Existence of solution for fractional Langevin equation: variational approach

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Abstract. We consider the Dirichlet problem for the fractional Langevin equation with two fractional order derivatives

$$-_0D_t^\alpha(_0D_t^\alpha u(t)) = f(t, u(t), _0D_t^\alpha u(t)), \quad t \in [0,1],$$
$$u(0) = u(1) = 0.$$

The existence of a nontrivial solution is stated through an iterative method based on mountain pass techniques.

Keywords: Riemann–Liouville fractional operator, fractional Langevin equation, critical point theory, variational method.

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1 Introduction

The Langevin equation was proposed by Langevin [18] in 1908 to give an elaborate description of Brownian motion. In his work, Newton's second law was applied to a Brownian particle to invent the "$F = ma$" of stochastic physics which is now called "Langevin equation". On the other hand, Einstein's method of studying Brownian motion is based on the Fokker–Planck equation governing the time evolution of the Brownian particle's probability density. Langevin's approach is more simple than Einstein's at the cost of forcing into existence new mathematical objects (Gaussian white noise and the stochastic differential equation) with unusual properties. For a long time, the Langevin equation was widely used to describe the dynamical processes taking place in fluctuating environments [11]. However, for systems in disordered or fractal medium, some interesting phenomena such as anomalous transport [15] are observed. In these cases, the ordinary Langevin equation cannot give a correct description of the dynamics any more. Thus, the generalized Langevin equation (GLE) was introduced by Kubo [17] in 1966, where a fractional memory kernel was incorporated into the Langevin equation to describe the fractal and memory properties. The generalization of the Langevin equation has since become a hot research topic.

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As the intensive development of fractional derivative, a natural generalization of the Langevin equation is to replace the ordinary derivative by a fractional derivative to yield fractional Langevin equation (FLE), which can be considered as a particular case of the GLE. FLE was introduced by Mainardi and collaborators [22, 23] in earlier 1990s. The literature on this respect is huge, several different types of FLE were studied in [5, 6, 9, 10, 30, 19, 20, 21]. The usual FLE involving only one fractional order was studied in [9, 21]; the Langevin equation containing both fractional memory kernel and fractional derivative was studied in [10, 30]; the nonlinear Langevin equation involving two fractional orders was studied in [5, 6, 19, 20]. We focus on a particular case of the last type of FLE proposed first by Lim et al. [19] in 2008:

$$_0D_t^\beta(_0D_t^\alpha + \lambda)u(t) = f(t, u(t)).$$

More precisely, we study the Dirichlet boundary value problem of the Langevin equation with two fractional orders derivatives given by

$$-_0D_t^\alpha(_0D_t^\alpha u(t)) = f(t, u(t), _0D_t^\alpha u(t)), \ t \in [0, 1], \tag{1.1}$$
$$u(0) = u(1) = 0,$$

where $\frac{1}{2} < \alpha < 1$, $_0D_t^\alpha u$ is the Riemann–Liouville fractional derivative and $f \in C([0, 1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$. In particular, if $f(t, u(t), _0D_t^\alpha u(t)) = \lambda_0 D_t^\alpha u(t) - g(t, u(t))$, we recover a model of the nonlinear fractional Langevin equation

$$_0D_t^\alpha(_0D_t^\alpha + \lambda)u(t) = g(t, u(t)). \tag{1.2}$$

In recent years, the boundary value problem of fractional order differential equations have emerged as an important area of research, since these problems have applications in various disciplines of science and engineering such as mechanics, electricity, chemistry, biology, economics, control theory, signal and image processing, polymer rheology, regular variation in thermodynamics, biophysics, aerodynamics, viscoelasticity and damping, electrodynamics of complex medium, wave propagation, and blood flow phenomena [16, 26, 28, 29]. Many researchers have studied the existence theory for nonlinear fractional differential equations with a variety of boundary conditions; for instance, see the papers [1, 2, 3, 4, 8, 25, 35] and the references therein. However, as to the nonlinear Langevin equation involving two different fractional orders, the research work is still in its infancy and is focused on boundary value problems. The Dirichlet boundary value problem was studied in [5], while the three-point boundary value problem was studied in [6], both of them by using fixed point theorem.

It should be noted that critical point theory and variational methods have also turned out to be very effective tools in determining the existence of solutions for integer and fractional order differential equations. The idea behind them is trying to find solutions of a given boundary value problem by looking for critical points of a suitable energy functional defined on an appropriate function space. In the last 30 years, the critical point theory has become a wonderful tool in studying the existence of solutions to differential equations with variational structures, we refer the reader to the books due to Mawhin and Willem [24], Rabinowitz [27], Schechter [31] and the papers [12, 13, 14, 32, 33, 34, 36].

Motivated by these previous works, we consider the solvability of the Dirichlet problem (1.1) by using variational methods and iterative technique. For that purpose, we say a function $u \in E^\alpha$ is a weak solution of problem (1.1) if

$$-\int_0^1 (_0D_t^\alpha u(t), _tD_1^\alpha v(t)) \, dt - \int_0^1 f(t, u(t), _0D_t^\alpha u(t)) v(t) \, dt = 0$$

for all $v(t) \in E^\alpha$, (see Section 2 for the definition of E^α).

Since the nonlinearity f depends on the $_0D_t^\alpha$ of the solution, solving (1.1) is not variational. In fact the well developed critical point theory cannot be applied directly. We follow the ideas of Xie, Xiao and Luo [36] to overcome this difficulty. That is, we associate to problem (1.1) a family of fractional differential equations with no dependence on $_0D_t^\alpha$ of the solution. Namely, for each $w \in E^\alpha$, we consider the problem

$$-_0D_t^\alpha(_0D_t^\alpha u(t)) = f(t, u(t), _0D_t^\alpha w(t)), \ t \in [0,1], \tag{1.3}$$
$$u(0) = u(1) = 0.$$

This problem is variational and we can treat it by variational methods.

Associated to the boundary value problem (1.3), for given $w(t) \in E^\alpha$, we have the functional $I_w \colon E^\alpha \to \mathbb{R}$ defined by

$$I_w(u) = -\frac{1}{2}\int_0^1 (_0D_t^\alpha u(t), _tD_1^\alpha \varphi(t))\, dt - \int_0^1 F(t, u(t), _0D_1^\alpha w(t))\, dt, \tag{1.4}$$

where $F(t, x, \xi) = \int_0^x f(t, s, \xi)\, ds$. By continuity hypothesis on f we have $I_w \in C^1(E^\alpha, \mathbb{R})$ and $\forall v \in E^\alpha$

$$I_w'(u)v = -\frac{1}{2}\int_0^1 (_0D_t^\alpha u(t), _tD_1^\alpha v(t)) + (_0D_t^\alpha v(t), _tD_1^\alpha u(t))\, dt$$
$$- \int_0^1 f(t, u(t), _0D_t^\alpha w(t))v(t)\, dt. \tag{1.5}$$

Moreover, critical points of I_w are weak solutions of (1.3). Therefore, for each $w \in E^\alpha$, we can find a solution $u_w \in E^\alpha$ with some bounds. Next, by iterative methods we can show that there exists a solution for problem (1.1).

Before stating our results, we make precise assumptions on the nonlinear term $f \colon [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$:

(H_1) $\lim_{x \to 0} \frac{f(t,x,\xi)}{x} = 0$ uniformly for $t \in [0,1]$ and $\xi \in \mathbb{R}$.

(H_2) There are positive constants c and $p > 1$ such that

$$|f(t, x, \xi)| \le c(1 + |x|^p), \quad \text{for all } t \in [0,1], \ x, \xi \in \mathbb{R}.$$

(H_3) There exist $\mu > 2$ and $M \ge 0$ such that

$$0 < \mu F(t, x, \xi) \le x f(t, x, \xi) \quad \text{for every } t \in [0,1], \ |x| \ge M, \ \xi \in \mathbb{R},$$

where

$$F(t, x, \xi) = \int_0^x f(t, s, \xi)\, ds.$$

(H_4) There exist constants $c_1, c_2 > 0$ such that

$$F(t, x, \xi) \ge c_1|x|^\mu - c_2, \quad \text{for all } t \in [0,1], \ x, \xi \in \mathbb{R}.$$

(H_5) The function f satisfies the following conditions:

$$|f(t, x, \xi) - f(t, x_1, \xi)| \le L_1|x - x_1|, \quad \forall t \in [0,1], \ x, x_1 \in [-\rho_1, \rho_1], \ \xi \in \mathbb{R}$$
$$|f(t, x, \xi) - f(t, x, \xi_1)| \le L_2|\xi - \xi_1|, \quad \forall t \in [0,1], \ x \in [-\rho_1, \rho_1], \ \xi, \xi_1 \in \mathbb{R}$$

where ρ_1 is a positive constant, which is given below.

Now we are in a position to state our main existence theorem

Theorem 1.1. *Assume that (H_1)–(H_5) hold, and the constant*

$$l := \frac{L_2 \Gamma(\alpha + 1)}{|\cos(\pi\alpha)|[\Gamma(\alpha + 1)]^2 - L_1}$$

satisfies $0 < l < 1$. Then problem (1.1) *has one nontrivial solution.*

The rest of the paper is organized as follows: in Section 2 we present preliminaries on fractional calculus and we introduce the functional setting of the problem. In Section 3 we prove Theorem 1.1.

2   Fractional calculus

In this section we introduce some basic definitions of fractional calculus which are used further in this paper. For the proof see [16], [26] and [29].

Definition 2.1 (Left and right Riemann–Liouville fractional integral)**.** Let u be a function defined on $[a, b]$. The left (right) Riemann–Liouville fractional integral of order $\alpha > 0$ for function u is defined by

$$_a I_t^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - s)^{\alpha-1} u(s)\, ds, \quad t \in [a, b],$$

$$_t I_b^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (s - t)^{\alpha-1} u(s)\, ds, \quad t \in [a, b],$$

provided in both cases that the right-hand side is pointwise defined on $[a, b]$.

Definition 2.2 (Left and right Riemman–Liouville fractional derivative)**.** Let u be a function defined on $[a, b]$. The left and right Riemann–Liouville fractional derivatives of order $\alpha > 0$ for function u denoted by $_a D_t^\alpha u(t)$ and $_t D_b^\alpha u(t)$, respectively, are defined by

$$_a D_t^\alpha u(t) = \frac{d^n}{dt^n}\, _a I_t^{n-\alpha} u(t),$$

$$_t D_b^\alpha u(t) = (-1)^n \frac{d^n}{dt^n}\, _t I_b^{n-\alpha} u(t),$$

where $t \in [a, b]$, $n - 1 \leq \alpha < n$ and $n \in \mathbb{N}$.

The left and right Caputo fractional derivatives are defined via the above Riemann–Liouville fractional derivatives [16]. In particular, they are defined for the functions belonging to the space of absolutely continuous functions.

Definition 2.3. If $\alpha \in (n - 1, n)$ and $u \in AC^n[a, b]$, then the left and right Caputo fractional derivative of order α for function u denoted by $_a^c D_t^\alpha u(t)$ and $_t^c D_b^\alpha u(t)$, respectively, are defined by

$$_a^c D_t^\alpha u(t) = {_a I_t^{n-\alpha}} u^{(n)}(t) = \frac{1}{\Gamma(n - \alpha)} \int_a^t (t - s)^{n-\alpha-1} u^n(s)\, ds,$$

$$_t^c D_b^\alpha u(t) = (-1)^n {_t I_b^{n-\alpha}} u^{(n)}(t) = \frac{(-1)^n}{\Gamma(n - \alpha)} \int_t^b (s - t)^{n-\alpha-1} u^{(n)}(s)\, ds$$

The Riemann–Liouville fractional derivative and the Caputo fractional derivative are connected with each other by the following relations.

Theorem 2.4. *Let $n \in \mathbb{N}$ and $n-1 < \alpha < n$. If u is a function defined on $[a,b]$ for which the Caputo fractional derivatives ${}^c_a D^\alpha_t u(t)$ and ${}^c_t D^\alpha_b u(t)$ of order α exists together with the Riemann–Liouville fractional derivatives ${}_a D^\alpha_t u(t)$ and ${}_t D^\alpha_b u(t)$, then*

$$
{}^c_a D^\alpha_t u(t) = {}_a D^\alpha_t u(t) - \sum_{k=0}^{n-1} \frac{u^{(k)}(a)}{\Gamma(k-\alpha+1)}(t-a)^{k-\alpha}, \quad t \in [a,b],
$$

$$
{}^c_t D^\alpha_b u(t) = {}_t D^\alpha_b u(t) - \sum_{k=0}^{n-1} \frac{u^{(k)}(b)}{\Gamma(k-\alpha+1)}(b-t)^{k-\alpha}, \quad t \in [a,b].
$$

In particular, when $0 < \alpha < 1$, we have

$$
{}^c_a D^\alpha_t u(t) = {}_a D^\alpha_t u(t) - \frac{u(a)}{\Gamma(1-\alpha)}(t-a)^{-\alpha}, \quad t \in [a,b] \tag{2.1}
$$

and

$$
{}^c_t D^\alpha_b u(t) = {}_t D^\alpha_b u(t) - \frac{u(b)}{\Gamma(1-\alpha)}(b-t)^{-\alpha}, \quad t \in [a,b]. \tag{2.2}
$$

Now we remember some properties of the Riemann–Liouville fractional integral and derivative operators.

Theorem 2.5.

$$
{}_a I^\alpha_t \left({}_a I^\beta_t u(t) \right) = {}_a I^{\alpha+\beta}_t u(t) \quad \text{and}
$$

$$
{}_t I^\alpha_b \left({}_t I^\beta_b u(t) \right) = {}_t I^{\alpha+\beta}_b u(t) \quad \forall \alpha, \beta > 0,
$$

in any point $t \in [a,b]$ for continuous function u and for almost every point in $[a,b]$ if the function $u \in L^1[a,b]$.

Theorem 2.6 (Left inverse). *Let $u \in L^1[a,b]$ and $\alpha > 0$,*

$$
{}_a D^\alpha_t \left({}_a I^\alpha_t u(t) \right) = u(t), \quad a.e. \ t \in [a,b] \ \text{ and}
$$

$$
{}_t D^\alpha_b \left({}_t I^\alpha_b u(t) \right) = u(t), \quad a.e. \ t \in [a,b].
$$

Theorem 2.7. *For $n-1 \le \alpha < n$, if the left and right Riemann–Liouville fractional derivatives ${}_a D^\alpha_t u(t)$ and ${}_t D^\alpha_b u(t)$ of the function u are integral on $[a,b]$, then*

$$
{}_a I^\alpha_t \left({}_a D^\alpha_t u(t) \right) = u(t) - \sum_{k=}^{n} [{}_a I^{k-\alpha}_t u(t)]_{t=a} \frac{(t-a)^{\alpha-k}}{\Gamma(\alpha-k+1)},
$$

$$
{}_t I^\alpha_b \left({}_t D^\alpha_b u(t) \right) = u(t) - \sum_{k=1}^{n} [{}_t I^{k-\alpha}_n u(t)]_{t=b} \frac{(-1)^{n-k}(b-t)^{\alpha-k}}{\Gamma(\alpha-k+1)},
$$

for $t \in [a,b]$.

Theorem 2.8 (Integration by parts).

$$
\int_a^b [{}_a I^\alpha_t u(t)] v(t)\, dt = \int_a^b u(t) \, {}_t I^\alpha_b v(t)\, dt, \quad \alpha > 0, \tag{2.3}
$$

provided that $u \in L^p[a,b]$, $v \in L^q[a,b]$ and

$$p \geq 1, \ q \geq 1 \quad and \quad \frac{1}{p} + \frac{1}{q} < 1 + \alpha \quad or \quad p \neq 1, \ q \neq 1 \quad and \quad \frac{1}{p} + \frac{1}{q} = 1 + \alpha.$$

$$\int_a^b [_a D_t^\alpha u(t)] v(t) \, dt = \int_a^b u(t) {}_t D_b^\alpha v(t) \, dt, \quad 0 < \alpha \leq 1, \tag{2.4}$$

provided the boundary conditions

$$u(a) = u(b) = 0, \ u' \in L^\infty[a,b], \ v \in L^1[a,b] \quad or$$
$$v(a) = v(b) = 0, \ v' \in L^\infty[a,b], \ u \in L^1[a,b]$$

are fulfilled.

2.1 Fractional derivative space

In order to establish a variational structure for BVP (1.1), it is necessary to construct appropriate function spaces. For this setting we take some results from [13].

Let us recall that for any fixed $t \in [0, T]$ and $1 \leq p < \infty$,

$$\|u\|_{L^p[0,t]} = \left(\int_0^t |u(s)|^p \, ds \right)^{1/p},$$

$$\|u\|_{L^p} = \left(\int_0^T |u(s)|^p \, ds \right)^{1/p} \quad and$$

$$\|u\|_\infty = \max_{t \in [0,T]} |u(t)|.$$

Definition 2.9. Let $0 < \alpha \leq 1$ and $1 < p < \infty$. The fractional derivative spaces $E_0^{\alpha,p}$ are defined by

$$E_0^{\alpha,p} = \{ u \in L^p[0,T] \mid {}_0 D_t^\alpha u \in L^p[0,T] \quad and \quad u(0) = u(T) = 0 \}$$
$$= \overline{C_0^\infty[0,T]}^{\|\cdot\|_{\alpha,p}}.$$

where $\|\cdot\|_{\alpha,p}$ is defined by

$$\|u\|_{\alpha,p}^p = \int_0^T |u(t)|^p \, dt + \int_0^T |_0 D_t^\alpha u(t)|^p \, dt. \tag{2.5}$$

Remark 2.10. For any $u \in E_0^{\alpha,p}$, noting the fact that $u(0) = 0$, we have ${}_0^c D_t^\alpha u(t) = {}_0 D_t^\alpha u(t)$, $t \in [0, T]$ according to (2.1).

Proposition 2.11 ([13]). *Let $0 < \alpha \leq 1$ and $1 < p < \infty$. The fractional derivative space $E_0^{\alpha,p}$ is a reflexive and separable Banach space.*

The following result yields the boundedness of the Riemann–Liouville fractional integral operators from the space $L^p[0, T]$ to the space $L^p[0, T]$, where $1 \leq p < \infty$.

Lemma 2.12 ([13]). *Let $0 < \alpha \leq 1$ and $1 \leq p < \infty$. For any $u \in L^p[0, T]$ we have*

$$\|_0 I_\xi^\alpha u\|_{L^p[0,t]} \leq \frac{t^\alpha}{\Gamma(\alpha+1)} \|u\|_{L^p[0,t]}, \quad for \ \xi \in [0,t], \ t \in [0,T]. \tag{2.6}$$

Proposition 2.13 ([14]). *Let $0 < \alpha \leq 1$ and $1 < p < \infty$. For all $u \in E_0^{\alpha,p}$, if $\alpha > 1/p$ we have*

$$_0I_t^\alpha \left(_0D_t^\alpha u(t) \right) = u(t).$$

Moreover, $E_0^{\alpha,p} \in C[0,T]$.

Proposition 2.14 ([14]). *Let $0 < \alpha \leq 1$ and $1 < p < \infty$. For all $u \in E_0^{\alpha,p}$, if $\alpha > 1/p$ we have*

$$\|u\|_{L^p} \leq \frac{T^\alpha}{\Gamma(\alpha+1)} \|_0D_t^\alpha u\|_{L^p}. \tag{2.7}$$

If $\alpha > 1/p$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\|u\|_\infty \leq \frac{T^{\alpha-1/p}}{\Gamma(\alpha)((\alpha-1)q+1)^{1/q}} \|_0D_t^\alpha u\|_{L^p}. \tag{2.8}$$

Remark 2.15. Let $1/2 < \alpha \leq 1$, if $u \in E_0^{\alpha,p}$, then $u \in L^q[0,T]$ for $q \in [p, +\infty]$. In fact

$$\int_0^T |u(t)|^q \, dt = \int_0^T |u(t)|^{q-p} |u(t)|^p \, dt$$
$$\leq \|u\|_\infty^{q-p} \|u\|_{L^p}^p.$$

In particular the embedding $E_0^{\alpha,p} \hookrightarrow L^q[0,T]$ is continuos for all $q \in [p, +\infty]$.

According to (2.7), we can consider in $E_0^{\alpha,p}$ the following norm

$$\|u\|_{\alpha,p} = \|_0D_t^\alpha u\|_{L^p}, \tag{2.9}$$

and (2.9) is equivalent to (2.5).

Proposition 2.16 ([14]). *Let $0 < \alpha \leq 1$ and $1 < p < \infty$. Assume that $\alpha > \frac{1}{p}$ and $\{u_k\} \rightharpoonup u$ in $E_0^{\alpha,p}$. Then $u_k \to u$ in $C[0,T]$, i.e.*

$$\|u_k - u\|_\infty \to 0, \ k \to \infty.$$

We denote by $E^\alpha = E_0^{\alpha,2}$, this is a Hilbert space with respect to the norm $\|u\|_\alpha = \|u\|_{\alpha,2}$ given by (2.9).

Now we consider the functional

$$u \to \int_0^T \left(_0D_t^\alpha u(t), _tD_T^\alpha u(t) \right)$$

on E^α. The following estimates are useful for our further discussion.

Proposition 2.17 ([13]). *If $1/2 < \alpha \leq 1$, then for any $u \in E^\alpha$, we have*

$$|\cos(\pi\alpha)| \|u\|_\alpha^2 \leq - \int_0^T \left(_0D_t^\alpha u(t), _tD_T^\alpha u(t) \right) dt \leq \frac{1}{|\cos(\pi\alpha)|} \|u\|_\alpha^2 \tag{2.10}$$

3 Proof of Theorem 1.1

In order to prove Theorem 1.1, we proceed by three steps.

Proof. **Step 1:** Let $w \in E^\alpha$, we show that I_w has a nontrivial critical point in E^α by the mountain pass theorem.

Firstly, it follows from (H_1) and (H_2) that, given ϵ, with $\epsilon \in (0, |\cos(\pi\alpha)|[\Gamma(\alpha+1)]^2)$, there exists a positive constant C_ϵ, independent of w, such that

$$|f(t, x, \xi)| \leq \epsilon|x| + C_\epsilon|x|^p, \tag{3.1}$$

and

$$|F(t, x, \xi)| \leq \frac{\epsilon}{2}|x|^2 + \frac{C_\epsilon}{p+1}|x|^{p+1}. \tag{3.2}$$

By (2.10) and (3.2)

$$
\begin{aligned}
I_w(u) &= -\frac{1}{2}\int_0^1 ({}_0D_t^\alpha u(t), {}_tD_1^\alpha u(t))\, dt - \int_0^1 F(t, u(t), {}_0D_t^\alpha w(t))\, dt \\
&\geq \frac{|\cos(\pi\alpha)|}{2}\|u\|_\alpha^2 - \frac{\epsilon}{2}\int_0^1 |u(t)|^2\, dt - \frac{C_\epsilon}{p+1}\int_0^1 |u(t)|^{p+1}\, dt \\
&\geq \frac{|\cos(\pi\alpha)|}{2}\|u\|_\alpha^2 - \frac{\epsilon}{2[\Gamma(\alpha+1)]^2}\int_0^1 |{}_0D_t^\alpha u(t)|^2\, dt \\
&\quad - \frac{C_\epsilon}{(p+1)[\Gamma(\alpha)\sqrt{\Gamma(\alpha+1)}]^{p+1}}\left(\int_0^1 |{}_0D_t^\alpha u(t)|^2\, dt\right)^{\frac{p+1}{2}} \\
&= \left[\frac{|\cos(\pi\alpha)|}{2} - \frac{\epsilon}{2[\Gamma(\alpha+1)]^2} - \frac{C_\epsilon}{(p+1)[\Gamma(\alpha)\sqrt{\Gamma(\alpha+1)}]^{p+1}}\|u\|_\alpha^{p-1}\right]\|u\|_\alpha^2
\end{aligned}
$$

we can choose $\rho > 0$ such that

$$\frac{|\cos(\pi\alpha)|}{2} - \frac{\epsilon}{2[\Gamma(\alpha+1)]^2} > \frac{C_\epsilon}{(p+1)[\Gamma(\alpha)\sqrt{\Gamma(\alpha+1)}]^{p+1}}\rho^{p-1},$$

hence, let $u \in E^\alpha$ with $\|u\|_\alpha = \rho$, we know that there exists $\beta > 0$, such that for $\|u\|_\alpha = \rho$, $I_w(u) \geq \beta$ uniformly for $w \in E^\alpha$.

Secondly, for given $\overline{u} \in E^\alpha$ with $\|\overline{u}\|_\alpha = 1$, by (2.10) and (H_4) we have that for $\tau > 0$,

$$
\begin{aligned}
I_w(\tau\overline{u}) &= -\frac{\tau^2}{2}\int_0^1 ({}_0D_t^\alpha\overline{u}(t), {}_tD_1^\alpha\overline{u}(t))\, dt - \int_0^1 F(t, \tau\overline{u}(t), {}_0D_t^\alpha w(t))\, dt \\
&\leq \frac{\tau^2}{2|\cos(\pi\alpha)|}\|\overline{u}\|_\alpha^2 - \int_0^1 F(t, \tau\overline{u}, {}_0D_t^\alpha w(t))\, dt \\
&\leq \frac{\tau^2}{2|\cos(\pi\alpha)|} - c_1\tau^\mu\int_0^1 |\overline{u}(t)|^\mu\, dt + c_2
\end{aligned}
$$

Since $\mu > 2$, taking τ large enough and let $e = \tau\overline{u}$, then $I_w(e) < 0$ with $\|e\|_\alpha > \rho$.

Thirdly, we show that I_w satisfies the Palais–Smale condition. Let $\{u_k\} \in E^\alpha$ such that

$$|I_w(u_k)| \leq K, \quad \lim_{k\to\infty} I_w'(u_k) = 0, \quad \text{for some } K > 0. \tag{3.3}$$

We have

$$I_w(u_k) = -\frac{1}{2}\int_0^1 ({}_0D_t^\alpha u_k(t), {}_tD_1^\alpha u_k(t))\, dt - \int_0^1 F(t, u_k(t), {}_0D_t^\alpha w(t))\, dt,$$

and
$$I'_w(u_k)u_k = -\int_0^1 ({}_0D_t^\alpha u_k(t), {}_tD_1^\alpha u_k(t))\,dt - \int_0^1 f(t, u_k(t), {}_0D_t^\alpha w(t))u_k(t)\,dt.$$

Then by (3.2) and (H_3),

$$|\cos(\pi\alpha)|\left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_k\|_\alpha^2$$

$$\leq I_w(u_k) - \frac{1}{\mu}I'_w(u_k)u_k$$

$$+ \int_{\{|u_k|>M\}}\left[F(t, u_k(t), {}_0D_t^\alpha w(t)) - \frac{u_k}{\mu}f(t, u_k(t), {}_0D_t^\alpha w(t))\right]dt$$

$$+ \int_{\{|u_k|\leq M\}}\left[F(t, u_k(t), {}_0D_t^\alpha w(t)) - \frac{u_k}{\mu}f(t, u_k(t), {}_0D_t^\alpha w(t))\right]dt$$

$$\leq I_w(u_k) - \frac{1}{\mu}I'_w(u_k)u_k + c_3 \tag{3.4}$$

$$\leq K + \frac{1}{\mu}\|I'_w(u_k)\|\|u_k\|_\alpha + c_3,$$

where $c_3 > 0$. Combining with $I'_w(u_k) \to 0$, as $k \to \infty$, we know that $\{u_k\}$ is bounded in E^α.

Since E^α is a reflexive space, we can assume that $u_k \rightharpoonup u$ in E^α, according to Proposition 2.16, we have that $\{u_k\}$ is bounded in $C([0,1])$ and $\lim_{k\to\infty}\|u_k - u\|_\infty = 0$. By the assumption (H_2), we have

$$\int_0^1 [f(t, u_k(t), {}_0D_t^\alpha w(t)) - f(t, u(t), {}_0D_t^\alpha w(t))](u_k(t) - u(t))\,dt \to 0, \quad k \to \infty$$

Notice that

$$[I'_w(u_k) - I'_w(u)](u_k - u) = I'_w(u_k)(u_k - u) - I'_w(u)(u_k - u)$$
$$\leq \|I'_w(u_k)\|\|u_k - u\|_\alpha - I'_w(u)(u_k - u)$$
$$\to 0, \quad \text{as } k \to \infty$$

Moreover,

$$|\cos(\pi\alpha)|\|u_k - u\|_\alpha^2 \leq -\int_0^1 ({}_0D_t^\alpha(u_k - u), {}_tD_1^\alpha(u_k - u))\,dt$$
$$= \int_0^1 [f(t, u_k, {}_0D_t^\alpha w(t)) - f(t, u(t), {}_0D_t^\alpha w(t))](u_k - u)\,dt$$
$$+ [I'_w(u_k) - I'_w(u)](u_k - u),$$

so $\|u_k - u\|_\alpha \to 0$ as $k \to \infty$. That is, $\{u_k\}$ converges strongly to u in E^α.

Obviously, $I_w(0) = 0$, therefore, by the mountain pass theorem, I_w has a nontrivial critical point u_w in E^α, with

$$I_w(u_w) = \inf_{\gamma\in\Gamma}\max_{u\in\gamma([0,1])} I_w(u) \geq \beta > 0,$$

where $\Gamma = \{\gamma \in C([0,1], E^\alpha) : \gamma(0) = 0, \gamma(1) = e\}$.

Step 2: We construct an iterative sequence $\{u_n\}$ and estimate its norm in E^α.

We consider the solutions $\{u_n\}$ of the problem

$$-{}_0D_t^\alpha({}_0D_t^\alpha u_n) = f(t, u_n(t), {}_0D_t^\alpha u_{n-1}(t)), t \in [0,1] \tag{3.5}$$
$$u_n(0) = u_n(1) = 0,$$

starting with an arbitrary $u_0 \in E^\alpha$. By iterative technique, we can get a sequence of $\{u_n\}$, the nontrivial critical point obtained by Step 1.

In the following, we estimate the norm of $\{u_n\}$. Since u_n is the solution of problem (3.5), we have

$$- \int_0^1 \left({}_0D_t^\alpha u_n(t), {}_tD_1^\alpha u_n \right) dt = \int_0^1 f(t, u_n(t), {}_0D_t^\alpha u_{n-1}(t)) u_n(t) \, dt. \tag{3.6}$$

By (3.1), (3.6),

$$|\cos(\pi\alpha)| \|u_n(t)\|_\alpha^2 \leq - \int_0^1 \left({}_0D_t^\alpha u_n(t), {}_tD_1^\alpha u_n(t) \right) dt$$

$$\leq \epsilon \int_0^1 |u_n(t)|^2 dt + C_\epsilon \int_0^1 |u_n(t)|^{p+1} \, dt$$

$$\leq \frac{\epsilon}{[\Gamma(\alpha+1)]^2} \|u_n(t)\|_\alpha^2 + \frac{C_\epsilon}{[\Gamma(\alpha)\sqrt{2\alpha-1}]^{p+1}} \|u_n(t)\|_\alpha^{p+1},$$

that is,

$$\left(|\cos(\pi\alpha)| - \frac{\epsilon}{[\Gamma(\alpha+1)]^2} \right) \|u_n(t)\|_\alpha^2 \leq \frac{C_\epsilon}{[\Gamma(\alpha)\sqrt{2\alpha-1}]^{p+1}} \|u_n(t)\|_\alpha^{p+1},$$

and since $p + 1 > 2$ and $u_n(t) \neq 0$, then there exists $R_1 > 0$ such that

$$\|u_n(t)\|_\alpha \geq R_1 > 0. \tag{3.7}$$

On the other hand, by mountain pass characterization of the critical level, and (H_4), we have

$$|I_{u_{n-1}}(u_n)| \leq \max_{\tau \in [0,\infty)} I_{u_{n-1}}(\tau\overline{u})$$

$$\leq \frac{\tau^2}{2|\cos(\pi\alpha)|} - c_1\tau^\mu \int_0^1 |\overline{u}(t)|^\mu \, dt + c_2,$$

Let

$$H(\tau) = \frac{\tau^2}{2|\cos(\pi\alpha)|} - c_1\tau^\mu \int_0^1 |\overline{u}(t)|^\mu \, dt + c_2, \quad \tau \geq 0,$$

since $\mu > 2$, then $H(\tau)$ can achieve its maximum at some τ_0. Hence

$$|I_{u_{n-1}}(u_n)| \leq H(\tau_0),$$

by (3.4) and $I'_{u_{n-1}}(u_n)u_n = 0$, we have

$$|\cos(\pi\alpha)| \left(\frac{1}{2} - \frac{1}{\mu} \right) \|u_n\|_\alpha^2 \leq I_{u_{n-1}}(u_n) - \frac{1}{\mu} I'_{u_{n-1}}(u_n)u_n + c_3$$

$$\leq H(\tau_0) + c_3,$$

so

$$\|u_n\|_\alpha \leq \sqrt{\frac{H(\tau_0) + c_3}{|\cos(\pi\alpha)| \left(\frac{1}{2} - \frac{1}{\mu} \right)}} =: R_2$$

Step 3: We show that the iterative sequence $\{u_n\}$ constructed in Step 2 is convergent to a nontrivial solution of problem (3.5).

By Step 2, we know $0 < R_1 \leq \|u_n\|_\alpha \leq R_2$, therefore, there exists a positive constant ρ_1, such that

$$\|u_n\|_\infty \leq \rho_1. \tag{3.8}$$

By (1.5), and $I'_{u_n}(u_{n+1})(u_{n+1} - u_n) = 0$, $I'_{u_{n-1}}(u_n)(u_{n+1} - u_n) = 0$, we obtain

$$-\frac{1}{2} \int_0^1 ({}_0D_t^\alpha u_{n+1}, {}_tD_1^\alpha(u_{n+1} - u_n)) + ({}_0D_t^\alpha(u_{n+1} - u_n), {}_tD_1^\alpha u_{n+1})\, dt$$
$$= \int_0^1 f(t, u_{n+1}, {}_0D_t^\alpha u_n)(u_{n+1} - u_n)\, dt,$$

and

$$-\frac{1}{2} \int_0^1 ({}_0D_t^\alpha u_n, {}_tD_1^\alpha(u_{n+1} - u_n)) + ({}_0D_t^\alpha(u_{n+1} - u_n), {}_tD_1^\alpha u_n)\, dt$$
$$= \int_0^1 f(t, u_n, {}_0D_t^\alpha u_{n-1})(u_{n+1} - u_n)\, dt,$$

hence

$$-\int_0^1 ({}_0D_t^\alpha(u_{n+1} - u_n), {}_tD_1^\alpha(u_{n+1} - u_n))$$
$$= \int_0^1 [f(t, u_{n+1}, {}_0D_t^\alpha u_n) - f(t, u_n, {}_0D_t^\alpha u_{n-1})](u_{n+1} - u_n)\, dt,$$

so we have

$$|\cos(\pi\alpha)| \|u_{n+1} - u_n\|_\alpha^2$$
$$\leq \int_0^1 [f(t, u_{n+1}, {}_0D_t^\alpha u_n) - f(t, u_n, {}_0D_t^\alpha u_n)](u_{n+1} - u_n)\, dt$$
$$+ \int_0^1 [f(t, u_n, {}_0D_t^\alpha u_n) - f(t, u_n, {}_0D_t^\alpha u_{n-1})](u_{n+1} - u_n)\, dt$$
$$\leq L_1 \int_0^1 |u_{n+1} - u_n|^2 dt + L_2 \int_0^1 |{}_0D_t^\alpha(u_n - u_{n-1})| |u_{n+1} - u_n|\, dt$$
$$\leq \frac{L_1}{[\Gamma(\alpha+1)]^2} \|u_{n+1} - u_n\|_\alpha^2 + L_2 \|{}_0D_t^\alpha(u_n - u_{n-1})\|_{L^2} \|u_{n+1} - u_n\|_{L^2}$$
$$\leq \frac{L_1}{[\Gamma(\alpha+1)]^2} \|u_{n+1} - u_n\|_\alpha^2 + \frac{L_2}{\Gamma(\alpha+1)} \|u_n - u_{n-1}\|_\alpha \|u_{n+1} - u_n\|_\alpha,$$

hence

$$\left(|\cos(\pi\alpha)| - \frac{L_1}{[\Gamma(\alpha+1)]^2}\right) \|u_{n+1} - u_n\|_\alpha \leq \frac{L_2}{\Gamma(\alpha+1)} \|u_n - u_{n-1}\|_\alpha.$$

Since $0 < l < 1$, we know that $\{u_n\}$ is a cauchy sequence in E^α, so there exists a $u \in E^\alpha$ such that $\{u_n\}$ converges strongly to u in E^α, and by (3.7) we know that $u \neq 0$.

In order to show that u is a solution of problem (1.1) we need to prove that

$$-\int_0^1 ({}_0D_t^\alpha u(t), {}_tD_1^\alpha v(t))\, dt = \int_0^1 f(t, u(t), {}_0D_t^\alpha u(t))v(t)\, dt, \quad \forall v \in E^\alpha.$$

It suffices to show that

$$\int_0^1 f(t, u_n, {}_0D_t^\alpha u_{n-1})v(t)\, dt \to \int_0^1 f(t, u, {}_0D_t^\alpha u(t))v(t)\, dt, \quad \text{as } n \to \infty.$$

Indeed, it follows from the assumption (H_5) that

$$\int_0^1 [f(t, u_n(t), {}_0D_t^\alpha u_{n-1}(t)) - f(t, u(t), {}_0D_t^\alpha u(t))]v(t)\, dt$$

$$= \int_0^1 [f(t, u_n(t), {}_0D_t^\alpha u_{n-1}(t)) - f(t, u_n(t), {}_0D_t^\alpha u(t))]v(t)\, dt$$

$$+ \int_0^1 [f(t, u_n(t), {}_0D_t^\alpha u(t)) - f(t, u(t), {}_0D_t^\alpha u(t))]v(t)\, dt$$

$$\leq L_1 \int_0^1 |u_n(t) - u(t)||v(t)|\, dt + L_2 \int_0^1 |{}_0D_t^\alpha(u_n(t) - u_{n-1}(t))||v(t)|\, dt$$

$$\leq \left[\frac{L_1}{[\Gamma(\alpha+1)]^2}\|u_n - u\|_\alpha + \frac{L_2}{\Gamma(\alpha+1)}\|u_{n-1} - u\|_\alpha \right] \|v\|_\alpha$$

$$\to 0, \quad n \to \infty.$$

Therefore, we obtain a nontrivial solution of problem (1.1). □

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