Third order differential equations with delay

Petr Liška[✉]

Department of Mathematics and Statistics, Masaryk University,
Kotlářská 2, Brno, 611 37, Czech Republic
Department of Mathematics, Mendel University, Zemědělská 3, Brno, 616 00, Czech Republic

Abstract. In this paper, we study the oscillation and asymptotic properties of solutions of certain nonlinear third order differential equations with delay. In particular, we extend results of I. Mojsej (Nonlinear Analysis 68, 2008) and we improve conditions on the property B of N. Parhi and S. Padhi (Indian J. Pure Appl. Math. 33, 2002). Some examples are considered to illustrate our main results.

Keywords: oscillation, third-order differential equation, delay.

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1 Introduction

We consider the third order nonlinear equations with delay of the form

$$\left(\frac{1}{p(t)}\left(\frac{1}{r(t)}x'\right)'\right)' + q(t)\left|x\big(g(t)\big)\right|^{1/\lambda}\operatorname{sgn} x\big(g(t)\big) = 0 \qquad (\mathrm{N}, g)$$

and the adjoint equation

$$\left(\frac{1}{r(t)}\left(\frac{1}{p(t)}z'\right)'\right)' - q(t)\left|z\big(g(t)\big)\right|^{\lambda}\operatorname{sgn} z\big(g(t)\big) = 0. \qquad (\mathrm{N}^{\mathcal{A}}, g)$$

Throughout the paper we always assume that

(i) $p, r, q \in C\left([a, \infty), (0, \infty)\right)$,

(ii) $g \in C\left([a, \infty), \mathbb{R}\right)$, $g(t) < t$, $g(t)$ is nondecreasing, $g(t) \to \infty$ as $t \to \infty$,

(iii) $\int_a^\infty p(t)\,\mathrm{d}t = \int_a^\infty r(t)\,\mathrm{d}t = \infty$.

(iv) $0 < \lambda \leq 1$.

[✉]Email: petr.liska@mendelu.cz

We will denote by (L, g) and (L$^{\mathcal{A}}$, g) the linear versions of equations (N, g) and (N$^{\mathcal{A}}$, g), respectively, i.e.

$$\left(\frac{1}{p(t)}\left(\frac{1}{r(t)}x'\right)'\right)' + q(t)x\big(g(t)\big) = 0 \tag{L, g}$$

and the adjoint equation

$$\left(\frac{1}{r(t)}\left(\frac{1}{p(t)}z'\right)'\right)' - q(t)z\big(g(t)\big) = 0. \tag{L$^{\mathcal{A}}$, g}$$

Further, we denote by (L) and (L$^{\mathcal{A}}$) the corresponding linear equations without the delay. Prototypes of equations (L, g) and (L$^{\mathcal{A}}$, g) are

$$x'''(t) \pm q(t)x\big(g(t)\big) = 0, \qquad t \in [a, \infty]. \tag{E$\pm$}$$

The asymptotic behaviour of solutions of special types of the above equations have been studied by many authors. This paper benefits mostly from work of Kusano and Naito [8] and from papers written by Cecchi, Došlá, Marini [4, 5], Akin-Bohner, Došlá, Lawrence [3] or Mojsej [9], see also references there. Some other results are given in papers [2, 6, 10] or recently in [1]. The extensive survey can be found in the excellent book [11], see also references there. The equation (E$\pm$) has been studied in [12].

The aim of the paper is to extend some results from the paper by I. Mojsej [9] and to study the influence of the delayed argument on the oscillation of equations (N, g) and (N$^{\mathcal{A}}$, g). Some examples are considered to illustrate our results.

If x is a solution of (N, g) then functions

$$x^{[0]} = x, \qquad x^{[1]} = \frac{1}{r}x', \qquad x^{[2]} = \frac{1}{p}\left(\frac{1}{r}x'\right)' = \frac{1}{p}\big(x^{[1]}\big)'$$

are called quasiderivatives of x. Similarly, we can proceed for (N$^{\mathcal{A}}$, g).

A solution x of (N, g) is said to be proper if it exists on the interval $[a, \infty)$ and satisfies the condition

$$\sup\{|x(s)|: t \le s < \infty\} > 0 \quad \text{for any } t \ge a.$$

A proper solution is called oscillatory or nonoscillatory according to whether it does or does not have arbitrarily large zeros. Similar definitions hold for (N$^{\mathcal{A}}$, g).

Following [7], we define property A and property B by the following way.

Definition 1.1. The equation (N, g) is said to have property A if any proper solution x of (N, g) is either oscillatory or satisfies

$$\big|x^{[i]}(t)\big| \downarrow 0 \quad \text{as} \quad t \to \infty, \; i = 0, 1, 2.$$

Definition 1.2. The equation (N$^{\mathcal{A}}$, g) is said to have property B if any proper solution z of (N$^{\mathcal{A}}$, g) is either oscillatory or satisfies

$$\big|z^{[i]}(t)\big| \uparrow \infty \quad \text{as} \quad t \to \infty, \; i = 0, 1, 2.$$

The notation $y(t) \downarrow 0$ ($y(t) \uparrow \infty$) means that y monotonically decreases to zero as $t \to \infty$ (y monotonically increases to ∞ as $t \to \infty$).

From a slight modification of a lemma by Kiguradze (see [7]) nonoscillatory solutions x of (N, g) and (L, g) can be divided into the two classes:

$$\mathcal{N}_0 = \left\{ x \text{ solution, } \exists T_x \colon x(t)x^{[1]}(t) < 0, \ x(t)x^{[2]}(t) > 0 \text{ for } t \geq T_x \right\}$$
$$\mathcal{N}_2 = \left\{ x \text{ solution, } \exists T_x \colon x(t)x^{[1]}(t) > 0, \ x(t)x^{[2]}(t) > 0 \text{ for } t \geq T_x \right\}.$$

Similarly, nonoscillatory solutions z of $(N^{\mathcal{A}}, g)$ and $(L^{\mathcal{A}}, g)$ can be divided into the two following classes:

$$\mathcal{M}_1 = \left\{ z \text{ solution, } \exists T_z \colon z(t)z^{[1]}(t) > 0, \ z(t)z^{[2]}(t) < 0 \text{ for } t \geq T_z \right\}$$
$$\mathcal{M}_3 = \left\{ z \text{ solution, } \exists T_z \colon z(t)z^{[1]}(t) > 0, \ z(t)z^{[2]}(t) > 0 \text{ for } t \geq T_z \right\}.$$

It is clear, that (N, g) or (L, g) has property A if and only if all nonoscillatory solutions of (N, g), or (L, g), respectively, belong to the class $\mathcal{N}_0$ and $\lim_{t \to \infty} x^{[i]}(t) = 0$, $i = 0, 1, 2$. Similarly, $(N^{\mathcal{A}}, g)$ or $(L^{\mathcal{A}}, g)$ has property B if and only if all nonoscillatory solutions of $(N^{\mathcal{A}}, g)$, or $(L^{\mathcal{A}}, g)$, respectively, belong to the class $\mathcal{M}_3$ and $\lim_{t \to \infty} z^{[i]}(t) = \infty$, $i = 0, 1, 2$.

We will study the relationship between property A for (L, g) and for (N, g) and property B for $(L^{\mathcal{A}}, g)$ and $(N^{\mathcal{A}}, g)$. Our results complete recent ones in [9]. As a consequence, an equivalence result for property A for (L, g) and for property B for $(L^{\mathcal{A}}, g)$ is obtained. The paper is completed by some examples, which illustrate the role of function g.

2 Preliminary results

Results about relationship between the oscillation and properties A and B for linear equations without delay can be summarized as follows.

Theorem A ([4]). *The following assertions are equivalent:*

  (i) (L) *has property A.*

 (ii) $(L^{\mathcal{A}})$ *has property B.*

(iii) (L) *is oscillatory and it holds*

$$\int_a^\infty q(t) \int_a^t p(s) \int_a^s r(\tau) \, d\tau \, ds \, dt = \infty.$$

 (iv) $(L^{\mathcal{A}})$ *is oscillatory and it holds*

$$\int_a^\infty q(t) \int_a^t p(s) \int_a^s r(\tau) \, d\tau \, ds \, dt = \infty.$$

Under our assumptions Theorem 1 from [8] reads as follows.

Theorem B. *(i) If equation* (L, g) *has property A, then equation* (L) *has property A.*

 (ii) If equation $(L^{\mathcal{A}}, g)$ *has property B, then equation* $(L^{\mathcal{A}})$ *has property B.*

We can reformulate Theorem 3.1 in [9] as follows.

Theorem C. *Consider* (N, g) *and function* $\tau(t)$ *such that*

$$\tau \in C([a, \infty), \mathbb{R}), \qquad \tau(t) > t, \qquad g(\tau(t)) \leq t. \tag{2.1}$$

Assume that

$$\limsup_{t \to \infty} \int_t^{\tau(t)} q(s) \int_a^{g(s)} r(u) \int_a^u p(v) \, dv \, du \, ds > 1 \quad \text{if} \quad \lambda = 1 \tag{2.2}$$

or

$$\limsup_{t \to \infty} \int_t^{\tau(t)} q(s) \int_a^{g(s)} r(u) \int_a^u p(v) \, dv \, du \, ds > 0 \quad \text{if} \quad 0 < \lambda < 1.$$

If equation (L) *has property A, then equation* (N, g) *has property A.*

In particular, for (L, g) we have the following result.

Corollary 2.1. *Let* (2.1) *and* (2.2) *hold. If equation* (L) *has property A, then equation* (L, g) *has property A.*

From the previous results we have the following corollary.

Corollary 2.2. *Let* (2.1) *and* (2.2) *hold. Then*

$$(L) \text{ has property } A \Longleftrightarrow (L, g) \text{ has property } A$$
$$\Updownarrow$$
$$(L^{\mathcal{A}}) \text{ has property } B \Longleftarrow (L^{\mathcal{A}}, g) \text{ has property } B$$

In [12] there are criteria for the equation (E−) to have property B, which can be summarized as follows.

Theorem D. *The equation* (E−) *has property B if any of the following conditions hold*

i) $\int_a^\infty q(t) \, dt < \infty$ *and* $\int_a^\infty t q(t) \, dt = \infty$,

ii) *for every* $T \geq a$

$$\limsup_{t \to \infty} (t - T)^2 \int_{2g^{-1}(t)}^\infty q(s) \, ds > 2. \tag{2.3}$$

Our aim is the extension of Theorem C for the equation (N$^{\mathcal{A}}$, g) and property B. In particular, the question is whether or not we can complete the diagram in Corollary 2.2 with the last implication.

3 Main results

First we prove a slight modification of Theorem 2.1 from [3].

Theorem 3.1. *Let*

$$\int_a^\infty q(s) \left(\int_a^{g(s)} p(\tau) \int_a^\tau r(v) \, dv \, d\tau \right)^\lambda ds = \infty. \tag{3.1}$$

Then every solution z of (N$^{\mathcal{A}}$, g) *from the class* $\mathcal{M}_3$ *satisfies*

$$\lim_{t \to \infty} z^{[i]}(t) = \infty, \quad \text{for} \quad i = 0, 1, 2.$$

Proof. We rewrite $(\mathrm{N}^{\mathcal{A}}, g)$ as a system

$$\begin{cases} z'(t) = p(t)y(t) \\ y'(t) = r(t)x(t) \\ x'(t) = q(t)\,|z(g(t))|^{\lambda}\,\mathrm{sgn}\,z(g(t)). \end{cases} \tag{3.2}$$

Let $z(t)$ be a solution of $(\mathrm{N}^{\mathcal{A}}, g)$ from the class $\mathcal{M}_3$. Then the vector $(z(t), y(t), x(t))$, where $y(t) = \frac{1}{p(t)}z'(t)$ and $x(t) = \frac{1}{r(t)}y'(t)$, is a solution of system (3.2) such that

$$\mathrm{sgn}\,x(t) = \mathrm{sgn}\,y(t) = \mathrm{sgn}\,z(t) \quad \text{for large } t.$$

We prove that

$$\lim_{t\to\infty}|x(t)| = \lim_{t\to\infty}|y(t)| = \lim_{t\to\infty}|z(t)| = \infty.$$

There exists $T \geq a$ such that $x(t) > 0$, $y(t) > 0$, $z(t) > 0$ for $t \geq T$. As $y(t)$ is eventually increasing, there exists $T_1 \geq T$ and $K > 0$ such that

$$z'(t) = p(t)y(t) \geq p(t)K \quad \text{for } t \geq T_1,$$

so integrating in $[T_1, t]$ we get

$$z(t) \geq K \int_{T_1}^{t} p(s)\,\mathrm{d}s.$$

Using the assumption $\int^{\infty} p(t)\,\mathrm{d}t = \infty$ we get $\lim_{t\to\infty} z(t) = \infty$.

Since $x(t)$ is eventually increasing, there exists $T_2 \geq T_1$ and $L > 0$ such that

$$y'(t) = r(t)x(t) \geq r(t)L \quad \text{for } t \geq T_2,$$

and integrating in $[T_2, t]$

$$y(t) \geq L \int_{T_2}^{t} r(s)\,\mathrm{d}s. \tag{3.3}$$

Using the assumption $\int^{\infty} r(t)\,\mathrm{d}t = \infty$ we get $\lim_{t\to\infty} y(t) = \infty$.

Integrating the first equation of (3.2) from T_1 to $g(t)$ and using (3.3) we obtain

$$z(g(t)) \geq \int_{T_1}^{g(t)} p(s)y(s)\,\mathrm{d}s \geq L \int_{T_1}^{g(t)} p(s) \int_{T_1}^{s} r(u)\,\mathrm{d}u\,\mathrm{d}s. \tag{3.4}$$

Using the third equation of (3.2) and (3.3), there exists $T_2 \geq T_1$ such that

$$x'(t) = q(t)\,(z(g(t)))^{\lambda} \geq q(t)\left(L \int_{T_1}^{g(t)} p(s) \int_{T_1}^{s} r(u)\,\mathrm{d}u\,\mathrm{d}s\right)^{\lambda}.$$

Integrating the last inequality from T_2 to t gives

$$x(t) \geq L^{\lambda} \int_{T_2}^{t} q(s)\left(\int_{T_2}^{g(s)} p(\tau) \int_{T_1}^{\tau} r(v)\,\mathrm{d}v\,\mathrm{d}\tau\right)^{\lambda}\,\mathrm{d}s$$

and using the (3.1) we have $\lim_{t\to\infty} x(t) = \infty$. □

In order to the equation $(\mathrm{N}^{\mathcal{A}}, g)$ having the property B we establish sufficient condition for $\mathcal{M}_1 = \emptyset$. To this aim the following lemma will be needed.

Lemma 3.2. *Assume that z is a solution of* $(\mathrm{N}^{\mathcal{A}}, g)$ *such that* $z \in \mathcal{M}_1$. *Then*

$$\lim_{t \to \infty} z^{[2]}(t) = 0.$$

Proof. We rewrite $(\mathrm{N}^{\mathcal{A}}, g)$ as a system (3.2) and apply Lemma 4.2 from [3]. □

Theorem 3.3. *Let* (2.1) *hold and assume that*

$$\limsup_{t \to \infty} \int_t^{\tau(t)} q(s) \left(\int_a^{g(s)} r(u) \int_a^u p(v) \, dv \, du \right)^{\lambda} ds > 1. \tag{3.5}$$

Then $\mathcal{M}_1 = \varnothing$ *for* $(\mathrm{N}^{\mathcal{A}}, g)$.

Proof. Without loss of generality we suppose that there exists $T \geq a$ such that $z(t) > 0$ for $t \geq T$. Let $z \in \mathcal{M}_1$. As z is a positive nonoscillatory solution of $(\mathrm{N}^{\mathcal{A}}, g)$ in class $\mathcal{M}_1$, there exists $T_1 \geq T$ such that $z(t) > 0$, $z^{[1]}(t) > 0$ and $z^{[2]}(t) < 0$ for $t \geq T_1$. Let $T_2 \geq T_1$ be such that $g(t) \geq T_1$ for $t \geq T_2$. Because $(z^{[2]}(t))' = q(t)z^{\lambda}(g(t)) > 0$ for $t \geq T_2$, $z^{[2]}(t)$ is a negative increasing function, so we have

$$0 \leq -z^{[2]}(t) < \infty.$$

Integrating the equation $(\mathrm{N}^{\mathcal{A}}, g)$ in $[t, \infty)$ we get

$$z^{[2]}(\infty) - z^{[2]}(t) = \int_t^{\infty} q(s) z^{\lambda}(g(s)) \, ds$$

and using the fact that $0 \leq -z^{[2]}(\infty) < \infty$ we obtain the inequality

$$-z^{[2]}(t) \geq \int_t^{\infty} q(s) z^{\lambda}(g(s)) \, ds. \tag{3.6}$$

Integrating the identity $-z^{[2]}(t) = -z^{[2]}(t)$ twice, for the first time in $[t, \infty)$ and for the second time in $[T_1, t]$, we obtain

$$z(t) \geq \int_{T_1}^t p(s) \int_s^{\infty} r(u)(-z^{[2]}(u)) \, du \, ds.$$

By changing the order of integration we get

$$z(t) \geq \int_{T_1}^t r(s)(-z^{[2]}(s)) \int_{T_1}^s p(u) \, du \, ds \quad \text{for } t \geq T_1$$

and therefore

$$z(g(t)) \geq \int_{T_1}^{g(t)} r(s)(-z^{[2]}(s)) \int_{T_1}^s p(u) \, du \, ds \quad \text{for } t \geq T_2. \tag{3.7}$$

Using (3.7) in (3.6) we have

$$-z^{[2]}(t) \geq \int_t^{\infty} q(s) \left(\int_{T_1}^{g(s)} r(u)(-z^{[2]}(u)) \int_{T_1}^u p(v) \, dv \, du \right)^{\lambda} ds.$$

Considering the fact that $-z^{[2]}(t)$ is decreasing and $-z^{[2]}(g(t))$ is nonincreasing, we get

$$-z^{[2]}(t) \geq \left(-z^{[2]}(g(\tau(t))) \right)^{\lambda} \int_t^{\tau(t)} q(s) \left(\int_{T_1}^{g(s)} r(u) \int_{T_1}^u p(v) \, dv \, du \right)^{\lambda} ds.$$

Since $-z^{[2]}(t)$ is decreasing, $\lim_{t\to\infty} z^{[2]}(t) = 0$, $\lambda \leq 1$ and (3.5) holds we have

$$1 \geq \frac{-z^{[2]}(t)}{\left(-z^{[2]}(g(\tau(t)))\right)^{\lambda}} \geq \int_t^{\tau(t)} q(s) \left(\int_{T_1}^{g(s)} r(u) \int_{T_1}^u p(v)\,dv\,du\right)^{\lambda} ds > 1,$$

which is a contradiction. $\qquad\square$

Lemma 3.4. *If* (2.1) *and* (3.5) *hold, then*

$$\int_a^\infty q(s) \left(\int_a^{g(s)} r(u) \int_a^u p(v)\,dv\,du\right)^{\lambda} ds = \infty.$$

Proof. By contradiction, let $\int_a^\infty q(s) \int_a^{g(s)} r(u) \int_a^u p(v)\,dv\,du\,ds < \infty$. Then there exists $t_0 > a$ such that $\int_{t_0}^\infty q(s) \int_a^{g(s)} r(u) \int_a^u p(v)\,dv\,du\,ds < 1$. For every $t > t_0$ we have

$$\begin{aligned}
1 &> \int_{t_0}^\infty q(s) \left(\int_a^{g(s)} r(u) \int_a^u p(v)\,dv\,du\right)^{\lambda} ds \\
&> \int_t^\infty q(s) \left(\int_a^{g(s)} r(u) \int_a^u p(v)\,dv\,du\right)^{\lambda} ds \\
&> \int_t^{\tau(t)} q(s) \left(\int_a^{g(s)} r(u) \int_a^u p(v)\,dv\,du\right)^{\lambda} ds.
\end{aligned}$$

Passing $t \to \infty$ and using (3.5) we get the contradiction. $\qquad\square$

The main result is the following extension of Theorem C to property B.

Theorem 3.5. *Let* (2.1) *and* (3.5) *hold and assume that*

$$\limsup_{t\to\infty} \int_t^{\tau(t)} q(s) \left(\int_a^{g(s)} p(u) \int_a^u r(v)\,dv\,du\right)^{\lambda} ds > 1. \tag{3.8}$$

Then the equations $(\mathrm{L}^{\mathcal{A}})$ *and* $(\mathrm{N}^{\mathcal{A}}, g)$ *have property B.*

Proof. Since (2.1) and (3.8) hold, then by using Lemma 3.4, where functions r and p are exchanged, we get

$$\int_a^\infty q(t) \left(\int_a^{g(t)} p(s) \int_a^s r(\tau)\,d\tau\,ds\right)^{\lambda} dt = \infty.$$

As $g(t) < t$ and $0 < \lambda \leq 1$, we have

$$\int_a^\infty q(t) \int_a^t p(s) \int_a^s r(\tau)\,d\tau\,ds\,dt = \infty,$$

which, due to Theorem A, means that the equation $(\mathrm{L}^{\mathcal{A}})$ has property B.

Moreover, assumption (3.8) implies that (3.1) holds, so by Theorem 3.1, every solution $z(t)$ of $(\mathrm{N}^{\mathcal{A}}, g)$ from the class $\mathcal{M}_3$ satisfies

$$\lim_{t\to\infty} z^{[i]}(t) = \infty, \quad \text{for } i = 0, 1, 2.$$

According to Theorem 3.3 the condition (3.5) implies that $\mathcal{M}_1 = \varnothing$, thus $(\mathrm{N}^{\mathcal{A}}, g)$ has property B. $\qquad\square$

4 Applications and examples

(1) Now we can complete Corollary 2.2.

Corollary 4.1. *Let* (2.1), (3.5) *and* (3.8) *hold. Then*

$$
\begin{array}{ccc}
\text{(L) \textit{has property} A} & \Longleftrightarrow & \text{(L, } g\text{) \textit{has property} A} \\
\Updownarrow & & \Updownarrow \\
\text{(L}^{\mathcal{A}}\text{) \textit{has property} B} & \Longleftrightarrow & \text{(L}^{\mathcal{A}}\text{, } g\text{) \textit{has property} B}
\end{array}
$$

Proof. It follows from Theorem 3.5 and Corollary 2.2. $\square$

(2) Let us consider the equations (N, g) and (N$^{\mathcal{A}}$, g) with symmetrical operator, i.e. $r(t) = p(t)$

$$
\left(\frac{1}{p(t)} \left(\frac{1}{p(t)} x' \right)' \right)' + q(t) \left| x(g(t)) \right|^{1/\lambda} \operatorname{sgn} x(g(t)) = 0 \tag{S, g}
$$

and

$$
\left(\frac{1}{p(t)} \left(\frac{1}{p(t)} z' \right)' \right)' - q(t) \left| z(g(t)) \right|^{\lambda} \operatorname{sgn} z(g(t)) = 0. \tag{S$^{\mathcal{A}}$, g}
$$

Further, we denote (S) and (S$^{\mathcal{A}}$) corresponding linear equations without the deviating argument, i.e. equations (S, g) and (S$^{\mathcal{A}}$, g), where $g(t) = t$ and $\lambda = 1$.

Corollary 4.2. *Let* (2.1) *hold and assume that*

$$
\limsup_{t \to \infty} \int_t^{\tau(t)} q(s) \left(\int_a^{g(s)} p(u) \int_a^u p(v) \, dv \, du \right)^{\lambda} ds > 1. \tag{4.1}
$$

Then the following holds.

(a) Equations (S) *and* (S, g) *have property A.*

(b) Equations (S$^{\mathcal{A}}$) *and* (S$^{\mathcal{A}}$, g) *have property B.*

Proof. As the condition (4.1) holds, Lemma 3.4 implies that

$$
\int_a^{\infty} q(t) \int_a^t p(s) \int_a^s p(\tau) \, d\tau \, ds \, dt = \infty,
$$

i.e. equation (S) or equation (S$^{\mathcal{A}}$) has property A or property B, respectively.

Due to Theorem C, equation (S, g) has property A. By Lemma 3.4, condition (4.1) implies (3.1). Thus applying Theorem 3.3 and Theorem 3.1 we get the assertion. $\square$

The following examples illustrate our results.

Example 4.3. Consider the equation

$$
z''' - q(t) z^{\lambda}(g(t)) = 0, \qquad \lambda \le 1. \tag{4.2}
$$

Let τ satisfy (2.1). We have

$$
\int_t^{\tau(t)} q(s) \left(\int_a^{g(s)} (u - a) \, du \right)^{\lambda} ds = \frac{1}{2^{\lambda}} \int_t^{\tau(t)} \left[(q(s) - a)^2 \right]^{\lambda} ds,
$$

thus condition (4.1) gives

$$\limsup_{t \to \infty} \int_t^{\tau(t)} q(s) \, (g(s) - a)^{2\lambda} \, ds > 2^\lambda.$$ (4.3)

By Corollary 4.2, if (4.3) holds, then equation (4.2) has property B.

In particular, the equation

$$z''' - \frac{1}{t^{2\lambda}} z^\lambda (t - \tau) = 0, \qquad (t \geq 1)$$

has property B if $\tau > 2^\lambda$. Indeed, if we take $\tau(t) = t + \tau$, then condition (2.1) is fulfilled and (4.3) gives $\tau > 2^\lambda$.

Example 4.4. Consider the equations

$$x''' + \frac{\mu}{t^3} x(kt) = 0, \qquad (t \geq 1)$$ (4.4)

and

$$z''' - \frac{\mu}{t^3} z(kt) = 0, \qquad (t \geq 1),$$ (4.5)

where $k < 1$.

If we take $\tau(t) = \frac{t}{k}$, then condition (2.1) is fulfilled. We have

$$\int_t^{\frac{t}{k}} \frac{\mu}{s^3} (ks - 1)^2 \geq \mu k^2 \left(\ln \frac{t}{k} - \ln t \right) = \mu k^2 \ln \frac{1}{k}.$$

Passing $t \to \infty$ condition (4.3) gives

$$- \mu k^2 \ln k > 2.$$ (4.6)

Thus, by Corollary 4.2, if (4.6) holds, then equation (4.4) has property A and equation (4.5) has property B.

In particular, condition (4.6) is satisfied for the following equations

$$
\begin{aligned}
x''' + \frac{160}{t^3} x\left(\frac{t}{3}\right) = 0, & \qquad z''' - \frac{160}{t^3} z\left(\frac{t}{3}\right) = 0, \\
x''' + \frac{44}{t^3} x\left(\frac{3t}{5}\right) = 0, & \qquad z''' - \frac{44}{t^3} z\left(\frac{3t}{5}\right) = 0, \\
x''' + \frac{82}{t^3} x\left(\frac{3t}{4}\right) = 0, & \qquad z''' - \frac{82}{t^3} z\left(\frac{3t}{4}\right) = 0, \\
x''' + \frac{46}{t^3} x\left(\frac{10t}{11}\right) = 0, & \qquad z''' - \frac{46}{t^3} z\left(\frac{10t}{11}\right) = 0,
\end{aligned}
$$ (4.7)

hence all these equations have property A or property B, respectively.

In the book [11], see Section 6.3, or in [12] oscillation of equations (4.4) and (4.5) has been investigated in the terms of property $\bar{A}$ and property B. There are given some sufficient conditions for equation (4.4) to have property $\bar{A}$ and for equation (4.5) to have property B. In general, property $\bar{A}$ is weaker than property A and means that every nonoscillatory solution of (4.4) is in the class $\mathcal{N}_0$.

Observe that equations (4.7) appear in [11, 12], where various criteria are used to verify that equations of the type (4.4) have property $\bar{A}$. As far as property B is concerned, conditions

from [12] are summarized in Theorem D. The first condition can not be applied and condition (2.3) gives the following

$$\limsup_{t \to \infty} (t - T)^2 \int_{\frac{2t}{k}}^{\infty} \frac{\mu}{s^3} \, ds = \lim_{t \to \infty} \frac{\mu(t - T)^2 k^2}{8t^2} = \frac{\mu k^2}{8} > 2,$$

i.e.

$$\mu k^2 > 16, \qquad T \geq 1. \tag{4.8}$$

For example, if we take $k = \frac{1}{3}$, condition (4.8) gives that equation (4.5) has property B if $\mu > 144$ while our (4.6) condition gives that equation (4.5) has property B for $\mu > \frac{18}{\log 3} \doteq 16.38$. Hence, we can say that our condition (4.6) improves those mentioned there.

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