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The Foam Drainage Equation with Time- and Space-Fractional Derivatives Solved by The Adomian Method

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Abstract

In this paper, by introducing the fractional derivative in the sense of Caputo, we apply the Adomian decomposition method for the foam drainage equation with time- and space-fractional derivative. As a result, numerical solutions are obtained in a form of rapidly convergent series with easily computable components.

Keywords Caputo Fractional derivative, Adomian method, Foam Drainage Equation, Evolution Equations.
AMS Subject Classifications[2000]: 26A33, 35F25

1 Introduction

Since the introduction by Adomian of the decomposition method [4, 5] at the begin of 1980s, the algorithm has been widely used for obtaining analytic solutions of physically significant equations [4, 5, 6, 16, 26, 27, 28, 29, 30, 31]. With this method, we can easily obtain approximate solutions in the form of a rapidly convergent infinite series with each term computed conveniently. As

it is known, for the nonlinear equations with derivatives of integer order, many methods are used to derive approximation solutions [3, 4, 5, 7, 10]. However, for the fractional differential equations, there are only limited approaches, such as Laplace transform method [21], the Fourier transform method [17], the iteration method [22] and the operational method [20, 23].
In recent years, the fractional differential equations have attracted great attention; they are used in many areas of physics and engineering [9, 15, 32], like phenomena in electromagnetic theory, acoustics, electrochemistry and material science [11, 21, 22, 32].
The study to foam drainage equation is very significant for that the equation is a simple model of the flow of liquid through channels (Plateau borders [34]) and nodes (intersection of four channels) between the bubbles, driven by gravity and capillarity [18, 33]. It has been studied by many authors [12, 13, 24]. However, as we know, the study for the foam drainage equation with time-and space-fractional derivatives of this form

$$D_t^\alpha u - \frac{1}{2}uu_{xx} + 2u^2 D_x^\beta u - (D_x^\beta u)^2 = 0, \tag{1}$$

by the Adomian method (ADM) has not been investigated. Here α and β are the parameters standing for the order of the fractional time and space derivatives, respectively and they satisfy $0 < \alpha, \beta \leq 1$ and $x > 0$. In fact, different response equations can be obtained when at lest one of the parameters varies. When $\alpha = \beta = 1$, the fractional equation reduces to the foam drainage equation of the form

$$u_t - \frac{1}{2}uu_{xx} + 2u^2 u_x - (u_x)^2 = 0. \tag{2}$$

We introduce Caputo fractional derivative and apply the ADM to derive numerical solutions of the foam drainage equation with time-and space-fractional derivatives.
The paper is organized as follows. In Sec. II, some necessary details on the fractional calculus are provided. In Sec. III, the foam drainage equation with time and space- fractional derivative is studied with the ADM. Finally, conclusions follow.

2 Description of Fractional Calculus

There are several mathematical definitions about fractional derivative [21, 22]. Here, we adopt the two usually used definitions: the Caputo and its reverse operator Riemann-Liouville. That is because Caputo fractional derivative allows traditional initial condition assumption and boundary conditions. More details one can consults [21]. In the following, we will give the necessary notation and basic definitions.

Definition 2.1 *A real valued function $f(x), x > 0$ is said to be in the space $C_\mu, \mu \in \mathbb{R}$ if there exists a real number $p > \mu$ such that $f(x) = x^p f_1(x)$ where*

$f_1(x) \in C([0,\infty))$.

<u>Definition</u> 2.2 *A function $f(x), x>0$ is said to be in the space $C^n_\mu, n \in \mathbb{N}$, if $f^{(n)} \in C_\mu$.*

<u>Definition</u> 2.3 *The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, for a function $f \in C_\mu$,*
$(\mu \geq -1)$ is defined as

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt; \quad \alpha > 0, x > 0 \tag{3}$$
$$J^0 f(x) = f(x).$$

For the convenience of establishing the results for the fractional foam drainage equation, we give one basic property

$$J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x). \tag{4}$$

For the expression (3), when $f(x) = x^\beta$ we get another expression that will be used later:

$$J^\alpha x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} x^{\alpha+\beta}. \tag{5}$$

<u>Definition</u> 2.4 *The fractional derivative of $f \in C^n_{-1}$ in the Caputo's sense is defined as*

$$D^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, & n-1 < \alpha < n, n \in \mathbb{N}^*, \\ \frac{d^n}{dt^n} f(t), & \alpha = n. \end{cases} \tag{6}$$

According to the Caputo's derivative, we can easily obtain the following expressions:

$$D^\alpha K = 0; \quad K \text{ is a constant}, \tag{7}$$

$$D^\alpha t^\beta = \begin{cases} \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta-1)} t^{\beta-\alpha}, & \beta > \alpha - 1, \\ 0, & \beta \leq \alpha - 1. \end{cases} \tag{8}$$

Details on Caputo's derivative can be found in [21].

<u>Remark</u> 2.1 *In this paper, we consider equation (1) with time-and space-fractional derivative. When $\alpha \in \mathbb{R}^+$, we have:*

$$D_t^\alpha u(x,t) = \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} \frac{\partial^n u(x,\tau)}{\partial \tau^n} d\tau, & n-1 < \alpha < n \\ \frac{\partial^n u(x,t)}{\partial t^n}, & \alpha = n. \end{cases} \tag{9}$$

The form of the space fractional derivative is similar to the above and we just omit it here.

3 Applications of the ADM Method

Consider the foam drainage equation with time and space-fractional derivatives Eq.(1).
In order to solve numerical solutions for this equation by using ADM method, we rewrite it in the operator form

$$D_t^\alpha u = \frac{1}{2} u u_{xx} - 2u^2 D_x^\beta u + (D_x^\beta u)^2; \;\; 0 < \alpha \leq 1, \;\; 0 < \beta \leq 1, \tag{10}$$

where the operators D_t^α and D_x^β stand for the fractional derivative and are defined as in (6).
Take the initial condition as

$$u(x, 0) = f(x). \tag{11}$$

Applying the operator J^α, the inverse of D^α on corresponding sub-equation of Eq.(10), using the initial condition (11), yields:

$$u(x,t) = f(x) - 2J^\alpha \Phi_1(u(x,t)) + \frac{1}{2} J^\alpha \Phi_2(u(x,t)) + J^\alpha \Phi_3(u(x,t)), \tag{12}$$

where $\Phi_1(u) = u^2 D_x^\beta u$, $\Phi_2(u) = u u_{xx}$ and $\Phi_3(u) = (D_x^\beta u)^2$. Following Adomian decomposition method [4, 5], the solution is represented as infinite series like

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t). \tag{13}$$

The nonlinear operators $\Phi_1(u)$, $\Phi_2(u)$ and $\Phi_3(u)$ are decomposed in these forms

$$\Phi_1(u) = \sum_{n=0}^{\infty} A_n, \quad \Phi_2(u) = \sum_{n=0}^{\infty} B_n, \;\; \Phi_3(u) = \sum_{n=0}^{\infty} C_n, \tag{14}$$

where A_n, B_n and C_n are the so-called Adomian polynomials and have the form

$$\begin{aligned} A_n &= \frac{1}{n!}\frac{d^n}{d\lambda^n}\Big[\Phi_1\Big(\sum_{k=0}^{\infty}\lambda^k u_k\Big)\Big]_{\lambda=0} = \frac{1}{n!}\frac{d^n}{d\lambda^n}\Big[\Big(\sum_{k=0}^{\infty}\lambda^k u_k\Big)^2 D_x^\beta\Big(\sum_{k=0}^{\infty}\lambda^k u_k\Big)\Big]_{\lambda=0}, \\ B_n &= \tfrac{1}{n!}\tfrac{d^n}{d\lambda^n}\Big[\Phi_2\Big(\textstyle\sum_{k=0}^{\infty}\lambda^k u_k\Big)\Big]_{\lambda=0} = \tfrac{1}{n!}\tfrac{d^n}{d\lambda^n}\Big[\Big(\textstyle\sum_{k=0}^{\infty}\lambda^k u_k\Big)\Big(\textstyle\sum_{k=0}^{\infty}\lambda^k u_{kxx}\Big)\Big]_{\lambda=0}, \\ C_n &= \tfrac{1}{n!}\tfrac{d^n}{d\lambda^n}\Big[\Phi_3\Big(\textstyle\sum_{k=0}^{\infty}\lambda^k u_k\Big)\Big]_{\lambda=0} = \tfrac{1}{n!}\tfrac{d^n}{d\lambda^n}\Big[\Big(D_x^\beta(\textstyle\sum_{k=0}^{\infty}\lambda^k u_k)\Big)^2\Big]_{\lambda=0}. \end{aligned} \tag{15}$$

In fact, these Adomian polynomials can be easily calculated. Here we give the first three components of these polynomials:

$$
\begin{aligned}
A_0 &= u_0^2 D_x^\beta u_0, \\
A_1 &= 2u_0u_1 D_x^\beta u_0 + u_0^2 D_x^\beta u_1, \\
A_2 &= u_1^2 D_x^\beta u_0 + 2u_0u_2 D_x^\beta u_0 + 2u_0u_1 D_x^\beta u_1 + u_0^2 D_x^\beta u_2, \\
A_3 &= u_1^2 D_x^\beta u_1 + 2u_1u_2 D_x^\beta u_0 + 2u_0u_2 D_x^\beta u_1 + 2u_0u_1 D_x^\beta u_2 \\
&\quad + 2u_0u_3 D_x^\beta u_0 + u_0^2 D_x^\beta u_0^2 D_x^\beta u_3.
\end{aligned} \tag{16}
$$

The first three components of B_n are

$$
\begin{aligned}
B_0 &= u_0 u_{0xx}, \\
B_1 &= u_0 u_{1xx} + u_1 u_{0xx}, \\
B_2 &= u_2 u_{0xx} + u_0 u_{2xx} + u_1 u_{1xx}, \\
B_3 &= u_3 u_{0xx} + u_0 u_{3xx} + u_2 u_{1xx} + u_1 u_{2xx},
\end{aligned} \tag{17}
$$

and those of C_n are given by

$$
\begin{aligned}
C_0 &= (D_x^\beta u_0)^2, \\
C_1 &= 2D_x^\beta u_0 D_x^\beta u_1, \\
C_2 &= (D_x^\beta u_1)^2 + 2D_x^\beta u_0 D_x^\beta u_2, \\
C_3 &= 2D_x^\beta u_1 D_x^\beta u_2 + 2D_x^\beta u_0 D_x^\beta u_3.
\end{aligned} \tag{18}
$$

Other polynomials can be generated in a like manner.
Substituting the decomposition series (13) and (14) into Eq.(12), yields the following recursive formula:

$$
\begin{aligned}
u_0(x,t) &= f(x), \\
u_{n+1}(x,t) &= -2J^\alpha\Big(A_n\Big) + \tfrac{1}{2}J^\alpha\Big(B_n\Big) + J^\alpha\Big(C_n\Big); \;\; n \geq 0.
\end{aligned} \tag{19}
$$

The Adomian decomposition method converges generally very quickly. Details about its convergence and convergence speed can be found in [1, 2, 8, 14]. Here, according to the above steps, we will derive the numerical solution for the equation with time and space-fractional derivative in details.

3.1 Numerical Solutions of Time-Fractional Foam Drainage Equation

Consider the following form of the time-fractional equation

$$
D_t^\alpha u = \frac{1}{2} u u_{xx} - 2u^2 u_x + u_x^2; \;\; 0 < \alpha \leq 1, \tag{20}
$$

with the initial condition

$$
u(x,0) = f(x) = -\sqrt{c}\tanh\sqrt{c}(x). \tag{21}
$$

where c is the velocity of wavefront [23].
The exact solution of (20) for the special case $\alpha = \beta = 1$ is

$$
u(x,t) = \begin{cases} -\sqrt{c}\tanh(\sqrt{c}(x-ct)); & x \leq ct \\ 0; & x > ct. \end{cases} \tag{22}
$$

In order to obtain numerical solution of equation (20), substituting the initial condition (21) and using the Adomian polynomials (16,17,18) into the expression (19), we can compute the results. For simplicity, we only give the first few terms of series:

$$\begin{gathered} u_0 = f(x), \\ u_1 = -2J^\alpha(A_0) + \tfrac{1}{2}J^\alpha(B_0) + J^\alpha(C_0) \\ = -2J^\alpha(u_0^2 u_{0x}) + \tfrac{1}{2}J^\alpha(u_0 u_{0xx}) + J^\alpha(u_{0x}^2) = f_1 \tfrac{t^\alpha}{\Gamma(\alpha+1)}, \\ u_2 = -2J^\alpha(A_1) + \tfrac{1}{2}J^\alpha(B_1) + J^\alpha(C_1) \\ = -2J^\alpha(2u_0 u_1 u_{0x} + u_0^2 u_{1x}) + \tfrac{1}{2}J^\alpha(u_0 u_{1xx} + u_1 u_{0xx}) + J^\alpha(2u_{0x}u_{1x}) \\ = f_2 \tfrac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \end{gathered} \tag{23}$$

where

$$\begin{gathered} f(x) = -\sqrt{c}\tanh(\sqrt{c}x), \quad f_1(x) = -2f^2 f_x + \frac{1}{2} f f_{xx} + f_x^2, \\ f_2(x) = -2(2ff_xf_1 + f^2 f_{1x}) + \tfrac{1}{2} f f_{1xx} + \tfrac{1}{2} f_1 f_{xx} + 2f_x f_{1x}, \\ f_3(x) = -2f_x f_1^2 \tfrac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)} - 4ff_xf_2 - 4ff_1f_{1x}\tfrac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)} - 2f^2 f_{2x} + \tfrac{1}{2} f_{xx} f_2 \\ +\tfrac{1}{2} f f_{2xx} + \tfrac{1}{2} f_1 f_{1xx} \tfrac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)} + f_{1x}^2 \tfrac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)} + 2f_x f_{2x}. \end{gathered} \tag{24}$$

Then we can have the numerical solution of time-fractional equation (20) under the series form

$$u(x,t) = f(x) + f_1(x)\frac{t^\alpha}{\Gamma(\alpha+1)} + f_2(x)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + f_3(x)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + ... \tag{25}$$

In order to check the efficiency of the proposed ADM for the equation (20), we draw figures for the numerical solutions with $\alpha = \frac{1}{2}$ as well as the exact solution (22) when $\alpha = \beta = 1$. Figure 1(a) stands for the numerical solution of (25). Figure 1(b) shows the exact solution of equation (22). From these figures, we can appreciate how closely are the two solutions. This is to say that good approximations are achieved using the ADM method.

3.2 Numerical Solutions of Space-Fractional Foam Drainage Equation

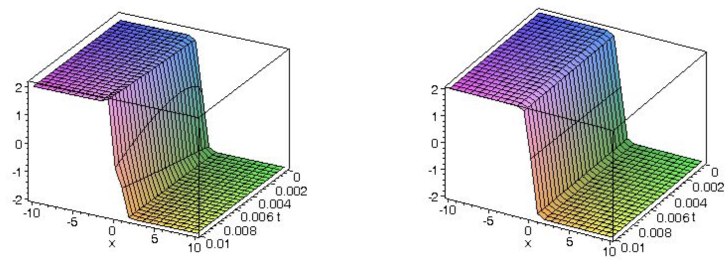
In this section, we will take the space-fractional equation as another example to illustrate the efficiency of the method. As the main computation method is the same as the above, we will omit the heavy calculation and only give some necessary expressions.

Considering the operator form of the space-fractional equation

$$u_t = \frac{1}{2}uu_{xx} - 2u^2 D_x^\beta u + (D_x^\beta u)^2; \quad 0 < \beta \leq 1. \tag{26}$$

Assuming the condition as

$$u(x,0) = f(x) = x^2. \tag{27}$$



(a) **(b)**

Figure 1: Representing time-fractional solutions of Eq.(20). In (a), solution obtained by the Adomian method. In (b), the exact solution (22).

Initial condition has been taken as the above polynomial to avoid heavy calculation of fractional differentiation.

In order to estimate the numerical solution of equation (26), substituting (14), (15) and the initial condition (27) into (19), we get the Adomian solution. Here, we give the first few terms of the series solution:

$$\begin{array}{c} u_0 = x^2, \\ u_1 = -2J(A_0) + \frac{1}{2}J(B_0) + J(C_0) \\ = -2J(u_0^2 D_x^\beta u_0) + \frac{1}{2}J(u_0 u_{0xx}) + J(D_x^\beta u_0)^2 \\ = (f_1 x^{6-\beta} + x^2 + f_2 x^{4-2\beta})t, \\ u_2 = -2J(A_1) + \frac{1}{2}J(B_1) + J(C_1) \\ = -2J(2u_0 u_1 D_x^\beta u_0 + u_0^2 D_x^\beta u_1) + \frac{1}{2}J(u_0 u_{1xx} + u_1 u_{0xx}) + J(2D_x^\beta u_0 D_x^\beta u_1) \\ = (f_3 x^{10-2\beta} + f_4 x^{8-3\beta} + f_5 x^{6-\beta} + f_6 x^{6-4\beta} + f_7 x^{4-2\beta} + x^2)\frac{t^2}{2}, \end{array} \tag{28}$$

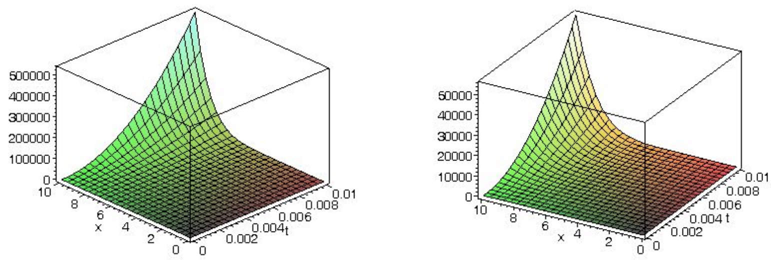
where

$$\begin{array}{c} f(x) = x^2, \quad f_1 = -2\dfrac{\Gamma(3)}{\Gamma(3-\beta)}, \quad f_2 = \left[\dfrac{\Gamma(3)}{\Gamma(3-\beta)}\right]^2, \\ f_3 = \left[\frac{\Gamma(7-\beta)}{\Gamma(7-2\beta)} - 4\frac{\Gamma(3)}{\Gamma(3-\beta)}\right] f_1, \\ f_4 = (\frac{\Gamma(5-2\beta)}{\Gamma(5-3\beta)} - 4\frac{\Gamma(3)}{\Gamma(3-\beta)})f_2 + 2(\frac{\Gamma(3)}{\Gamma(3-\beta)}\frac{\Gamma(7-\beta)}{\Gamma(7-2\beta)})f_1 \\ f_5 = ((6-\beta)(5-\beta)+2)\frac{f_1}{2} - 3\frac{\Gamma(3)}{\Gamma(3-\beta)}, \\ f_6 = 2\frac{\Gamma(3)}{\Gamma(3-\beta)}\frac{\Gamma(5-2\beta)}{\Gamma(5-3\beta)} f_2, \\ f_7 = ((4-2\beta)(3-2\beta)+6)\frac{f_2}{2}. \end{array} \tag{29}$$

Then we obtain a numerical solution of space-fractional Eq.(26) in series form

$$\begin{array}{c} u(x,t) = x^2 + (f_1 x^{6-\beta} + f_2 x^{4-2\beta} + x^2)t + \\ +(f_3 x^{10-2\beta} + f_4 x^{8-3\beta} + f_5 x^{6-\beta} + f_6 x^{6-4\beta} + f_7 x^{4-2\beta} + x^2)\frac{t^2}{2} + ... \end{array} \tag{30}$$

Figures 2(a, b) show, respectively, the numerical solutions given by expression (30) for space-fractional Eq.(26) with $\beta = \frac{1}{2}$ and $\beta = 1$. From these figures, we can appreciate the convergence rapidity of Adomian solutions.



(a) **(b)**

Figure 2: Representing space-fractional solutions of Eq.(26). In (a), solution obtained by the Adomian method for $\beta = 1/2$. In (b), solution obtained by the Adomian method for $\beta = 1$.

4 Conclusion

In this paper, the ADM has been successfully applied to derive explicit numerical solutions for the time-and space-fractional foam drainage equation. The above procedure shows that the ADM method is efficient and powerful in solving wide classes of equations in particular evolution fractional order equations.

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