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Oscillatory behavior of third order nonlinear difference equation with mixed neutral terms

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Abstract. In this paper, we obtain some new sufficient conditions for the oscillation of all solutions of third order nonlinear neutral difference equation of the form

$$\Delta^3 \left(x_n + b_n x_{n-\tau_1} + c_n x_{n+\tau_2}\right)^\alpha = q_n x^\beta_{n-\sigma_1} + p_n x^\gamma_{n+\sigma_2}, \quad n \ge n_0,$$

where α, β, and γ are the ratios of odd positive integers. Examples are given to illustrate the main results.

Keywords: third order, nonlinear, difference equation, mixed neutral terms, oscillation.
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1 Introduction

In this paper, we study the oscillation of all solutions of the third order nonlinear difference equation with mixed neutral terms of the form

$$\Delta^3 \left(x_n + b_n x_{n-\tau_1} + c_n x_{n+\tau_2}\right)^\alpha = q_n x^\beta_{n-\sigma_1} + p_n x^\gamma_{n+\sigma_2}, \quad n \ge n_0, \tag{1.1}$$

where n_0 is a nonnegative integer, subject to the following conditions:

(C1) α, β and γ are the ratios of odd positive integers;

(C2) τ_1, τ_2, σ_1 and σ_2 are positive integers;

(C3) $\{q_n\}$ and $\{p_n\}$ are sequences of nonnegative real numbers;

(C4) $\{b_n\}$ and $\{c_n\}$ are nonnegative real sequences, and there exist constants b and c such that $0 \le b_n \le b < \infty$ and $0 \le c_n \le c < \infty$.

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Let $\theta = \max\{\sigma_1, \tau_1\}$. By a solution of equation (1.1), we mean a real valued sequence $\{x_n\}$ defined for all $n \geq n_0 - \theta$ and satisfying the equation (1.1) for all $n \geq n_0$. As customary, a nontrivial solution $\{x_n\}$ of equation (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is called nonoscillatory.

Recently, there has been much interest in studying the oscillatory behavior of neutral type difference equations, see, for example [1,2,6,8–10,12–14] and the references cited therein. This is because such type has various applications in natural sciences and engineering. Regarding mixed type neutral difference equations, the authors Agarwal, Grace and Bohner [3], Ferreira and Pinelas [4], Grace [5], and Grace and Dontha [7] considered several third order neutral difference equations with mixed arguments and established sufficient conditions for the oscillation of all solutions. It is to be noted that all the results are obtained only for the linear equations, and the paper dealing with the oscillation of nonlinear equation is by Thandapani and Kavitha [15]. In [15], the authors considered equation of the form (1.1) with the sequences $\{q_n\}$ and $\{p_n\}$ are non-positive. The purpose of this paper is to obtain some new sufficient conditions for the oscillation of all solutions of equation (1.1) when the sequences $\{q_n\}$ and $\{p_n\}$ are non-negative. In Section 2, we obtain some new sufficient conditions for the oscillation of all solutions of equation (1.1), and in Section 3, we provide some examples in support of our main results. Thus, the results obtained in this paper extend and complement to that of in [2,6,9,13–15].

2 Oscillation results

For the convenience of the reader, in what follows, we use the notation without further mention:

$$Q_n = \min\{q_n, q_{n-\sigma_1}, q_{n-\tau_1}\}, \qquad P_n = \min\{p_n, p_{n-\sigma_1}, p_{n-\tau_1}\},$$

and

$$z_n = (x_n + b_n x_{n-\tau_1} + c_n x_{n+\tau_2})^\alpha .$$

Throughout this paper we prove the results for the positive solution only since the proof for the other case is similar.

We start with the following lemmas.

Lemma 2.1. *Assume $A \geq 0$, and $B \geq 0$. If $0 < \delta \leq 1$ then*

$$A^\delta + B^\delta \geq (A+B)^\delta , \tag{2.1}$$

and if $\delta \geq 1$ then

$$A^\delta + B^\delta \geq \frac{1}{2^{\delta-1}} (A+B)^\delta . \tag{2.2}$$

Proof. The proof can be found in Lemma 2.1 and Lemma 2.2 of [11]. □

Lemma 2.2. *If $\{x_n\}$ is a positive solution of equation (1.1), then the corresponding sequence $\{z_n\}$ satisfies only one of the following two cases:*

$$\text{(I)}\ \ z_n > 0, \ \ \Delta z_n > 0, \ \ \Delta^2 z_n > 0, \ \ \text{and} \ \ \Delta^3 z_n > 0, \tag{2.3}$$

$$\text{(II)}\ \ z_n > 0, \ \ \Delta z_n > 0, \ \ \Delta^2 z_n < 0, \ \ \text{and} \ \ \Delta^3 z_n > 0. \tag{2.4}$$

Proof. Assume that $\{x_n\}$ is a positive solution of equation (1.1). Then there exists an integer $n_1 \geq n_0$ such that $x_n > 0$, $x_{n-\sigma_1} > 0$, and $x_{n-\tau_1} > 0$ for all $n \geq n_1$. By the definition of z_n, we have $z_n > 0$ for all $n \geq n_1$. From the equation (1.1), we have $\Delta^3 z_n > 0$ for all $n \geq n_1$. Then $\{\Delta^2 z_n\}$ is strictly increasing and both $\Delta^2 z_n$ and Δz_n are of one sign for all $n \geq n_1$. We shall prove that $\Delta z_n > 0$ for all $n \geq n_1$. Otherwise there exists an integer $n_2 \geq n_1$, and a negative constant M such that $\Delta z_n < M$ for all $n \geq n_2$. Summing the last inequality from n_2 to $n-1$, we obtain

$$z_n < z_{n_2} + M(n - n_2).$$

Letting $n \to \infty$ in the above inequality we see that $z_n \to -\infty$, which is a contradiction to the positivity of z_n. This contradiction proves the lemma. □

Theorem 2.3. *Assume* $0 < \beta = \gamma \leq 1$,*and* $\sigma_1 > \max\{\tau_1, \tau_2\}$. *If the second order difference inequalities*

$$\Delta^2 y_n - P_n \frac{(\sigma_1 - \tau_2)^{\beta/\alpha}}{(1 + b^\beta + c^\beta)^{\beta/\alpha}} y^{\beta/\alpha}_{n-\sigma_1+\sigma_2} \geq 0, \tag{2.5}$$

and

$$\Delta^2 y_n - Q_n \frac{(\sigma_1 - \tau_1)^{\beta/\alpha}}{(1 + b^\beta + c^\beta)^{\beta/\alpha}} y^{\beta/\alpha}_{n-\sigma_1+\tau_1} \geq 0 \tag{2.6}$$

have no positive increasing solution, and no positive decreasing solution, respectively, then every solution of equation (1.1) *is oscillatory.*

Proof. Suppose $\{x_n\}$ is a nonoscillatory solution of equation (1.1). Without loss of generality, we may assume that $\{x_n\}$ is a positive solution of equation (1.1). Then there exists an integer $N_1 \geq n_0$ such that $x_n > 0$, $x_{n-\sigma_1} > 0$, and $x_{n-\tau_1} > 0$ for all $n \geq N_1$. Set

$$y_n = z_n + b^\beta z_{n-\tau_1} + c^\beta z_{n+\tau_2} \tag{2.7}$$

for all $n \geq n_1 \geq N_1$. Then $y_n > 0$ for all $n \geq n_1$, and

$$\begin{aligned}
\Delta^3 y_n &= \Delta^3 z_n + b^\beta \Delta^3 z_{n-\tau_1} + c^\beta \Delta^3 z_{n+\tau_2} \\
&= q_n x^\beta_{n-\sigma_1} + p_n x^\beta_{n+\sigma_2} + b^\beta \left[q_{n-\tau_1} x^\beta_{n-\tau_1-\sigma_1} + p_{n-\tau_1} x^\beta_{n-\tau_1+\sigma_2} \right] \\
&\quad + c^\beta \left[q_{n+\tau_2} x^\beta_{n+\tau_2-\sigma_1} + p_{n+\tau_2} x^\beta_{n+\tau_2+\sigma_2} \right] \\
&\geq Q_n \left[x^\beta_{n-\sigma_1} + b^\beta x^\beta_{n-\tau_1-\sigma_1} + c^\beta x^\beta_{n+\tau_2-\sigma_1} \right] \\
&\quad + P_n \left[x^\beta_{n+\sigma_2} + b^\beta x^\beta_{n-\tau_1+\sigma_2} + c^\beta x^\beta_{n+\tau_2+\sigma_2} \right].
\end{aligned}$$

Now using (2.1) in the right hand side of the last inequality, we obtain

$$\Delta^3 y_n \geq Q_n z^{\beta/\alpha}_{n-\sigma_1} + P_n z^{\beta/\alpha}_{n+\sigma_2}, \; n \geq n_1. \tag{2.8}$$

Since $\{x_n\}$ is a positive solution of equation (1.1), we have two cases for $\{z_n\}$ as given in Lemma 2.2.

Case (I). Suppose there exists an integer $n_2 \geq n_1$ such that $\Delta z_n > 0$, $\Delta^2 z_n > 0$, and $\Delta^3 z_n > 0$ for all $n \geq n_2$. Then from the definition of y_n, we have $\Delta y_n > 0$, $\Delta^2 y_n > 0$ and $\Delta^3 y_n > 0$ for all $n \geq n_3 \geq n_2$. From (2.8), we have

$$\Delta^3 y_n \geq P_n z^{\beta/\alpha}_{n+\sigma_2}, \quad \text{for all } n \geq n_3. \tag{2.9}$$

Using the monotonicity of Δz_n, we have

$$\Delta y_n = \Delta z_n + b^\beta \Delta z_{n-\tau_1} + c^\beta \Delta z_{n+\tau_2} \leq \left(1 + b^\beta + c^\beta\right) \Delta z_{n+\tau_2},$$

and

$$z_{n+\sigma_1-\tau_2} = z_n + \sum_{s=n}^{n+\sigma_1-\tau_2-1} \Delta z_s \geq (\sigma_1 - \tau_2)\Delta z_n. \tag{2.10}$$

Combining (2.9), (2.10) and (2.10), we obtain

$$\Delta^3 y_n \geq P_n \frac{(\sigma_1 - \tau_2)^{\beta/\alpha}}{\left(1 + b^\beta + c^\beta\right)^{\beta/\alpha}} \left(\Delta y_{n-\sigma_1+\sigma_2}\right)^{\beta/\alpha} \tag{2.11}$$

for all $n \geq n_3$. Define $w_n = \Delta y_n$ for all $n \geq n_3$. Then $w_n > 0$ and $\Delta w_n > 0$ for all $n \geq n_3$. Now from the inequality (2.11), we obtain

$$\Delta^2 w_n \geq P_n \frac{(\sigma_1 - \tau_2)^{\beta/\alpha}}{\left(1 + b^\beta + c^\beta\right)^{\beta/\alpha}} w_{n-\sigma_1+\sigma_2}^{\beta/\alpha}$$

for all $n \geq n_3$. Thus $\{w_n\}$ is a positive increasing solution of the inequality (2.5), which is a contradiction.

Case (II). Suppose there exists an integer $n_2 \geq n_1$ such that $\Delta z_n > 0$, $\Delta^2 z_n < 0$, and $\Delta^3 z_n > 0$ for all $n \geq n_2$. From the definition of y_n, we have $\Delta y_n > 0$, $\Delta^2 y_n < 0$ for all $n \geq n_3 \geq n_2$. Now from the inequality (2.8), we have

$$\Delta^3 y_n \geq Q_n z_{n-\sigma_1}^{\beta/\alpha} \tag{2.12}$$

for all $n \geq n_3$. By the monotonicity of Δz_n, we have

$$\Delta y_n = \Delta z_n + b^\beta \Delta z_{n-\tau_1} + c^\beta \Delta z_{n+\tau_2} \leq \left(1 + b^\beta + c^\beta\right) \Delta z_{n-\tau_1},$$

and

$$z_n = z_{n-\sigma_1+\tau_1} + \sum_{s=n-(\sigma_1-\tau_1)}^{n-1} \Delta z_s \geq (\sigma_1 - \tau_1)\Delta z_n. \tag{2.13}$$

Combining (2.12), (2.13) and (2.13), we obtain

$$\Delta^3 y_n \geq Q_n \frac{(\sigma_1 - \tau_1)^{\beta/\alpha}}{\left(1 + b^\beta + c^\beta\right)^{\beta/\alpha}} \left(\Delta y_{n-\sigma_1+\tau_1}\right)^{\beta/\alpha}$$

for all $n \geq n_3$. By setting $w_n = \Delta y_n$, we see that $w_n > 0$, $\Delta w_n = \Delta^2 y_n < 0$, and

$$\Delta^2 w_n \geq Q_n \frac{(\sigma_1 - \tau_1)^{\beta/\alpha}}{\left(1 + b^\beta + c^\beta\right)^{\beta/\alpha}} w_{n-\sigma_1+\tau_1}^{\beta/\alpha}$$

for all $n \geq n_3$. That is, $\{w_n\}$ is a positive decreasing solution of the inequality (2.6), which is a contradiction. Now the proof is complete. □

Theorem 2.4. *Assume $\beta = \gamma \geq 1$, and $\sigma_1 > \max\{\tau_1, \tau_2\}$. If the second order difference inequalities*

$$\Delta^2 y_n - \frac{P_n (\sigma_1 - \tau_2)^{\beta/\alpha}}{4^{\beta-1}\left(1 + b^\beta + \frac{c^\beta}{2^{\beta-1}}\right)^{\beta/\alpha}} y_{n-\sigma_1+\sigma_2}^{\beta/\alpha} \geq 0, \tag{2.14}$$

and

$$\Delta^2 y_n - \frac{Q_n(\sigma_1-\tau_1)^{\beta/\alpha}}{4^{\beta-1}\left(1+b^\beta+\frac{c^\beta}{2^{\beta-1}}\right)^{\beta/\alpha}} y_{n-\sigma_1+\tau_1}^{\beta/\alpha} \geq 0 \tag{2.15}$$

have no positive increasing solution, and no positive decreasing solution, respectively, then every solution of equation (1.1) is oscillatory.

Proof. The proof is similar to that of Theorem 2.3, and so the details are omitted. □

Theorem 2.5. *Assume $0<\beta\leq 1$, $\gamma\geq 1$, $b\leq 1$, $c\leq 1$, and $\sigma_1>\max\{\tau_1,\tau_2\}$. If the second order difference inequalities*

$$\Delta^2 y_n - \frac{P_n(\sigma_1-\tau_2)^{\gamma/\alpha}}{4^{\gamma-1}\left(1+b^\beta+c^\beta\right)^{\gamma/\alpha}} y_{n-\sigma_1+\sigma_2}^{\gamma/\alpha} \geq 0, \tag{2.16}$$

and

$$\Delta^2 y_n - \frac{Q_n(\sigma_1-\tau_1)^{\beta/\alpha}}{\left(1+b^\beta+c^\beta\right)^{\beta/\alpha}} y_{n-\sigma_1+\tau_1}^{\beta/\alpha} \geq 0, \tag{2.17}$$

have no positive increasing solution, and no positive decreasing solution, respectively, then every solution of equation (1.1) is oscillatory.

Proof. Let $\{x_n\}$ be a nonoscillatory solution of equation (1.1). Without loss of generality, we may assume that $\{x_n\}$ is a positive solution of equation (1.1). Then there exists an integer $N_1\geq n_0$ such that $x_{n-\theta}>0$, for all $n\geq N_1$. Define

$$y_n = z_n + b^\beta z_{n-\tau_1} + c^\beta z_{n+\tau_2} \tag{2.18}$$

for all $n\geq n_1\geq N_1$. Then $y_n>0$, and

$$\begin{aligned}
\Delta^3 y_n &= \Delta^3 z_n + b^\beta \Delta^3 z_{n-\tau_1} + c^\beta z_{n+\tau_2}\\
&= q_n x_{n-\sigma_1}^\beta + p_n x_{n+\sigma_2}^\gamma + b^\beta\left[q_{n-\tau_1}x_{n-\tau_1-\sigma_1}^\beta + p_{n-\tau_1}x_{n-\tau_1+\sigma_2}^\gamma\right]\\
&\quad + c^\beta\left[q_{n+\tau_2}x_{n+\tau_2-\sigma_1}^\beta + p_{n+\tau_2}x_{n+\tau_2+\sigma_2}^\gamma\right]\\
&\geq Q_n\left[x_{n-\sigma_1}^\beta + b^\beta x_{n-\tau_1-\sigma_1}^\beta + c^\beta x_{n+\tau_2-\sigma_1}^\beta\right]\\
&\quad + P_n\left[x_{n+\sigma_2}^\gamma + b^\beta x_{n-\tau_1+\sigma_2}^\gamma + c^\beta x_{n+\tau_2+\sigma_2}^\gamma\right]
\end{aligned}$$

for all $n\geq n_2\geq n_1$. Now using (2.1) twice on the first part of right hand side of last inequality, we have

$$\Delta^3 y_n \geq Q_n z_{n-\sigma_1}^{\beta/\alpha} + P_n\left[x_{n+\sigma_2}^\gamma + b^\beta x_{n-\tau_1+\sigma_2}^\gamma + c^\beta x_{n+\tau_2+\sigma_2}^\gamma\right]. \tag{2.19}$$

Since $b\leq 1$, $c\leq 1$, $\gamma\geq 1$, and $0<\beta\leq 1$, we have by (2.2) that

$$x_{n+\sigma_2}^\gamma + b^\beta x_{n-\tau_1+\sigma_2}^\gamma + c^\gamma x_{n+\tau_2+\sigma_2}^\gamma \geq x_{n+\sigma_2}^\gamma + b^\gamma x_{n-\tau_1+\sigma_2}^\gamma + c^\gamma x_{n+\tau_2+\sigma_2}^\gamma \geq \frac{1}{4^{\gamma-1}} z_{n+\sigma_2}^{\gamma/\alpha}.$$

Using (2.20) in (2.19), we have

$$\Delta^3 y_n \geq Q_n z_{n-\sigma_1}^{\beta/\alpha} + \frac{P_n}{4^{\gamma-1}} z_{n+\sigma_2}^{\gamma/\alpha}. \tag{2.20}$$

Now we consider the two cases for $\{z_n\}$ as stated in Lemma 2.2.
Case (I). Suppose there exists an integer $n_3 \geq n_2$ such that $\Delta z_n > 0$, $\Delta^2 z_n > 0$, and $\Delta^3 z_n > 0$ for all $n \geq n_3$. From the inequality (2.20), we have

$$\Delta^3 y_n \geq \frac{P_n}{4^{\gamma-1}} z_{n+\sigma_2}^{\gamma/\alpha} \tag{2.21}$$

for all $n \geq n_3$. By the monotonicity of Δz_n, we obtain

$$\Delta y_n = \Delta z_n + b^\beta \Delta z_{n-\tau_1} + c^\beta \Delta z_{n+\tau_2} \leq \left(1 + b^\beta + c^\beta\right) \Delta z_{n+\tau_2}$$

for all $n \geq n_3$, and

$$z_{n+\sigma_1-\tau_2} = z_n + \sum_{s=n}^{n+\sigma_1-\tau_2-1} \Delta z_s \geq (\sigma_1 - \tau_2)\Delta z_n. \tag{2.22}$$

Using (2.22) and (2.22) in (2.21), we obtain

$$\Delta^3 y_n \geq \frac{P_n(\sigma_1-\tau_2)^{\gamma/\alpha}}{4^{\gamma-1}\left(1+b^\beta+c^\beta\right)^{\gamma/\alpha}} \left(\Delta y_{n-\sigma_1+\sigma_2}\right)^{\gamma/\alpha}.$$

By taking $w_n = \Delta y_n$, we see that $w_n > 0$, $\Delta w_n = \Delta^2 y_n > 0$, and

$$\Delta^2 w_n \geq \frac{P_n(\sigma_1-\tau_2)^{\gamma/\alpha}}{4^{\gamma-1}\left(1+b^\beta+c^\beta\right)^{\gamma/\alpha}} w_{n-\sigma_1+\sigma_2}^{\gamma/\alpha}$$

for all $n \geq n_3$. Thus $\{w_n\}$ is a positive increasing solution of the inequality (2.16), which is a contradiction.
Case (II). In this case, we have $\Delta z_n > 0$, $\Delta^2 z_n < 0$, and $\Delta^3 z_n > 0$ for all $n \geq n_2$. Therefore $\Delta y_n > 0$, $\Delta^2 y_n < 0$, and $\Delta^3 y_n > 0$ for all $n \geq n_3 \geq n_2$. From the inequality (2.20), we have

$$\Delta^3 y_n \geq Q_n z_{n-\sigma_1}^{\beta/\alpha} \tag{2.23}$$

for all $n \geq n_3$. By the monotonicity of Δz_n, we obtain

$$\Delta y_n = \Delta z_n + b^\beta \Delta z_{n-\tau_1} + c^\beta \Delta z_{n+\tau_2} \leq \left(1 + b^\beta + c^\beta\right) \Delta z_{n-\tau_1}$$

for all $n \geq n_3$, and

$$z_n = z_{n-\sigma_1+\tau_1} + \sum_{s=n-(\sigma_1-\tau_1)}^{n-1} \Delta z_s \geq (\sigma_1 - \tau_1)\Delta z_n \tag{2.24}$$

for all $n \geq n_3$. Combining (2.23), (2.24) and (2.24), we obtain

$$\Delta^3 y_n \geq \frac{Q_n(\sigma_1-\tau_1)^{\beta/\alpha}}{\left(1+b^\beta+c^\beta\right)^{\beta/\alpha}} \left(\Delta y_{n-\sigma_1+\tau_1}\right)^{\beta/\alpha}$$

for all $n \geq n_3$. Setting $w_n = \Delta y_n$, we see that $\{w_n\}$ is a positive decreasing solution of the inequality (2.17), which is a contradiction. This completes the proof. □

Theorem 2.6. *Assume* $0 < \gamma \leq 1$, $\beta \geq 1$, $b \leq 1$, $c \leq 1$, *and* $\sigma_1 > \max\{\tau_1, \tau_2\}$. *If the second order difference inequalities*

$$\Delta^2 y_n - \frac{P_n(\sigma_1-\tau_2)^{\beta/\alpha}}{4^{\beta-1}\left(1+b^\beta+c^\beta\right)^{\beta/\alpha}} y_{n-\sigma_1+\sigma_2}^{\beta/\alpha} \geq 0, \tag{2.25}$$

and

$$\Delta^2 y_n - \frac{Q_n(\sigma_1-\tau_1)^{\gamma/\alpha}}{(1+b^\beta+c^\beta)^{\gamma/\alpha}} y^{\gamma/\alpha}_{n-\sigma_1+\tau_1} \geq 0 \tag{2.26}$$

have no positive increasing solution, and no positive decreasing solution, respectively, then every solution of equation (1.1) is oscillatory.

Proof. The proof is similar to that of Theorem 2.5, and hence the details are omitted. □

Theorem 2.7. *Assume $\beta \geq 1$, $0 < \gamma \leq 1$, $b \geq 1$, $c \geq 1$, and $\sigma_1 > \max\{\tau_1, \tau_2\}$. If the second order difference inequalities*

$$\Delta^2 y_n - \frac{P_n(\sigma_1-\tau_2)^{\gamma/\alpha}}{\left(1+b^\beta+\frac{c^\beta}{2^{\gamma-1}}\right)^{\gamma/\alpha}} y^{\gamma/\alpha}_{n-\sigma_1+\sigma_2} \geq 0, \tag{2.27}$$

and

$$\Delta^2 y_n - \frac{Q_n(\sigma_1-\tau_1)^{\beta/\alpha}}{4^{\beta-1}\left(1+b^\beta+\frac{c^\beta}{2^{\gamma-1}}\right)^{\beta/\alpha}} y^{\beta/\alpha}_{n-\sigma_1+\tau_1} \geq 0 \tag{2.28}$$

have no positive increasing solution, and no positive decreasing solution, respectively, then every solution of equation (1.1) is oscillatory.

Proof. Assume that $\{x_n\}$ is a nonoscillatory solution of equation (1.1). Without loss of generality, we may assume that $\{x_n\}$ is a positive solution of equation (1.1). Then there exists an integer $n_1 \geq n_0$ such that $x_{n-\theta} > 0$ for all $n \geq n_1$. Set

$$y_n = z_n + b^\beta z_{n-\tau_1} + \frac{c^\beta}{2^{\gamma-1}} z_{n+\tau_2} \tag{2.29}$$

for all $n \geq n_2 \geq n_1$. Then $\Delta y_n > 0$, and

$$\begin{aligned}
\Delta^3 y_n &= \Delta^3 z_n + b^\beta \Delta^3 z_{n-\tau_1} + \frac{c^\beta}{2^{\gamma-1}} \Delta^3 z_{n+\tau_2} \\
&= q_n x^\beta_{n-\sigma_1} + p_n x^\gamma_{n+\sigma_2} + b^\beta \left[q_{n-\tau_1} x^\beta_{n-\tau_1-\sigma_1} + p_{n-\tau_1} x^\gamma_{n-\tau_1+\sigma_2}\right] \\
&\quad + \frac{c^\beta}{2^{\gamma-1}} \left[q_{n+\tau_2} x^\beta_{n+\tau_2-\sigma_1} + p_{n+\tau_2} x^\gamma_{n+\tau_2+\sigma_2}\right] \\
&\geq Q_n \left[x^\beta_{n-\sigma_1} + b^\beta x^\beta_{n-\tau_1-\sigma_1} + \frac{c^\beta}{2^{\gamma-1}} x^\beta_{n+\tau_2-\sigma_1}\right] \\
&\quad + P_n \left[x^\gamma_{n+\sigma_2} + b^\beta x^\gamma_{n-\tau_1+\sigma_2} + \frac{c^\beta}{2^{\gamma-1}} x^\beta_{n+\tau_2+\sigma_2}\right].
\end{aligned}$$

Since $b \geq 1$, $c \geq 1$, $\gamma \leq 1$ and $\beta \geq 1$, we have from the last inequality

$$\Delta^3 y_n \geq Q_n \left[x^\beta_{n-\sigma_1} + b^\beta x^\beta_{n-\tau_1-\sigma_1} + \frac{c^\beta}{2^{\beta-1}} x^\beta_{n+\tau_2-\sigma_1}\right] + P_n \left[x^\gamma_{n+\sigma_2} + b^\gamma x^\gamma_{n+\sigma_2-\tau_1} + c^\gamma x^\gamma_{n+\tau_2+\sigma_2}\right].$$

Now using (2.1) and (2.2) in the right hand side of the last inequality, we obtain

$$\Delta^3 y_n \geq \frac{Q_n}{4^{\beta-1}} z^{\beta/\alpha}_{n-\sigma_1} + P_n z^{\gamma/\alpha}_{n+\sigma_2} \tag{2.30}$$

for all $n \geq n_2$. In the following we consider the two cases for $\{z_n\}$ as stated in Lemma 2.2.
Case (I). In this case, we have $\Delta z_n > 0$, $\Delta^2 z_n > 0$, and $\Delta^3 z_n > 0$ for all $n \geq n_3 \geq n_2$. From the inequality (2.30), we have

$$\Delta^3 y_n \geq P_n z_{n+\sigma_2}^{\gamma/\alpha} \tag{2.31}$$

for all $n \geq n_3$. Now applying the monotonicity of Δz_n, we obtain

$$\Delta y_n = \Delta z_n + b^\beta \Delta z_{n-\tau_1} + \frac{c^\beta}{2^{\gamma-1}} \Delta z_{n+\tau_2} \leq \left(1 + b^\beta + \frac{c^\beta}{2^{\gamma-1}}\right) \Delta z_{n+\tau_2}$$

for all $n \geq n_3$, and

$$z_{n+\sigma_1-\tau_2} = z_n + \sum_{s=n}^{n+\sigma_1-\tau_2-1} \Delta z_s \geq (\sigma_1 - \tau_2)\Delta z_n \tag{2.32}$$

for all $n \geq n_3$. Combining (2.31), (2.32) and (2.32), we obtain

$$\Delta^3 y_n \geq \frac{P_n(\sigma_1 - \tau_2)^{\gamma/\alpha}}{\left(1 + b^\beta + \frac{c^\beta}{2^{\gamma-1}}\right)^{\gamma/\alpha}} \left(\Delta y_{n-\sigma_1+\sigma_2}\right)^{\gamma/\alpha}$$

for all $n \geq n_3$. By setting $w_n = \Delta y_n$, we have $w_n > 0$, $\Delta w_n > 0$, and

$$\Delta^2 w_n \geq \frac{P_n(\sigma_1 - \tau_2)^{\gamma/\alpha}}{\left(1 + b^\beta + \frac{c^\beta}{2^{\gamma-1}}\right)^{\gamma/\alpha}} w_{n-\sigma_1+\sigma_2}^{\gamma/\alpha}$$

for all $n \geq n_3$. This implies that $\{w_n\}$ is a positive increasing solution of the inequality (2.27), which is a contradiction.
Case (II). In this case, we have $\Delta z_n > 0$, $\Delta^2 z_n < 0$, and $\Delta^3 z_n > 0$ for all $n \geq n_3 \geq n_2$. Using the monotonicity of Δz_n, we have

$$\Delta y_n = \Delta z_n + b^\beta \Delta z_{n-\tau_1} + \frac{c^\beta}{2^{\gamma-1}} \Delta z_{n+\tau_2} \leq \left(1 + b^\beta + \frac{c^\beta}{2^{\gamma-1}}\right) \Delta z_{n-\tau_1}$$

for all $n \geq n_3$, and

$$z_n = z_{n-\sigma_1+\tau_1} + \sum_{s=n-(\sigma_1-\tau_1)}^{n-1} \Delta z_s \geq (\sigma_1 - \tau_1)\Delta z_n \tag{2.33}$$

for all $n \geq n_3$. Again from (2.30), we have

$$\Delta^3 y_n \geq \frac{Q_n}{4^{\beta-1}} z_{n-\sigma_1}^{\beta/\alpha} \tag{2.34}$$

for all $n \geq n_3$. Using (2.33) and (2.33) in (2.34), we obtain

$$\Delta^2 y_n \geq \frac{Q_n(\sigma_1 - \tau_1)^{\beta/\alpha}}{4^{\beta-1}\left(1 + b^\beta + \frac{c^\beta}{2^{\gamma-1}}\right)^{\beta/\alpha}} \left(\Delta y_{n-\sigma_1+\tau_1}\right)^{\beta/\alpha}$$

for all $n \geq n_3$. By setting $w_n = \Delta y_n$, we see that $\{w_n\}$ is a positive decreasing solution of the inequality (2.28), which is a contradiction. This completes the proof. □

Theorem 2.8. *Assume $\gamma \geq 1$, $0 < \beta \leq 1$, $b \geq 1$, $c \geq 1$, and $\sigma_1 > \max\{\tau_1, \tau_2\}$. If the second order difference inequality*

$$\Delta^2 y_n - \frac{P_n(\sigma_1 - \tau_2)^{\gamma/\alpha}}{4^{\gamma-1}\left(1 + \frac{b^\gamma}{2^{\beta-1}} + c^\gamma\right)^{\gamma/\alpha}} y^{\gamma/\alpha}_{n-\sigma_1+\sigma_2} \geq 0 \tag{2.35}$$

has no positive increasing solution, and if the second order difference inequality

$$\Delta^2 y_n - \frac{Q_n(\sigma_1 - \tau_1)^{\beta/\alpha}}{\left(1 + \frac{b^\gamma}{2^{\beta-1}} + c^\gamma\right)^{\beta/\alpha}} y^{\beta/\alpha}_{n-\sigma_1+\tau_1} \geq 0 \tag{2.36}$$

has no positive decreasing solution, then every solution of equation (1.1) is oscillatory.

Proof. The proof is similar to that of Theorem 2.7, and hence the details are omitted. □

Corollary 2.9. *Let $\alpha = \beta = \gamma \geq 1$, and $\sigma_2 > \sigma_1 + 2$ with $\sigma_1 > \max\{\tau_1, \tau_2\}$. If*

$$\limsup_{n\to\infty} \sum_{s=n}^{n-\sigma_1+\sigma_2-2} (n - \sigma_1 + \sigma_2 - s - 1)\, P_s > \frac{\left(1 + b^\alpha + \frac{c^\alpha}{2^{\alpha-1}}\right) 4^{\alpha-1}}{(\sigma_1 - \tau_2)}, \tag{2.37}$$

and

$$\limsup_{n\to\infty} \sum_{s=n-(\sigma_1-\tau_1)}^{n} (n - s + 1) Q_s > \frac{\left(1 + b^\alpha + \frac{c^\alpha}{2^{\alpha-1}}\right) 4^{\alpha-1}}{(\sigma_1 - \tau_1)} \tag{2.38}$$

then every solution of equation (1.1) is oscillatory.

Proof. By Lemma 7.6.15 of [1], conditions (2.37) and (2.38) ensure that the inequalities (2.14) and (2.15) have no positive increasing solution and no positive decreasing solution, respectively. Now the conclusion follows from Theorem 2.4. □

Corollary 2.10. *Let $0 < \beta \leq 1$, $\gamma \geq 1$ with $\beta < \alpha < \gamma$, $b \leq 1$, $c \leq 1$, and $\sigma_2 > \sigma_1 + 2$ with $\sigma_1 > \max\{\tau_1, \tau_2\}$. If*

$$\sum_{n=n_0}^{\infty} \sum_{s=n+\sigma_1-\sigma_2+1}^{n-1} P_s = \infty, \tag{2.39}$$

and

$$\sum_{n=n_0}^{\infty} \sum_{s=n}^{n+\sigma_1-\tau_1} Q_s = \infty, \tag{2.40}$$

then every solution of equation (1.1) is oscillatory.

Proof. By Lemmas 2.2 and 2.3 of [16], conditions (2.39) and (2.40) ensure that the inequalities (2.16) and (2.17) have no positive increasing solution, and no positive decreasing solution, respectively. Now the conclusion follows from Theorem 2.5. □

Corollary 2.11. *Let $\beta \geq 1$, $0 < \gamma \leq 1$, with $\gamma < \alpha < \beta$, $b \leq 1$, $c \leq 1$, and $\sigma_2 > \sigma_1 + 2$ with $\sigma_1 > \max\{\tau_1, \tau_2\}$. If*

$$\sum_{n=n_0}^{\infty} \sum_{s=n+\sigma_1-\sigma_2+1}^{n-1} P_s = \infty, \tag{2.41}$$

and

$$\sum_{n=n_0}^{\infty} \sum_{s=n}^{n+\sigma_1-\tau_1} Q_s = \infty, \tag{2.42}$$

then every solution of equation (1.1) is oscillatory.

Proof. By Lemmas 2.2 and 2.3 of [16], conditions (2.41) and (2.42) ensure that the difference inequalities (2.25) and (2.26) have no positive increasing, and no positive decreasing solution, respectively. Then the conclusion follows from Theorem 2.6. □

3 Examples

In this section, we present three examples to illustrate the main results.

Example 3.1. Consider the following third order difference equation

$$\Delta^3 \left(x_n + 2x_{n-1} + 3x_{n+2}\right)^3 = 64(n+1)x_{n-3}^3 + 64nx_{n+6}^3, \quad n \geq 3. \tag{3.1}$$

Here, $b = 2$, $c = 3$, $\alpha = \beta = \gamma = 3$, $\tau_1 = 1$, $\tau_2 = 2$, $\sigma_1 = 3$, $\sigma_2 = 6$, $q_n = 64(n+1)$, $p_n = 64n$, $Q_n = 64(n-2)$, $P_n = 64(n-3)$. Then it is easy to see that all the conditions of Corollary 2.9 are satisfied. Therefore every solution of equation (3.1) is oscillatory. In fact $\{(-1)^n\}$ is one such oscillatory solution of equation (3.1).

Example 3.2. Consider the following third order difference equation

$$\Delta^3 \left(x_n + \frac{1}{2}x_{n-1} + \frac{1}{3}x_{n+2}\right) = \frac{25}{3}x_{n-3}^{\frac{1}{3}} + \frac{5}{3}x_{n+6}^3, \quad n \geq 5. \tag{3.2}$$

Here, $b = \frac{1}{2}$, $c = \frac{1}{3}$, $\alpha = 1$, $\beta = \frac{1}{3}$, $\gamma = 3$, $\tau_1 = 1$, $\tau_2 = 2$, $\sigma_1 = 3$, $\sigma_2 = 6$, $q_n = \frac{25}{3}$, $p_n = \frac{5}{3}$, $Q_n = \frac{25}{3}$, and $P_n = \frac{5}{3}$. Then it is easy to see that all the conditions of Corollary 2.10 are satisfied. Therefore every solution of equation (3.2) is oscillatory. In fact $\left\{(-1)^{3n}\right\}$ is one such oscillatory solution of equation (3.2).

Example 3.3. Consider the following third order difference equation

$$\Delta^3 \left(x_n + \frac{1}{2}x_{n-1} + x_{n+2}\right) = (n+12)x_{n-5}^3 + nx_{n+8}^{\frac{1}{3}}, \quad n \geq 5. \tag{3.3}$$

Here, $b = \frac{1}{2}$, $c = 1$, $\alpha = 1$, $\beta = 3$, $\gamma = \frac{1}{3}$, $\tau_1 = 1$, $\tau_2 = 2$, $\sigma_1 = 5$, $\sigma_2 = 8$, $q_n = n + 12$, $p_n = n$, $Q_n = n + 7$, and $P_n = n - 5$. Then it is easy to see that all the conditions of Corollary 2.11 are satisfied. Therefore every solution of equation (3.3) is oscillatory. In fact $\left\{(-1)^{3n}\right\}$ is one such oscillatory solution of equation (3.3).

We conclude this paper with the following remark.

Remark 3.4. The results obtained in this paper extend and complement to that of in [2, 6, 9, 10, 13–15]. Further if $c_n = 0$ and $p_n = 0$ for all $n \geq n_0$, then our results reduced to some of the results in [1, 5, 7, 13, 14]. It would be interesting to study the oscillatory behavior of the equation

$$\Delta(a_n\Delta^2 \left(x_n + b_n x_{n-\tau_1} + c_n x_{n+\tau_2}\right)^\alpha) = q_n x_{n-\sigma_1}^\beta + p_n x_{n+\sigma_2}^\gamma, \quad n \geq n_0,$$

when $\sum_{n=n_0}^{\infty} \frac{1}{a_n} = \infty$ or $\sum_{n=n_0}^{\infty} \frac{1}{a_n} < \infty$.

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References

[1] R. P. Agarwal, M. Bohner, S. R. Grace, D. O'Regan, *Discrete oscillation theory*, Hindawi Publ. Corp., New York, 2005. MR2179948; url

[2] R. P. Agarwal, S. R. Grace, Oscillation of certain third-order difference equations, *Comput. Math. Appl.* **42**(2001), 379–384. MR1837999; url

[3] R. P. Agarwal, S. R. Grace, E. A. Bohner, On the oscillation of higher order neutral difference equations of mixed type, *Dynam. Systems Appl.* **11**(2002), 459–470. MR1946136

[4] J. F. Ferreira, S. Pinelas, Oscillatory mixed difference systems, *Adv. Difference Equ.* **2006**, Art. ID 92923, 1–18. MR2238984

[5] S. R. Grace, Oscillation of certain difference equations of mixed type, *J. Math. Anal. Appl.* **224**(1998), 241–254. MR1637453; url

[6] S. R. Grace, R. P. Agarwal, J. R. Graef, Oscillation criteria for certain third order nonlinear difference equation, *Appl. Anal. Discrete Math.* **3**(2009), 27–38. MR2499304; url

[7] S. R. Grace, S. Dontha, Oscillation of higher order neutral difference equations of mixed type, *Dynam. Systems Appl.* **12**(2003), 521–532. MR2020481

[8] B. Karpuz, R. N. Rath, S. K. Rath, On oscillation and asymptotic behaviour of a higher order functional difference equation of neutral type, *Int. J. Difference Equ.* **4**(2009), 69–96. MR2553889

[9] S. H. Saker, Oscillation and asymptotic behavior of third-order nonlinear neutral delay difference equations, *Dynam. Systems Appl.* **15**(2006), 549–567. MR2367663

[10] B. Smith, Oscillatory and asymptotic behavior in certain third order difference equations, *Rocky Mountain J. Math.* **17**(1987), 597–606. MR908266; url

[11] E. Thandapani, M. Vijaya, T. Li, On the oscillation of third order half linear neutral type difference equations, *Electron. J. Qual. Theory Differ. Equ.* **2011**, No. 76., 1–13. MR2838504

[12] E. Thandapani, K. Mahalingam, Oscillatory properties of third order neutral delay difference equations, *Demonstratio Math.* **35**(2002), 325–336. MR1907305

[13] E. Thandapani, S. Selvarangam, Oscillation of third-order half-linear neutral difference equations, *Math. Bohem.* **138**(2013), 87–104. MR3076223

[14] E. Thandapani, S. Selvarangam, Oscillation results for third order halflinear neutral difference equations, *Bull. Math. Anal. Appl.* **4**(2012), 91–102. MR2955923

[15] E. Thandapani, N. Kavitha, Oscillatory behavior of solutions of certain third order mixed neutral difference equations, *Acta Math. Sci. Ser. B Engl. Ed.* **33**(2013), 218–226. MR39A10; url

[16] E. Thandapani, N. Kavitha, Oscillation theorems for second order nonlinear neutral difference equations of mixed type, *J. Math. Comput. Sci.* **1**(2011), 89–102. MR2913381