory solution x of (N), such that either

$$x \in \mathcal{N}_2 \cup \cdots \cup \mathcal{N}_{n-1} \tag{21}$$

or

$$x \in \mathcal{N}_0 \text{ and } \lim_{t\to\infty} x(t) \neq 0. \tag{22}$$

Without loss of generality, assume x eventually positive. Hence, in both cases, there exists $c_x > 0$ such that (19) holds. Consider, for $t \geq T_x$, the linearized equation (L_F): because $v \equiv x$ is an its nonoscillatory solution and either (21) or (22) holds, (L_F) does not have property A. In view of (19), there exists k such that (20) holds for $t \geq T_x$. Using the same argument as in claim (a), we get a contradiction. Now we prove that ($\mathrm{N}^{\mathcal{A}}$) has property B. If all the proper solutions of ($\mathrm{N}^{\mathcal{A}}$) are oscillatory, then ($\mathrm{N}^{\mathcal{A}}$) has property B. Assume that ($\mathrm{N}^{\mathcal{A}}$) has nonoscillatory solutions, that is, in view of (C),

$$\mathcal{N}_1 \cup \mathcal{N}_3 \cup \cdots \cup \mathcal{N}_n \neq \emptyset.$$

Suppose that ($\mathrm{N}^{\mathcal{A}}$) does not have property B. Taking into account Lemma 2, there exists a proper nonoscillatory solution u of ($\mathrm{N}^{\mathcal{A}}$) such that either

$$u \in \mathcal{N}_1 \cup \mathcal{N}_3 \cup \cdots \cup \mathcal{N}_{n-2}$$

or

$$u \in \mathcal{N}_n,\ |u^{[n-1]}(\infty)| < \infty.$$

Consider, for $t \geq T_x$, the linearized equation ($\mathrm{L}_G^{\mathcal{A}}$): using the same linearization method as above and taking into account Theorem 1, we obtain a contradiction with the fact that (L_k) has property A for every $k > 0$.

Claims (c), (d). We proceed in a similar way as above. The details are omitted. $\square$

Proposition 1, Theorem 2-(a) and Theorem 3 yield the following result:

Corollary 2. *Assume (C) and (H2).*

(a) Let n be odd and $p(t) > 0$. If every nonoscillatory solution of (L_k) is Kneser for any $k > 0$ and

$$\lim_{t\to\infty} I_n(t, T; p, a_{n-1}, \dots, a_1) = \infty, \tag{23}$$

then (L) and (N) have property A and ($N^{\mathcal{A}}$) has property B.

(b) Let n be even and $p(t) < 0$. Assume that every nonoscillatory solution of (L_k) is either Kneser or strongly monotone for any $k > 0$. If (23) *and*

$$\lim_{t\to\infty} I_n(t, T; p, a_1, \dots, a_{n-1}) = \infty, \tag{24}$$

hold, then (L), (N) and ($N^{\mathcal{A}}$) have property B.

(c) Let n be odd and $p(t) < 0$. If every nonoscillatory solution of (L_k) is strongly monotone for any $k > 0$ and (23) *holds, then (L) and (N) have property B and ($N^{\mathcal{A}}$) has property A.*

Proof. First let us remark that if (C) and (23) hold, then for any positive constant k

$$\lim_{t\to\infty} I_n(t, T; a_i, \dots, a_1, kp, a_{n-1}, \dots, a_{i+1}) = \infty, \quad i = 1, \dots, n-1. \tag{25}$$

Similarly, if (C) and (24) hold, then for any positive constant k

$$\lim_{t\to\infty} I_n(t, T; a_i, \dots, a_{n-1}, kp, a_1, \dots, a_{i+1}) = \infty, \quad i = 1, \dots, n-1. \tag{26}$$

Claim (a). By Proposition 1, every Kneser solution y of (L_k) satisfies $y^{[i]}(\infty) = 0, i = 0, 1, ..n-1$. Taking into account that every nonoscillatory solution of (L_k) is Kneser, we get that (L_k) has property A for any $k > 0$. Taking into account Theorem 3–(b), we obtain that (N) has property A and ($\mathrm{N}^{\mathcal{A}}$) has property B.

Claim (b). By Proposition 1, the condition (25) implies that every Kneser solution y of (L_k) satisfies $y^{[i]}(\infty) = 0, i = 0, 1, ..n-1$ and the condition (26) implies that every strongly monotone solution z of (L_k) satisfies $|z^{[i]}(\infty)| = \infty$, $i = 0, 1, ..n-1$. Reasoning as above we obtain that (L_k) has property B for any $k > 0$. Applying Theorem 3-(c), we get the assertion.

Claim (c). Reasoning as above and using Proposition 3, we obtain that ($\mathrm{L}_k^{\mathcal{A}}$) has property A for any $k > 0$. Applying to ($\mathrm{L}_k^{\mathcal{A}}$) Theorem 3-(b), we get the assertion. $\square$

Remark 3. When n is even and $p(t) < 0$ [n is odd and $p(t) < 0$], integral conditions posed on a_i and p and ensuring that for any $k > 0$ every nonoscillatory solution of (L_k) is strongly monotone [Kneser or strongly monotone] are given in [16, Theorems 1, 2].

When n is odd and $p(t) > 0$, integral conditions posed on a_i and p and ensuring that for any $k > 0$ every nonoscillatory solution of (L_k) is Kneser are given in [9, Theorem 2].

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Author's addresses:

M. Cecchi and M. Marini, Depart. of Electr. Eng., University of Florence, Via S. Marta 3, 50139 Firenze, Italy
e-mail: marini@ing.unifi.it

Z. Došlá, Depart. of Mathematics, Masaryk University, Janáčkovo nám. 2a, 66295 Brno, Czech Republic
e-mail: dosla@math.muni.cz