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Asymptotic integration of a linear fourth order differential equation of Poincaré type

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Abstract. This article deals with the asymptotic behavior of nonoscillatory solutions of fourth order linear differential equation where the coefficients are perturbations of constants. We define a change of variable and deduce that the new variable satisfies a third order nonlinear differential equation. We assume three hypotheses. The first hypothesis is related to the constant coefficients and set up that the characteristic polynomial associated with the fourth order linear equation has simple and real roots. The other two hypotheses are related to the behavior of the perturbation functions and establish asymptotic integral smallness conditions of the perturbations. Under these general hypotheses, we obtain four main results. The first two results are related to the application of a fixed point argument to prove that the nonlinear third order equation has a unique solution. The next result concerns with the asymptotic behavior of the solutions of the nonlinear third order equation. The fourth main theorem is introduced to establish the existence of a fundamental system of solutions and to precise the formulas for the asymptotic behavior of the linear fourth order differential equation. In addition, we present an example to show that the results introduced in this paper can be applied in situations where the assumptions of some classical theorems are not satisfied.

Keywords: Poincaré–Perron problem, asymptotic behavior, nonoscillatory solutions.

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1 Introduction

1.1 Scope

Linear fourth-order differential equations appear as the most basic mathematical models in several areas of science and engineering. These simplified equations arise from different linearization approaches used to give an ideal description of the physical phenomenon or to analyze (analytically solve or numerically simulate) the corresponding nonlinear governing equations. For example in the following cases, the one-dimensional model of Euler–Bernoulli

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in linear theory of elasticity [1,34], the optimization of quadratic functionals in optimization theory [1], the mathematical model in viscoelastic flows [7,22], and the biharmonic equations in radial coordinates in harmonic analysis [16,21].

An important family of linear fourth order differential equations is given by the equations of the following type

$$y^{(\mathrm{iv})} + [a_3 + r_3(t)]y''' + [a_2 + r_2(t)]y'' + [a_1 + r_1(t)]y' + [a_0 + r_0(t)]y = 0, \tag{1.1}$$

where a_i are constants and r_i are real-valued functions. Note that (1.1) is a perturbation of the following constant coefficient equation:

$$y^{(\mathrm{iv})} + a_3y''' + a_2y'' + a_1y' + a_0y = 0. \tag{1.2}$$

We recall that the study of perturbed equations of the type (1.1), in the general case of n-order equations, was motivated by Poincaré [30]. Thus (1.1) is also known as the scalar linear differential equation of Poincaré type. Moreover, we recall that the classical analysis introduced in the seminal work [30] is mainly focused on two questions: the existence of a fundamental system of solutions for (1.1) and the characterization of the asymptotic behavior of its solutions. Later on, equations of the Poincaré type (1.1) (of different orders) have been investigated by several authors with a long and rich history of results [4,10,18,19]. Even though this is an old problem, it is still an issue which does not lose its topicality and importance in the research community. For instance, in the case of asymptotic behavior of third order equations, there are the following newer results [14,15,28,32,33].

In this contribution, we address the question of the asymptotic behavior of (1.1) under new general hypotheses for the perturbation functions.

1.2 A brief review of asymptotic behavior results in linear ordinary differential equations

Historically, some landmarks in the analysis of the asymptotic behavior in linear ordinary differential equations are given by the works of Poincaré [30], Perron [25], Levinson [23], Hartman–Wintner [20] and Harris and Lutz [17,18]. The works of Poincaré and Perron are focused in the scalar case and the contributions of Levinson, Hartman–Wintner and Harris and Lutz are centered on diagonal systems. Indeed, to be more precise, let us briefly recall these results.

- Poincaré [30] assumes the following two hypotheses:

$$\left.\begin{array}{l}\mu \text{ is a simple characteristic root of (1.2) such that } \mathrm{Re}(\mu) \neq \mathrm{Re}(\mu_0)\\ \text{for any other characteristic root } \mu_0 \text{ of (1.2),}\end{array}\right\} \tag{1.3}$$

$$\text{the functions } r_i \text{ are rational functions such that } r_i(t) \to 0 \text{ when } t \to \infty. \tag{1.4}$$

Then, under (1.3)–(1.4), he deduces that (1.1) has a solution $y_\mu(t)$ satisfying the following asymptotic behavior:

$$\frac{y_\mu^{(\mathrm{iv})}(t)}{y_\mu(t)} \to \lambda^4, \qquad \frac{y_\mu'''(t)}{y_\mu(t)} \to \lambda^3, \qquad \frac{y_\mu''(t)}{y_\mu(t)} \to \lambda^2, \qquad \frac{y_\mu'(t)}{y_\mu(t)} \to \lambda, \tag{1.5}$$

when $t \to \infty$.

- Perron [25] extends the results of Poincaré by assuming (1.3) and considering instead of (1.4) the hypothesis that the perturbation functions r_i are continuous functions such that $r_i(t) \to 0$ when $t \to \infty$.

- Levinson [23] analyzes the non-autonomous system

$$\mathbf{x}'(t) = [\Lambda(t) + R(t)]\mathbf{x}(t)$$

where Λ is a diagonal matrix and R is the perturbation matrix. He assumes that the diagonal matrix satisfies a dichotomy condition and the perturbation function is continuous and belongs to $L^1([t_0, \infty[)$. Then, Levinson proves that a fundamental matrix X has the following asymptotic representation

$$X(t) = [I + o(1)] \exp\Big(\int_{t_0}^{t} \Lambda(s)\, ds \Big). \tag{1.6}$$

- Hartman and Wintner in [20] assume that the diagonal matrix Λ satisfies a condition that is stronger that the Levinson dichotomy condition and the perturbation function is continuous and belongs to $L^p([t_0, \infty[)$ for some $p \in]1, 2]$. Then, they prove that

$$X(t) = [I + o(1)] \exp\Big(\int_{t_0}^{t} (\Lambda(s) + \mathrm{diag}(R(s)))\, ds \Big).$$

- Harris and Lutz in [17] (see also [13, 18, 26, 27, 29]) find a change of variable to unify the results of Levinson and Hartman and Wintner.

The list is non-exhaustive and there are other important results, like the contributions given by Elias and Gingold [11] and Šimša [31]. We note that the application of Levinson, Hartman–Wintner and Harris and Lutz results to (1.1) are not direct. This fact is a practical disadvantage of this kind of results, since in general the explicit algebraic calculus of Λ and R in terms of the perturbations are difficult.

The important point here is that (1.3) can be stated equivalently as follows: $\mu \in \mathbb{R}$ is a simple characteristic root of (1.2), since the requirement "$\mathrm{Re}(\mu) \neq \mathrm{Re}(\mu_0)$ for any other characteristic root μ_0 of (1.2)" excludes to the conjugate root of $\overline{\mu}$, then $\mu \in \mathbb{R}$. Thus, in this paper we consider the following hypothesis:

$$\Big\{ \lambda_i, i = \overline{1,4} \;:\; \lambda_1 > \lambda_2 > \lambda_3 > \lambda_4 \Big\} \subset \mathbb{R} \text{ is the set of characteristic roots for (1.2).} \tag{1.7}$$

The hypothesis (1.7) is satisfied, for instance in the case of the biharmonic equation

$$\Delta^2 u(\mathbf{x}) = 0 \quad \text{in } \mathbb{R}^n, \text{ with } n \geq 5,$$

in radial coordinates. Indeed, considering $r = \|\mathbf{x}\|$ and $\phi(r) = u(\mathbf{x})$, the biharmonic equation may be rewritten as follows

$$\phi^{(\mathrm{iv})}(r) + \frac{2(n-1)}{r}\phi'''(r) + \frac{(n-1)(n-3)}{r^2}\phi''(r) - \frac{(n-1)(n-3)}{r^3}\phi'(r) = 0, \qquad r \in]0, \infty[.$$

Now, by introducing the change of variable $v(t) = e^{-4t/(p-1)}\phi(e^t)$ for some $p > \frac{(n+4)}{(n-4)}$, the differential equation for ϕ can be transformed in the following equivalent equation

$$v^{(\mathrm{iv})}(t) + K_3 v'''(t) + K_2 v''(t) + K_1 v'(t) + K_0 v(t) = 0, \qquad t \in \mathbb{R}, \tag{1.8}$$

where K_i are real constants depending of n and p, see [16] for further details. We note that the roots of the characteristic polynomial associated to the homogeneous equation are given by

$$\lambda_1 = 2\frac{p+1}{p-1} > \lambda_2 = \frac{4}{p-1} > 0 > \lambda_3 = \frac{4p}{p-1} - n > \lambda_4 = 2\frac{p+1}{p-1} - n. \tag{1.9}$$

Thus, the radial solutions of the biharmonic equation in a space of dimension $n \geq 5$ and with $p > (n+4)(n-4)^{-1}$ can be analysed by the linear fourth order differential equation (1.8) where the characteristic roots satisfy (1.9) which will be generalized. Thus the analysis of the a perturbed equation for (1.8) are relevant for instance, in the analysis of the equation

$$\Delta^2 u(\mathbf{x}) - \frac{\mu}{|\mathbf{x}|^{-4}} u(\mathbf{x}) = \lambda f(\mathbf{x}) u(\mathbf{x}), \qquad \lambda, \mu \in \mathbb{R}, \quad f \colon \mathbb{R}^n \to \mathbb{R},$$

which appears naturally in the weighted eigenvalue problems [35].

Nowadays, there are three big approaches to study the problem of asymptotic behavior of solutions for scalar linear differential equations of Poincaré type: the analytic theory, the nonanalytic theory and the scalar method. In a broad sense, we recall that the essence of the analytic theory consists of the assumption of some representation of the coefficients and of the solution, for instance power series representation (see [5] for details). In relation to the nonanalytic theory, we know that the methods are procedures that consist of two main steps: first, a change of variable to transform the scalar perturbed linear differential equation in a system of first order of Poincaré type and then a diagonalization process (for further details, consult [6, 10, 12, 24]). Meanwhile, in the scalar method [2–4, 14, 15, 28, 32, 33] the asymptotic behavior of solutions for scalar linear differential equations of Poincaré type is obtained by a change of variable which reduces the order and transforms the perturbed linear differential equation in a nonlinear equation. Then, the results for the original problem are derived by analyzing the asymptotic behavior of this nonlinear equation.

1.3 The scalar method

In this paper, we are interested in the development of a modified version of the original scalar method. Indeed, in order to contextualize the basic ideas of this methodology, we recall the work of Bellman [3]. In [3], Bellman presents the analysis of the second order differential equation $u'' - (1 + f(t))u = 0$ by introducing the new variable $v = u'/u$ which transforms the linear perturbed equation in the following Riccati equation $v' + v^2 - (1 + f(t)) = 0$. Then, by assuming several conditions on the regularity and integrability of f, he obtains the formulas for characterization of the asymptotic behavior of u. For example, in the case that $f(t) \to 0$ when $t \to \infty$, Bellman proves that there are two linearly independent solutions u_1 and u_2, such that

$$\frac{u_1'(t)}{u_1(t)} \to 1, \tag{1.10}$$

$$\frac{u_2'(t)}{u_2(t)} \to -1, \tag{1.11}$$

$$\exp\Big(t - \int_{t_0}^{t} |f(\tau)| d\tau\Big) \leq u_1(t) \leq \exp\Big(t + \int_{t_0}^{t} |f(\tau)|\, d\tau\Big), \tag{1.12}$$

$$\exp\Big(-t - \int_{t_0}^{t} |f(\tau)| d\tau\Big) \leq u_2(t) \leq \exp\Big(-t + \int_{t_0}^{t} |f(\tau)|\, d\tau\Big). \tag{1.13}$$

More details and a summarization of the results of the application of the scalar method to a special second order equation are given in [4]. Note that (1.10)–(1.11) correspond to (1.5) and (1.12)–(1.13) to (1.6).

We reorganize and reformulate the original scalar method. Indeed, in Section 3, the presented scalar method distinguishes three big steps. First, for each characteristic root μ of (1.2), we introduce the change of variable $z = y'/y - \mu$ and deduce that z is a solution of an equation of the following type

$$\begin{aligned} z''' + \hat{a}_2 z'' + \hat{a}_1 z' + \hat{a}_0 z = \hat{r}_0(t) + \hat{r}_1(t) z z'' + \hat{r}_2(t) z z' + \hat{r}_3(t) z^2 z' \\ + \hat{r}_4(t)(z')^2 + \hat{r}_5(t) z^2 + \hat{r}_6(t) z^3 + \hat{r}_7(t) z^4, \end{aligned} \tag{1.14}$$

where $\hat{a}_k$ are real constants and $\hat{r}_k$ are real-valued functions, see equation (3.2). Second, we prove the existence, uniqueness and the asymptotic behavior of the solution of (1.14). Finally, in a third step, we deduce the existence of a fundamental system of solutions for (1.1) and conclude the process with the formulation and proof of the asymptotic integration formulas for the solutions of (1.1). Basically, we translate the properties of z (the solution of (1.14)) to y (the solution of (1.1)) via the relation $y(t) = \exp(\int_{t_0}^{t} (z(s) + \mu)\, ds)$.

1.4 Aim and results of the paper

The main purpose of this paper is to describe the asymptotic behavior of nonoscillatory solutions of equation (1.1) by the application of the scalar method. Then, our results are the following.

(i) If $\mu = \lambda_i$ in (1.14), we prove that $\lambda_j - \lambda_i$ with $j \in \{1,2,3,4\} \backslash \{i\}$ are the roots of the characteristic polynomial associated to the linear part of (1.14), see Proposition 3.1.

(ii) Assuming that (1.7) is satisfied and that (1.14) has a solution z_i for $\mu = \lambda_i$, we establish that

$$y_i(t) = \exp\Big(\int_{t_0}^{t} (\lambda_i + z_i(s))\, ds \Big), \qquad i = 1, \ldots, 4, \tag{1.15}$$

defines a fundamental system of solutions for (1.1), see Lemma 3.2.

(iii) We consider a third order nonlinear differential equation of the following type

$$z''' + b_2 z'' + b_1 z' + b_0 z = \mathbb{P}(t, z, z', z''), \tag{1.16}$$

where b_i are real constants and $\mathbb{P}$ is a given function. Then, we prove that (1.16) has a unique solution $z \in C_0^2([t_0, \infty[)$ by assuming three hypotheses: (a) the function $\mathbb{P}$ admits a especial decomposition, (b) the roots of the linear part of (1.16) are simple, and (c) the coefficients satisfy the decomposition of $\mathbb{P}$ and an integral smallness condition, see Theorem 3.3.

(iv) We assume that the constants a_i are such that (1.7) is satisfied and the perturbation functions are asymptotically small in the following sense

$$\lim_{t\to\infty} \left| \int_{t_0}^{\infty} g(t,s) p(\lambda_i, s)\, ds \right| + \left| \int_{t_0}^{\infty} \frac{\partial g}{\partial t}(t,s) p(\lambda_i, s)\, ds \right| + \left| \int_{t_0}^{\infty} \frac{\partial^2 g}{\partial t^2}(t,s) p(\lambda_i, s)\, ds \right| = 0, \tag{1.17}$$

$$\lim_{t\to\infty} \int_{t_0}^{\infty} \left[|g(t,s)| + \left| \frac{\partial g}{\partial t}(t,s) \right| + \left| \frac{\partial^2 g}{\partial t^2}(t,s) \right| \right] |r_j(s)|\, ds = 0, \qquad j = 1, \ldots, 4, \tag{1.18}$$

with g a Green function and p is given by

$$p(\lambda_i, s) = \lambda_i^3 r_3(s) + \lambda_i^2 r_2(s) + \lambda_i r_1(s) + r_0(s). \tag{1.19}$$

Then, by noticing that (1.14) is of the type (1.16), we prove that exists z_i a unique solution of (1.14) for $\mu = \lambda_i$, see Theorem 3.4

(v) We assume that (1.7) and (1.18) are satisfied. We prove the following asymptotic formulas

$$z_i(t), z_i'(t), z_i''(t) = \begin{cases} O\Big(\displaystyle\int_t^\infty e^{-\beta(t-s)}|p(\lambda_1, s)|ds\Big), & i=1, \ \ \beta \in [\lambda_2 - \lambda_1, 0[, \\ O\Big(\displaystyle\int_{t_0}^\infty e^{-\beta(t-s)}|p(\lambda_2, s)|ds\Big), & i=2, \ \ \beta \in [\lambda_3 - \lambda_2, 0[, \\ O\Big(\displaystyle\int_{t_0}^\infty e^{-\beta(t-s)}|p(\lambda_3, s)|ds\Big), & i=3, \ \ \beta \in [\lambda_4 - \lambda_3, 0[, \\ O\Big(\displaystyle\int_{t_0}^t e^{-\beta(t-s)}|p(\lambda_4, s)|ds\Big), & i=4, \ \ \beta \in]0, \lambda_3 - \lambda_4], \end{cases} \tag{1.20}$$

under different assumptions on the perturbation functions. To be more precise, we obtain (1.19)–(1.20), by assuming that the perturbations satisfy the following inequalities for $j = 0, \ldots, 3$

$$\int_t^\infty e^{-(\lambda_2-\lambda_1)(t-s)}|r_j(s)|\,ds \le \min\left\{\frac{1}{\lambda_1-\lambda_2}, \frac{1}{\sigma_1 A_1}\right\}, \qquad t \ge t_0, \ \ i=1, \tag{1.21}$$

$$\begin{aligned} &\int_{t_0}^t e^{-(\lambda_2-\lambda_1)(t-s)}|r_j(s)|\,ds + \int_t^\infty e^{-(\lambda_3-\lambda_2)(t-s)}|r_j(s)|\,ds \\ &\qquad \le \min\left\{\frac{1-e^{-(\lambda_1-\lambda_2)t_0}}{\lambda_1-\lambda_2} + \frac{1}{\lambda_2-\lambda_3}, \frac{1}{\sigma_2 A_2}\right\}, \qquad i=2, \end{aligned} \tag{1.22}$$

$$\begin{aligned} &\int_{t_0}^t e^{-(\lambda_2-\lambda_3)(t-s)}|r_j(s)|\,ds + \int_t^\infty e^{-(\lambda_4-\lambda_3)(t-s)}|r_i(s)|\,ds \\ &\qquad \le \min\left\{\frac{1-e^{-(\lambda_2-\lambda_3)t_0}}{\lambda_2-\lambda_3} + \frac{1}{\lambda_3-\lambda_4}, \frac{1}{\sigma_3 A_3}\right\}, \qquad i=3, \end{aligned} \tag{1.23}$$

$$\int_{t_0}^t e^{-(\lambda_2-\lambda_1)(t-s)}|r_j(s)|\,ds \le \min\left\{\frac{1}{\lambda_3-\lambda_4}, \frac{1}{\sigma_4 A_4}\right\}, \qquad i=4. \tag{1.24}$$

with σ_i and A_i some given constants, see (H_3) and Theorem 3.5 for details.

(vi) Under (1.7) and (1.18), we prove that (1.1) has a fundamental system of solutions given by (1.15) and that the asymptotic behavior (1.5) is valid. Note that (1.7) is a weaker condition than (1.11). Moreover, assuming (1.21), (1.22), (1.23) or (1.24), we prove that the following asymptotic behavior

$$y_i(t) = e^{\lambda_i(t-t_0)} \exp\left(\pi_i^{-1}\int_{t_0}^t \mathbb{P}(s, z_i(s), z_i'(s), z_i''(s))\,ds\right), \tag{1.25}$$

$$y_i''(t) = (\lambda_i + o(1)) e^{\lambda_i(t-t_0)} \exp\left(\pi_i^{-1}\int_{t_0}^t \mathbb{P}(s, z_i(s), z_i'(s), z_i''(s))\,ds\right), \tag{1.26}$$

$$y_i'''(t) = (\lambda_i^2 + o(1)) e^{\lambda_i(t-t_0)} \exp\left(\pi_i^{-1}\int_{t_0}^t \mathbb{P}(s, z_i(s), z_i'(s), z_i''(s))\,ds\right), \tag{1.27}$$

$$y_i^{(\mathrm{iv})}(t) = (\lambda_i^3 + o(1))e^{\lambda_i(t-t_0)} \exp\left(\pi_i^{-1}\int_{t_0}^t \mathbb{P}(s, z_i(s), z_i'(s), z_i''(s))\,ds\right), \tag{1.28}$$

holds, when $t \to \infty$ with z_i, z_i' and z_i'' given asymptotically by (1.19)–(1.20), see Theorem 3.6.

Moreover, we present an example of application of Theorem 3.6.

1.5 Outline of the paper

This paper is organized as follows. In Section 2 we present the general assumptions. In Section 3 we present the reformulated scalar method and the main results of this paper. Then, in Section 4 we present the proofs of Theorems 3.3, 3.5 and 3.6. Finally, in Section 5 we present an example.

2 General assumptions

The main general hypotheses about the coefficients and perturbation functions are (1.7), (1.17)–(1.18) and (1.21)–(1.24). Now, for convenience of the presentation, we introduce some notation and summarize these conditions in the following list

(H$_1$) $\left\{\lambda_i, i = \overline{1,4}\ :\ \lambda_1 > \lambda_2 > \lambda_3 > \lambda_4\right\} \subset \mathbb{R}$ is the set of characteristic roots for (1.2).

(H$_2$) Consider $p(\lambda_i, s)$ defined by (1.19). The perturbation functions are selected such that $\mathcal{G}(p(\lambda_i,\cdot))(t) \to 0$ and $\mathcal{L}(r_j)(t) \to 0$, $j = 0,1,2,3$, when $t \to \infty$, where $\mathcal{G}$ and $\mathcal{L}$ are the functionals defined as follows

$$\mathcal{G}(E)(t) = \left|\int_{t_0}^{\infty} g(t,s)E(s)\,ds\right| + \left|\int_{t_0}^{\infty} \frac{\partial g}{\partial t}(t,s)E(s)\,ds\right| + \left|\int_{t_0}^{\infty} \frac{\partial^2 g}{\partial t^2}(t,s)E(s)\,ds\right|, \tag{2.1}$$

$$\mathcal{L}(E)(t) = \int_{t_0}^{\infty}\left[|g(t,s)| + \left|\frac{\partial g}{\partial t}(t,s)\right| + \left|\frac{\partial^2 g}{\partial t^2}(t,s)\right|\right]|E(s)|\,ds. \tag{2.2}$$

(H$_3$) Let us introduce some notations. Consider the operators $\mathbb{F}_1, \mathbb{F}_2, \mathbb{F}_3$ and $\mathbb{F}_4$ defined as follows

$$\begin{aligned}
\mathbb{F}_1(E)(t) &= \int_t^{\infty} e^{-(\lambda_2-\lambda_1)(t-s)}|E(s)|\,ds,\\
\mathbb{F}_2(E)(t) &= \int_{t_0}^{t} e^{-(\lambda_1-\lambda_2)(t-s)}|E(s)|\,ds + \int_t^{\infty} e^{-(\lambda_3-\lambda_2)(t-s)}|E(s)|\,ds,\\
\mathbb{F}_3(E)(t) &= \int_{t_0}^{t} e^{-(\lambda_2-\lambda_3)(t-s)}|E(s)|\,ds + \int_t^{\infty} e^{-(\lambda_4-\lambda_3)(t-s)}|E(s)|\,ds,\\
\mathbb{F}_4(E)(t) &= \int_{t_0}^{t} e^{-(\lambda_3-\lambda_4)(t-s)}|E(s)|\,ds;
\end{aligned}$$

and σ_i, A_i defined by

$$\begin{aligned}
\sigma_i &= 3|\lambda_i|^2 + 5|\lambda_i| + 3\\
&\quad + \Big(19 + 7|\lambda_i| + |12\lambda_i + 3a_3| + |6\lambda_i^2 + 3\lambda_i a_3 + a_2|\Big)\eta, \qquad \eta \in\,]0, 1/2[,\\
A_i &= \frac{1}{|\Upsilon_i|}\sum_{(j,k,\ell)\in I_i} |\lambda_k - \lambda_\ell|\Big(1 + |\lambda_j - \lambda_i| + |\lambda_j - \lambda_i|^2\Big),
\end{aligned} \tag{2.3}$$

with

$$\Upsilon_i = \prod_{k>j}(\lambda_k - \lambda_j), \quad k,j \in \{1,2,3,4\} - \{i\},$$
$$I_i = \Big\{ (j,k,\ell) \in \{1,2,3,4\}^3 \ : \ (j,k,\ell) \neq (i,i,i), \ (k,\ell) \neq (j,j) \Big\}.$$

Then, the left inequalities in (1.21)–(1.24) are unified in the new notation as the following inequality $\mathbb{F}_i(r_j)(t) \leq \min\{\mathbb{F}_i(1)(t), (A_i\sigma_i)^{-1}\}$. Thus, defining the sets

$$\mathcal{F}_i([t_0,\infty[) = \left\{ E : [t_0,\infty[\to \mathbb{R} \ : \ \mathbb{F}_i(E)(t) \leq \rho_i := \min\left\{ \mathbb{F}_i(1)(t), \frac{1}{A_i\sigma_i} \right\} \right\}, \tag{2.4}$$

we assume that the perturbation functions $r_0, r_1, r_2, r_3 \in \mathcal{F}_i([t_0,\infty[)$.

3 Revisited scalar method and statement of the results

In this section we present the scalar method as a process of three steps. At each step we present the main results whose proofs are deferred to Section 4.

3.1 Change of variable and reduction of the order

We introduce a little bit different change of variable to those proposed by Bellman. Here, in this paper, the new variable z is of the following type

$$z(t) = \frac{y'(t)}{y(t)} - \mu \quad \text{or equivalently} \quad y(t) = \exp\left(\int_{t_0}^{t} (z(s) + \mu)\, ds \right), \tag{3.1}$$

where y is a solution of (1.1) and μ is an arbitrary root of the characteristic polynomial associated to (1.2). Then, by differentiation of $y(t)$ and by replacing the results in (1.1), we deduce that z is a solution of the following third order nonlinear equation

$$\begin{aligned} z''' &+ [4\mu + a_3]z'' + [6\mu^2 + 3a_3\mu + a_2]z' + [4\mu^3 + 3\mu^2 a_3 + 2\mu a_2 + a_1]z \\ &= -\Big\{ r_3(t)z'' + [3\mu r_3(t) + r_2(t)]z' + [3\mu^2 r_3(t) + 2\mu r_2(t) + r_1(t)]z + \mu^3 r_3(t) \\ &\qquad + \mu^2 r_2(t) + \mu r(t) + r_0(t) + 4zz'' + [12\mu + 3a_3 + 3r_3(t)]zz' + 6z^2 z' \\ &\qquad + 3[z']^2 + [6\mu^2 + 3\mu a_3 + a_2 + 3\mu r_3(t) + r_2(t)]z^2 + [4\mu + r_3(t)]z^3 + z^4 \Big\}. \end{aligned} \tag{3.2}$$

Thus, the analysis of original linear perturbed equation of fourth order (1.1) is translated to the analysis of a nonlinear third order equation (3.2).

We note that the characteristic polynomial associated to (1.2) and the third constant coefficient equation defined by the left hand side of (3.2) are related in the sense Proposition 3.1. Thus, noticing that the change of variable (3.1) can be applied by each characteristic root λ_i and assuming that the equation (3.2) with $\mu = \lambda_i$ has a solution, we can prove that (1.1) has a fundamental system of solutions, see Lemma 3.2.

Proposition 3.1. *If λ_i and λ_j are two distinct characteristic roots of the polynomial associated to* (1.2), *then $\lambda_j - \lambda_i$ is a root of the characteristic polynomial associated with the following differential equation*

$$z''' + [4\lambda_i + a_3]z'' + [6\lambda_i^2 + 3a_3\lambda_i + a_2]z' + [4\lambda_i^3 + 3\lambda_i^2 a_3 + 2\lambda_i a_2 + a_1]z = 0. \tag{3.3}$$

Proof. Considering $\lambda_i \neq \lambda_j$ satisfying the characteristic polynomial associated to (1.2), subtracting the equalities, dividing the result by $\lambda_j - \lambda_i$ and using the identities

$$\begin{aligned}\lambda_j^3 + \lambda_j^2\lambda_i + \lambda_j\lambda_i^2 + \lambda_i^3 &= (\lambda_j - \lambda_i)^3 + 4\lambda_i(\lambda_j - \lambda_i)^2 + 6\lambda_i^2(\lambda_j - \lambda_i) + 4\lambda_i^3\\ a_3(\lambda_j^2 + \lambda_j\lambda_i + \lambda_i^2) &= a_3(\lambda_j - \lambda_i)^2 + 3a_3\lambda_i(\lambda_j - \lambda_i) + 3a_3\lambda_i^2\\ a_2(\lambda_j - \lambda_i) &= a_2(\lambda_j - \lambda_i) + 2\lambda_i a_2,\end{aligned}$$

we deduce that $\lambda_j - \lambda_i$ is a root of the characteristic polynomial associated to (3.3). □

Lemma 3.2. *Consider that* (3.2) *has a solution for each* $\mu \in \{\lambda_1, \dots, \lambda_4\}$. *If the hypothesis* ($H_1$) *is satisfied, then the fundamental system of solutions of* (1.1) *is given by*

$$y_i(t) = \exp\Big(\int_{t_0}^{t} [\lambda_i + z_i(s)]\, ds\Big), \quad \textit{with } z_i \textit{ solution of } (3.2) \textit{ with } \mu = \lambda_i, \quad i \in \{1,2,3,4\}. \tag{3.4}$$

3.2 Well posedness and asymptotic behavior of (3.2)

In this second step, we obtain three results. The first result is related to the conditions for the existence and uniqueness of a more general equation of that given in (3.2), see Theorem 3.3. Then, we introduce a second result concerning to the well posedness of (3.2), see Theorem 3.4. Finally, we present the result of asymptotic behavior for (3.2), see Theorem 3.5. Indeed, to be precise these three results are the following theorems:

Theorem 3.3. *Let us introduce the notation* $C_0^2([t_0,\infty[)$ *for the following space of functions*

$$C_0^2([t_0,\infty[) = \Big\{z \in C^2([t_0,\infty[,\mathbb{R}) \;:\; z(t), z'(t), z''(t) \to 0 \textit{ when } t \to \infty\Big\}, \quad t_0 \in \mathbb{R},$$

and consider the equation

$$z''' + b_2 z'' + b_1 z' + b_0 z = \Omega(t) + F(t, z, z', z''), \tag{3.5}$$

where b_i *are real constants,* Ω *and* F *are given functions such that the following restrictions hold.*

($\mathcal{R}_1$) *There are functions* $\hat{F}_1, \hat{F}_2, \Gamma\colon \mathbb{R}^4 \to \mathbb{R}$; $\Lambda_1, \Lambda_2\colon \mathbb{R} \to \mathbb{R}^3$ *and* $\mathbf{C} \in \mathbb{R}^7$, *such that*

$$\begin{aligned}F &= \hat{F}_1 + \hat{F}_2 + \Gamma,\\ \hat{F}_1(t, x_1, x_2, x_3) &= \Lambda_1(t)\cdot(x_1, x_2, x_3),\\ \hat{F}_2(t, x_1, x_2, x_3) &= \Lambda_2(t)\cdot(x_1x_2, x_1^2, x_1^3),\\ \Gamma(t, x_1, x_2, x_3) &= \mathbf{C}\cdot(x_2^2, x_1x_2, x_1x_3, x_1^2, x_1^2x_2, x_1^3, x_1^4),\end{aligned}$$

where "$\cdot$" denotes the canonical inner product in $\mathbb{R}^n$.

($\mathcal{R}_2$) *The set of characteristic roots of* (3.5) *when* $\Omega = F = 0$ *is given by* $\{\gamma_1 > \gamma_2 > \gamma_3\} \subset \mathbb{R}$.

($\mathcal{R}_3$) *It is assumed that* $\mathcal{G}(\Omega)(t) \to 0$, $\mathcal{L}(\|\Lambda_1\|_1)(t) \to 0$ *and* $\mathcal{L}(\|\Lambda_2\|_1)(t)$ *is bounded, when* $t \to \infty$. *Here* $\|\cdot\|_1$ *denotes the norm of the sum in* $\mathbb{R}^n$, $\mathcal{G}$ *and* $\mathcal{L}$ *are the operators defined on* (2.1) *and* (2.2), *respectively.*

Then, there exists a unique $z \in C_0^2([t_0,\infty[)$ *solution of* (3.5).

Theorem 3.4. *Let us consider that the hypotheses* (H_1) *and* (H_2) *are satisfied. Then, the equation* (3.2) *with* $\mu = \lambda_i$ *has a unique solution* z_i *such that* $z_i \in C_0^2([t_0,\infty[)$.

Theorem 3.5. *Consider that the hypotheses (H$_1$),(H$_2$) and (H$_3$) are satisfied. Then z_i the solution of (3.2) with $\mu = \lambda_i$, has the following asymptotic behavior*

$$z_i(t), z_i'(t), z_i''(t) = \begin{cases} O\left(\int_t^\infty e^{-\beta(t-s)}|p(\lambda_1,s)|\,ds\right), & i=1, \quad \beta\in[\lambda_2-\lambda_1,0[,\\ O\left(\int_{t_0}^\infty e^{-\beta(t-s)}|p(\lambda_2,s)|\,ds\right), & i=2, \quad \beta\in[\lambda_3-\lambda_2,0[,\\ O\left(\int_{t_0}^\infty e^{-\beta(t-s)}|p(\lambda_3,s)|\,ds\right), & i=3, \quad \beta\in[\lambda_4-\lambda_3,0[,\\ O\left(\int_{t_0}^t e^{-\beta(t-s)}|p(\lambda_4,s)|\,ds\right), & i=4, \quad \beta\in]0,\lambda_3-\lambda_4], \end{cases} \tag{3.6}$$

where $p(\lambda_i,s) = \lambda_i^3 r_3(s) + \lambda_i^2 r_2(s) + \lambda_i r_1(s) + r_0(s)$.

3.3 Existence of a fundamental system of solutions for (1.1) and its asymptotic behavior

Here we translate the results for the behavior of z (see Theorem 3.4) to the variable y via the relation (3.1).

Theorem 3.6. *Let us assume that the hypotheses (H$_1$) and (H$_2$) are satisfied. Denote by $W[y_1,\ldots,y_4]$ the Wronskian of $\{y_1,\ldots,y_4\}$, by π_i the number defined as follows*

$$\pi_i = \prod_{k\in N_i}(\lambda_k-\lambda_i), \quad N_i = \{1,2,3,4\}-\{i\}, \quad i=1,\ldots,4,$$

by $p(\lambda_i,s)$ the function defined in Theorem 3.5 and by F the function defined in Theorem 3.3 with Λ_1,Λ_2 and $\mathbf{C}$ given in (4.14). Then, the equation (1.1) has a fundamental system of solutions given by (3.4). Moreover the following properties about the asymptotic behavior

$$\frac{y_i'(t)}{y_i(t)} = \lambda_i, \qquad \frac{y_i''(t)}{y_i(t)} = \lambda_i^2, \qquad \frac{y_i'''(t)}{y_i(t)} = \lambda_i^3, \qquad \frac{y_i^{(\mathrm{iv})}(t)}{y_i(t)} = \lambda_i^4, \tag{3.7}$$

$$W[y_1,\ldots,y_4] = \prod_{1\le k<\ell\le 4}(\lambda_\ell-\lambda_k)\; y_1y_2y_3y_4\;\big(1+o(1)\big), \tag{3.8}$$

are satisfied when $t\to\infty$. Furthermore, if (H$_3$) is satisfied, then

$$y_i(t) = e^{\lambda_i(t-t_0)}\exp\left(\pi_i^{-1}\int_{t_0}^t\Big[p(\lambda_i,s)+F(s,z_i(s),z_i'(s),z_i''(s))\Big]ds\right), \tag{3.9}$$

$$y_i'(t) = (\lambda_i+o(1))\,e^{\lambda_i(t-t_0)}\exp\left(\pi_i^{-1}\int_{t_0}^t\Big[p(\lambda_i,s)+F(s,z_i(s),z_i'(s),z_i''(s))\Big]ds\right), \tag{3.10}$$

$$y_i''(t) = (\lambda_i^2+o(1))\,e^{\lambda_i(t-t_0)}\exp\left(\pi_i^{-1}\int_{t_0}^t\Big[p(\lambda_i,s)+F(s,z_i(s),z_i'(s),z_i''(s))\Big]ds\right), \tag{3.11}$$

$$y_i'''(t) = (\lambda_i^3+o(1))\,e^{\lambda_i(t-t_0)}\exp\left(\pi_i^{-1}\int_{t_0}^t\Big[p(\lambda_i,s)+F(s,z_i(s),z_i'(s),z_i''(s))\Big]ds\right), \tag{3.12}$$

$$y_i^{(\mathrm{iv})}(t) = \Big(\lambda_i^4+o(1)\Big)\,e^{\lambda_i(t-t_0)}\exp\left(\pi_i^{-1}\int_{t_0}^t\Big[p(\lambda_i,s)+F(s,z_i(s),z_i'(s),z_i''(s))\Big]ds\right), \tag{3.13}$$

hold, when $t\to\infty$ with z_i, z_i' and z_i'' given asymptotically by (3.6).

4 Proof of the results

In this section we present the proofs of Theorems 3.3, 3.4, 3.5, and 3.6.

4.1 Proof of Theorem 3.3

Before presenting the proof, we need to define some notations about Green functions. First, let us consider the equation associated to (3.5) with $\Omega = F = 0$, i.e.

$$z''' + b_2 z'' + b_1 z' + b_0 z = 0, \tag{4.1}$$

and denote by γ_i, $i = 1, 2, 3$, the roots of the characteristic polynomial for (4.1). Then, the Green function for (4.1) is defined by

$$g(t,s) = \frac{1}{(\gamma_3-\gamma_2)(\gamma_3-\gamma_1)(\gamma_2-\gamma_1)} \begin{cases} g_1(t,s), & (\gamma_1,\gamma_2,\gamma_3) \in \mathbb{R}^3_{---}, \\ g_2(t,s), & (\gamma_1,\gamma_2,\gamma_3) \in \mathbb{R}^3_{+--}, \\ g_3(t,s), & (\gamma_1,\gamma_2,\gamma_3) \in \mathbb{R}^3_{++-}, \\ g_4(t,s), & (\gamma_1,\gamma_2,\gamma_3) \in \mathbb{R}^3_{+++}, \end{cases} \tag{4.2}$$

where

$$g_1(t,s) = \begin{cases} 0, & t \ge s, \\ (\gamma_2-\gamma_3)e^{-\gamma_1(t-s)} + (\gamma_3-\gamma_1)e^{-\gamma_2(t-s)} + (\gamma_1-\gamma_2)e^{-\gamma_3(t-s)}, & t \le s, \end{cases} \tag{4.3}$$

$$g_2(t,s) = \begin{cases} (\gamma_2-\gamma_3)e^{-\gamma_1(t-s)}, & t \ge s, \\ (\gamma_1-\gamma_2)e^{-\gamma_3(t-s)} + (\gamma_3-\gamma_1)e^{-\gamma_2(t-s)}, & t \le s, \end{cases} \tag{4.4}$$

$$g_3(t,s) = \begin{cases} (\gamma_2-\gamma_3)e^{-\gamma_1(t-s)} + (\gamma_3-\gamma_1)e^{-\gamma_2(t-s)}, & t \ge s, \\ (\gamma_2-\gamma_1)e^{-\gamma_3(t-s)}, & t \le s, \end{cases} \tag{4.5}$$

$$g_4(t,s) = \begin{cases} (\gamma_2-\gamma_3)e^{-\gamma_1(t-s)} + (\gamma_3-\gamma_1)e^{-\gamma_2(t-s)} + (\gamma_1-\gamma_2)e^{-\gamma_3(t-s)}, & t \ge s, \\ 0, & t \le s. \end{cases} \tag{4.6}$$

Further details on Green functions may be consulted in [4].

Now, we present the proof by noticing that, by the method of variation of parameters, the hypothesis $(\mathcal{R}_2)$, implies that the equation (3.5) is equivalent to the following integral equation

$$z(t) = \int_{t_0}^{\infty} g(t,s)\Big[\Omega(s) + F\Big(s, z(s), z'(s), z''(s)\Big)\Big]\, ds, \tag{4.7}$$

where g is the Green function defined on (4.2). Moreover, we recall that $C_0^2([t_0,\infty[)$ is a Banach space with the norm $\|z\|_0 = \sup_{t\ge t_0}[|z(t)| + |z'(t)| + |z''(t)|]$. Now, we define the operator T from $C_0^2([t_0,\infty[)$ to $C_0^2([t_0,\infty[)$ as follows

$$Tz(t) = \int_{t_0}^{\infty} g(t,s)\Big[\Omega(s) + F\Big(s, z(s), z'(s), z''(s)\Big)\Big]\, ds. \tag{4.8}$$

Then, we note that (4.7) can be rewritten as the operator equation

$$Tz = z \quad \text{over} \quad D_\eta := \Big\{ z \in C_0^2([t_0,\infty[) \ : \ \|z\|_0 \le \eta \Big\}, \tag{4.9}$$

where $\eta \in \mathbb{R}^+$ will be selected in order to apply the Banach fixed point theorem. Indeed, we have the following.

(a) *T is well defined from $C_0^2([t_0,\infty[)$ to $C_0^2([t_0,\infty[)$.* Let us consider an arbitrary $z \in C_0^2([t_0,\infty[)$. We note that

$$T'z(t) = \int_{t_0}^{\infty} \frac{\partial g}{\partial t}(t,s)\Big[\Omega(s) + F\Big(s, z(s), z'(s), z''(s)\Big)\Big]\, ds,$$
$$T''z(t) = \int_{t_0}^{\infty} \frac{\partial^2 g}{\partial t^2}(t,s)\Big[\Omega(s) + F\Big(s, z(s), z'(s), z''(s)\Big)\Big]\, ds.$$

Then, by the definition of g, we immediately deduce that $Tz, T'z, T''z \in C^2([t_0,\infty[,\mathbb{R})$. Furthermore, by the hypothesis $(\mathcal{R}_1)$, we can deduce the following estimates

$$\begin{aligned}
|z(t)| \le{}& \left|\int_{t_0}^{\infty} g(t,s)|\Omega(s)\, ds\right| + \int_{t_0}^{\infty} g(t,s)\Big[\Big|\hat{F}_1\Big(s,z(s),z'(s),z''(s)\Big)\Big| \\
&+ \Big|\hat{F}_2\Big(s,z(s),z'(s),z''(s)\Big)\Big| + \Big|\Gamma\Big(s,z(s),z'(s),z''(s)\Big)\Big|\Big]\, ds, \qquad (4.10)\\
|z'(t)| \le{}& \left|\int_{t_0}^{\infty} \frac{\partial g}{\partial t}(t,s)|\Omega(s)\, ds\right| + \int_{t_0}^{\infty} \frac{\partial g}{\partial t}(t,s)\Big[\Big|\hat{F}_1\Big(s,z(s),z'(s),z''(s)\Big)\Big| \\
&+ \Big|\hat{F}_2\Big(s,z(s),z'(s),z''(s)\Big)\Big| + \Big|\Gamma\Big(s,z(s),z'(s),z''(s)\Big)\Big|\Big]\, ds, \qquad (4.11)\\
|z''(t)| \le{}& \left|\int_{t_0}^{\infty} \frac{\partial^2 g}{\partial t^2}(t,s)|\Omega(s)\, ds\right| + \int_{t_0}^{\infty} \frac{\partial^2 g}{\partial t^2}(t,s)\Big[\Big|\hat{F}_1\Big(s,z(s),z'(s),z''(s)\Big)\Big| \\
&+ \Big|\hat{F}_2\Big(s,z(s),z'(s),z''(s)\Big)\Big| + \Big|\Gamma\Big(s,z(s),z'(s),z''(s)\Big)\Big|\Big]\, ds. \qquad (4.12)
\end{aligned}$$

Now, by application of the hypothesis $(\mathcal{R}_3)$, the properties of $\hat{F}_1, \hat{F}_2$ and Γ and the fact that $z \in C_0^2$, we have that the right hand sides of (4.11)–(4.12) tend to 0 when $t \to \infty$. Then, $Tz, T'z, T''z \to 0$ when $t \to \infty$ or equivalently $Tz \in C_0^2$ for all $z \in C_0^2$.

(b) *For all $\eta \in\,]0,1[$, the set D_η is invariant under T.* Let us consider $z \in D_\eta$. From (4.10)–(4.12), we can deduce the following estimate

$$\begin{aligned}
\|Tz\|_0 \le{}& \mathcal{G}(\Omega)(t) + \|z\|_0 \mathcal{L}\Big(\|\Lambda_1\|_1\Big)(t) + 2\Big(\|z\|_0\Big)^2 \mathcal{L}\Big(\|\Lambda_2\|_1\Big)(t) + \Big(\|z\|_0\Big)^3 \mathcal{L}\Big(\|\Lambda_2\|_1\Big)(t) \\
&+ \Big(\|z\|_0\Big)^2 \left(\sum_{i=1}^{4} |c_i| + \Big(|c_5| + |c_6|\Big)\|z\|_0 + |c_7|\Big(\|z\|_0\Big)^2\right) \mathcal{L}\Big(1\Big)(t) \\
\le{}& I_1(t) + I_2(t), \qquad (4.13)
\end{aligned}$$

where

$$\begin{aligned}
I_1(t) ={}& \mathcal{G}(\Omega)(t) \\
I_2(t) ={}& \|z\|_0 \Big\{ \mathcal{L}\Big(\|\Lambda_1\|_1\Big)(t) + \Big(2\mathcal{L}\Big(\|\Lambda_2\|_1\Big)(t) + \mathcal{L}\Big(\|\mathbf{C}\|_1\Big)(t)\Big)\|z\|_0 \\
&+ \Big(\mathcal{L}\Big(\|\Lambda_2\|_1\Big)(t) + \mathcal{L}\Big(\|\mathbf{C}\|_1\Big)(t)\Big)\Big(\|z\|_0\Big)^2 + \mathcal{L}\Big(\|\mathbf{C}\|_1\Big)(t)\Big(\|z\|_0\Big)^3 \Big\}.
\end{aligned}$$

Now, by $(\mathcal{R}_3)$ we deduce that $I_1(t) \to 0$ when $t \to \infty$. Similarly, by application of $(\mathcal{R}_3)$, we can prove that the inequality

$$\begin{aligned} I_2(t) &\le \eta^2 \Big\{ \Big(2\mathcal{L}\Big(\|\Lambda_2\|_1\Big)(t) + \mathcal{L}\Big(\|\mathbf{C}\|_1\Big)(t)\Big) + \Big(\mathcal{L}\Big(\|\Lambda_2\|_1\Big)(t) + \mathcal{L}\Big(\|\mathbf{C}\|_1\Big)(t)\Big)\eta \\ &\qquad + \mathcal{L}\Big(\|\mathbf{C}\|_1\Big)(t)\eta^2 \Big\} \\ &\le \eta, \end{aligned}$$

holds when $t \to \infty$ in a right neighborhood of $\eta = 0$. Hence, by (4.13) and $(\mathcal{R}_3)$, we prove that $Tz \in D_\eta$ for all $z \in D_\eta$.

(c) *T is a contraction for* $\eta \in]0, 1/2[$. Let $z_1, z_2 \in D_\eta$, by the hypothesis $(\mathcal{R}_1)$ and algebraic rearrangements, we obtain that

$$\begin{aligned} &\|Tz_1 - Tz_2\|_0 \\ &\quad \le \|z_1 - z_2\|_0 \int_{t_0}^{\infty} \left(|g(t,s)| + \left|\frac{\partial g}{\partial t}(t,s)\right| + \left|\frac{\partial^2 g}{\partial t^2}(t,s)\right| \right) \|\Lambda_1(s)\|_1 \, ds \\ &\qquad + \|z_1 - z_2\|_0 \max\left\{2\eta, 3\eta^2\right\} \int_{t_0}^{\infty} \left(|g(t,s)| + \left|\frac{\partial g}{\partial t}(t,s)\right| + \left|\frac{\partial^2 g}{\partial t^2}(t,s)\right| \right) \|\Lambda_2(s)\|_1 \, ds \\ &\qquad + \|z_1 - z_2\|_0 \max\left\{2\eta, 3\eta^2, 4\eta^3\right\} \int_{t_0}^{\infty} \left(|g(t,s)| + \left|\frac{\partial g}{\partial t}(t,s)\right| + \left|\frac{\partial^2 g}{\partial t^2}(t,s)\right| \right) \|\mathbf{C}\|_1 \, ds \\ &\quad \le \|z_1 - z_2\|_0 \left\{ \mathcal{L}\Big(\|\Lambda_1\|_1\Big) + \max\left\{2\eta, 3\eta^2\right\} \mathcal{L}\Big(\|\Lambda_2\|_1\Big) + \max\left\{2\eta, 3\eta^2, 4\eta^3\right\} \mathcal{L}\Big(\|\mathbf{C}\|_1\Big) \right\}. \end{aligned}$$

Then, by application of $(\mathcal{R}_3)$, we deduce that T is a contraction, since, for an arbitrary $\eta \in]0, 1/2[$, we have that $\max\left\{2\eta, 3\eta^2\right\} = \max\left\{2\eta, 3\eta^2, 4\eta^3\right\} = 2\eta < 1$.

Hence, from (a)–(c) and application of Banach fixed point theorem, we deduce that there are a unique $z \in D_\eta \subset C_0^2([t_0, \infty[)$ solution of (4.9).

4.2 Proof of Theorem 3.4

The proof of the Theorem 3.4 is followed by the application of Theorem 3.3. Indeed, in the next lines we verify the hypothesis $(\mathcal{R}_1)$–$(\mathcal{R}_3)$. First, the hypothesis $(\mathcal{R}_1)$ is satisfied since (3.4) can be rewritten as (3.5). More precisely, if λ_i denotes an arbitrary characteristic root of (1.2), we have that the constant coefficients b_j in (3.5) are defined by

$$b_0 = 4\lambda_i^3 + 3\lambda_i^2 a_3 + 2\lambda_i a_2 + a_1, \qquad b_1 = 6\lambda_i^2 + 3\lambda_i a_3 + a_2, \qquad b_2 = 4\lambda_i + a_3, \tag{4.14a}$$

the functions $\Omega \colon \mathbb{R} \to \mathbb{R}$ and $\Lambda_1, \Lambda_2 \colon \mathbb{R} \to \mathbb{R}^3$ and the constant $\mathbf{C} \in \mathbb{R}^7$ defining the function F are given by

$$\Omega(t) = -(\lambda_i^3 r_3(t) + \lambda_i^2 r_2(t) + \lambda_i r_1(t) + r_0(t)), \quad \Lambda_1(t) = (b(t), f(t), h(t)), \tag{4.14b}$$

$$\Lambda_2(t) = (q(t), f(t), h(t)), \quad \mathbf{C} = -\Big(3,\ 12\lambda_i + 3a_3,\ 4,\ 6\lambda_i^2 + 3\lambda_i a_3 + a_2,\ 6,\ 4\lambda_i, 1\Big), \tag{4.14c}$$

with

$$b(t) = -(3\lambda_i^2 r_3 + 2\lambda_i r_2 + r_1)(t), \quad f(t) = -(3\lambda_i r_3 + r_2)(t), \quad q(t) = 3h(t) = -3r_3(t). \tag{4.14d}$$

Second by the application of Proposition 3.1, we deduce that the hypothesis $(\mathcal{R}_2)$ is satisfied. Meanwhile, we note that (H_2) implies $(\mathcal{R}_3)$. Thus, we deduce that conclusion of the Theorem 3.4 is valid.

4.3 Proof of Theorem 3.5

We prove the formula (3.6) by analyzing an iterative sequence and using the properties of the operator T defined in (4.8).

4.3.1 Proof of (3.6) with $i = 1$

Let us denote by T the operator defined in (4.8) and by z_1 the solution of the equation (3.4) associated with the characteristic root λ_1 of (1.2). Now, on D_η with $\eta \in]0, 1/2[$, we define the sequence $\omega_{n+1} = T\omega_n$ with $\omega_0 = 0$, we have that $\omega_n \to z_1$ when $n \to \infty$. This fact is a consequence of the contraction property of T.

We note that the hypothesis (H_1) and Proposition (3.1) implies that all roots of the corresponding characteristic polynomial for (4.1) with b_i defined on (4.14a) are negative, since,

$$0 > \gamma_1 = \lambda_2 - \lambda_1 > \gamma_2 = \lambda_3 - \lambda_1 > \gamma_3 = \lambda_4 - \lambda_1. \tag{4.15}$$

Moreover, by (4.14b) we note that the identity $\Omega(s) = p(\lambda_1, s)$ is valid. Then, the Green function g defined on (4.2) is given in terms of g_1 and

$$(\gamma_2 - \gamma_1)(\gamma_3 - \gamma_2)(\gamma_3 - \gamma_1) = (\lambda_3 - \lambda_2)(\lambda_4 - \lambda_3)(\lambda_4 - \lambda_2) = \Upsilon_1.$$

Naturally, the operator T can be rewritten equivalently as follows

$$Tz(t) = \frac{1}{\Upsilon_1} \int_t^\infty g_1(t,s) \Big[p(\lambda_1, s) + F\Big(s, z(s), z'(s), z''(s)\Big)\Big] ds, \quad \text{for } t \geq t_0, \tag{4.16}$$

since $g_1(t,s) = 0$ for $s \in [t_0, t]$. Thus, the proof of (3.6) with $i = 1$ is reduced to prove that

$$\exists\, \Phi_n \in \mathbb{R}_+ \;:\; |\omega_n(t)| + |\omega_n'(t)| + |\omega_n''(t)| \leq \Phi_n \int_t^\infty e^{-\beta(t-\tau)} |p(\lambda_1, \tau)|\, d\tau, \qquad \forall\, t \geq t_0, \tag{4.17}$$

$$\exists\, \Phi \in \mathbb{R}_+ \;:\; \Phi_n \to \Phi, \text{ when } n \to \infty. \tag{4.18}$$

Hence, to complete the proof of (3.6) with $i = 1$, we proceed to prove (4.17) by mathematical induction on n and deduce that (4.18) is a consequence of the construction of the sequence $\{\Phi_n\}$. However, before to proceed with the details, we deduce a bound of the Green function g_1:

$$\begin{aligned}
&|g_1(t,s)| + \left|\frac{\partial g_1}{\partial t}(t,s)\right| + \left|\frac{\partial^2 g_1}{\partial t^2}(t,s)\right| \\
&\quad \leq \Big(|\gamma_3 - \gamma_2| + |\gamma_1 - \gamma_3| + |\gamma_2 - \gamma_1|\Big) e^{-\max\{\gamma_1,\gamma_2,\gamma_3\}(t-s)}, \\
&\qquad + \Big(|\gamma_3 - \gamma_2||\gamma_1| + |\gamma_1 - \gamma_3||\gamma_2| + |\gamma_2 - \gamma_1||\gamma_3|\Big) e^{-\max\{\gamma_1,\gamma_2,\gamma_3\}(t-s)}, \\
&\qquad + \Big(|\gamma_3 - \gamma_2||\gamma_1|^2 + |\gamma_1 - \gamma_3||\gamma_2|^2 + |\gamma_2 - \gamma_1||\gamma_3|^2\Big) e^{-\max\{\gamma_1,\gamma_2,\gamma_3\}(t-s)} \\
&\quad = \Big(|\lambda_4 - \lambda_3|\Big\{1 + |\lambda_2 - \lambda_1| + |\lambda_2 - \lambda_1|^2\Big\} + |\lambda_4 - \lambda_2|\Big\{1 + |\lambda_3 - \lambda_1| + |\lambda_3 - \lambda_1|^2\Big\} \\
&\qquad + |\lambda_3 - \lambda_2|\Big\{1 + |\lambda_4 - \lambda_1| + |\lambda_4 - \lambda_1|^2\Big\}\Big) e^{-\max\{\lambda_2-\lambda_1,\lambda_3-\lambda_1,\lambda_4-\lambda_1\}(t-s)} \\
&\quad = A_1 |\Upsilon_1| e^{-(\lambda_2 - \lambda_1)(t-s)}
\end{aligned} \tag{4.19}$$

Here A_1 is the notation defined on (2.3).

We now prove (4.17). Note that for $n = 1$ the estimate (4.17) is satisfied with $\Phi_1 = A_1$. Indeed, it can be proved immediately by the definition of the operator T given on (4.16), the property $F(s,0,0,0) = 0$, the estimate (4.19) and the hypothesis that $\beta \in [\lambda_2 - \lambda_1, 0[$, since

$$
\begin{aligned}
&|\omega_1(t)| + |\omega_1'(t)| + |\omega_1''(t)| \\
&\quad = |T\omega_0(t)| + |T'\omega_0(t)| + |T''\omega_0(t)| \\
&\quad = \frac{1}{|\Upsilon_1|}\int_t^{\infty}\left(|g_1(t,s)| + \left|\frac{\partial g_1}{\partial t}(t,s)\right| + \left|\frac{\partial^2 g_1}{\partial t^2}(t,s)\right|\right)|p(\lambda_1,s)|\,ds \\
&\quad \le A_1\int_t^{\infty} e^{-(\lambda_2-\lambda_1)(t-\tau)}|p(\lambda_1,\tau)|\,d\tau \\
&\quad \le A_1\int_t^{\infty} e^{-\beta(t-\tau)}|p(\lambda_1,\tau)|\,d\tau.
\end{aligned}
$$

Now, assuming that (4.17) is valid for $n = k$, we prove that (4.17) is also valid for $n = k+1$. However, before to prove the estimate (4.17) for $n = k+1$, we note that by the hypothesis (H_3) (i.e. the perturbations belong to $\mathcal{F}_1([t_0,\infty[)$), the notation (4.14) and the fact that $\max\{\eta,\eta^2,\eta^3\} = \eta$ for $\eta \in]0,1/2[$, we deduce the following estimates

$$
\begin{aligned}
&\Big|p(\lambda_1,s) + F\Big(s,\omega_k(s),\omega_k'(s),\omega_k''(s)\Big)\Big| \\
&\quad \le |p(\lambda_1,s)| + |b(s)||\omega_k(s)| + |f(s)||\omega_k'(s)| + |h(s)||\omega_k''(s)| \\
&\qquad + |q(s)||\omega_k'(s)||\omega_k(s)| + |f(s)||\omega_k(s)|^2 + |h(s)||\omega_k(s)|^3 \\
&\qquad + |C_1||\omega_k'(s)|^2 + |C_2||\omega_k(s)||\omega_k'(s)| + |C_3||\omega_k''| + |C_4||\omega_k(s)||\omega_k''(s)| \\
&\qquad + |C_5||\omega_k(s)|^2|\omega_k''(s)| + |C_6||\omega_k(s)|^3 + |C_7||\omega_k(s)|^4 \\
&\quad \le |p(\lambda_1,s)| + \Bigg[|b(s)| + |f(s)| + |h(s)| + |q(s)|\eta + |f(s)|\eta + |h(s)|\eta^2 \\
&\qquad\qquad + \Big(\sum_{i=1}^{4}|C_i|\Big)\eta + \Big(\sum_{i=5}^{6}|C_i|\Big)\eta^2\Bigg]\max\Big\{|\omega_k(s)|,|\omega_k'(s)|,|\omega_k''(s)|\Big\} \\
&\quad \le |p(\lambda_1,s)| + \Bigg[\|(b,f,h)(s)\|_1 + \Big(\|(q,f,h)(s)\|_1 + \|\mathbf{C}\|_1\Big)\eta\Bigg]\Big\|\Big(\omega_k,\omega_k',\omega_k''\Big)(s)\Big\|_1,
\end{aligned} \tag{4.20a}
$$

$$
\int_t^{\infty} e^{-(\lambda_2-\lambda_1)(t-s)}|b(s)|ds \le (3|\lambda_1|^2 + 2|\lambda_1| + 1)\rho_1, \tag{4.20b}
$$

$$
\int_t^{\infty} e^{-(\lambda_2-\lambda_1)(t-s)}|f(s)|ds \le (3|\lambda_1| + 1)\rho_1, \tag{4.20c}
$$

$$
\int_t^{\infty} e^{-(\lambda_2-\lambda_1)(t-s)}|q(s)|ds \le 3\rho_1, \qquad \int_t^{\infty} e^{-(\lambda_2-\lambda_1)(t-s)}|h(s)|ds \le \rho_1. \tag{4.20d}
$$

Using (4.16), the notation (4.14), the inductive hypothesis, the inequality (4.19) and the estimates (4.20) we have that

$$
\begin{aligned}
&|\omega_{k+1}(t)| + |\omega_{k+1}'(t)| + |\omega_{k+1}''(t)| \\
&\quad = |T\omega_k(t)| + |T'\omega_k(t)| + |T''\omega_k(t)| \\
&\quad = \frac{1}{|\Upsilon_1|}\Bigg\{\Bigg|\int_t^{\infty} g_1(t,s)\Big[p(\lambda_1,s) + F\Big(s,\omega_k(s),\omega_k'(s),\omega_k''(s)\Big)\Big]ds\Bigg|
\end{aligned}
$$

$$
\begin{aligned}
&+\left|\int_t^\infty \frac{\partial g_1}{\partial t}(t,s)\Big[p(\lambda_1,s)+F\Big(s,\omega_k(s),\omega_k'(s),\omega_k''(s)\Big)\Big]ds\right| \\
&+\left|\int_t^\infty \frac{\partial^2 g_1}{\partial t^2}(t,s)\Big[p(\lambda_1,s)+F\Big(s,\omega_k(s),\omega_k'(s),\omega_k''(s)\Big)\Big]ds\right|\Bigg\} \\
=&\ \frac{1}{|\Upsilon_1|}\int_t^\infty \left(|g_1(t,s)|+\left|\frac{\partial g_1}{\partial t}(t,s)\right|+\left|\frac{\partial^2 g_1}{\partial t^2}(t,s)\right|\right)\left|p(\lambda_1,s)+F\Big(s,\omega_k(s),\omega_k'(s),\omega_k''(s)\Big)\right|ds \\
\leq&\ A_1\int_t^\infty e^{-(\lambda_2-\lambda_1)(t-s)}\Bigg\{|p(\lambda_1,s)|+\left[\|(b,f,h)(s)\|_1+\Big(\|(p,f,h)(s)\|_1+\|\mathbf{C}\|_1\Big)\eta\right] \\
&\qquad\qquad \times\left\|\Big(\omega_k,\omega_k',\omega_k''\Big)(s)\right\|_1\Bigg\}ds \\
\leq&\ A_1\int_t^\infty e^{-(\lambda_2-\lambda_1)(t-s)}\Bigg\{|p(\lambda_1,s)|+\left[\|(b,f,h)(s)\|_1+\Big(\|(q,f,h)(s)\|_1+\|\mathbf{C}\|_1\Big)\eta\right] \\
&\qquad\qquad \times\Phi_k\int_s^\infty e^{-\beta(s-\tau)}|p(\lambda_1,\tau)|d\tau\Bigg\}ds \\
\leq&\ A_1\left\{1+\int_t^\infty e^{-(\lambda_2-\lambda_1)(t-s)}\left[\|(b,f,h)(s)\|_1+\Big(\|(q,f,h)(s)\|_1+\|\mathbf{C}\|_1\Big)\eta\right]\Phi_k\,ds\right\} \\
&\times\left\{\int_t^\infty e^{-\beta(t-\tau)}|p(\lambda_1,\tau)|d\tau\right\} \\
\leq&\ A_1\Big(1+\Phi_k\,\rho_1\,\sigma_1\Big)\int_t^\infty e^{-\beta(t-\tau)}|p(\lambda_1,\tau)|d\tau.
\end{aligned}
$$

Then, by the induction process, (4.17) is satisfied with $\Phi_n = A_1(1+\Phi_{n-1}\,\rho_1\,\sigma_1)$.

The proof of (4.18) is given as follows. Using recursively the definition of $\Phi_{n-2},\ldots,\Phi_2$, we can rewrite Φ_n as the sum of the terms of a geometric progression where the common ratio is given by $\rho_1 A_1\sigma_1$. Then, the hypothesis (H$_3$) implies the existence of Φ satisfying (4.18), since by the construction of ρ_1 we have that $\rho_1 A_1\sigma_1 \in]0,1[$ for $\eta\in\,]0,1/2[$. More precisely, we deduce that

$$
\lim_{n\to\infty}\Phi_n = A_1\lim_{n\to\infty}\sum_{i=0}^{n-1}\Big(\rho_1 A_1\sigma_1\Big)^i = A_1\lim_{n\to\infty}\frac{[(\rho_1 A_1\sigma_1)^n-1]}{\rho_1 A_1\sigma_1-1} = \frac{A_1}{1-\rho_1 A_1\sigma_1} = \Phi > 0.
$$

Hence, (4.17)-(4.18) are valid and the proof of (3.6) with $i=1$ is concluded by passing to the limit the sequence $\{\Phi_n\}$ when $n\to\infty$ in the topology of $C_0^2([t_0,\infty])$.

4.3.2 Proof of (3.6) with $i = 2$

Let us denote by z_2 the solution of the equation (3.4) associated with the characteristic root λ_2 of (1.2). Similarly to the case $i=1$ we define the sequence $\omega_{n+1} = T\omega_n$ with $\omega_0 = 0$ and, by the contraction property of T, we can deduce that $\omega_n \to z_2$ when $n\to\infty$. In this case, we note that $\Omega(s) = p(\lambda_2,s)$. Moreover, by Proposition (3.1) we have that $\gamma_1 = \lambda_1-\lambda_2 > 0 > \gamma_2 = \lambda_3-\lambda_2 > \gamma_3 = \lambda_4-\lambda_2$. Then the Green function g defined on (4.2) is given by $(\Upsilon_2)^{-1}g_2$.

Thereby, the operator T can be rewritten equivalently as follows

$$\begin{aligned} Tz(t) &= \frac{1}{\Upsilon_2}\int_{t_0}^{\infty} g_2(t,s)\Big[p(\lambda_2,s)+F\Big(s,z(s),z'(s),z''(s)\Big)\Big]ds \\ &= \frac{1}{\Upsilon_2}\Bigg\{\int_{t_0}^{t}(\lambda_3-\lambda_4)e^{-(\lambda_1-\lambda_2)(t-s)}\Big[p(\lambda_2,s)+F\Big(s,z(s),z'(s),z''(s)\Big)\Big]ds \\ &\qquad + \int_t^{\infty}\Big[(\lambda_1-\lambda_3)e^{-(\lambda_4-\lambda_2)(t-s)}+(\lambda_4-\lambda_1)e^{-(\lambda_3-\lambda_2)(t-s)}\Big] \\ &\qquad\qquad \times\Big[p(\lambda_2,s)+F\Big(s,z(s),z'(s),z''(s)\Big)\Big]ds\Bigg\}. \end{aligned} \tag{4.21}$$

Then, the proof of (3.6) with $i=2$ is reduced to prove

$$\exists\,\Phi_n\in\mathbb{R}_+ \;:\; |\omega_n(t)|+|\omega_n'(t)|+|\omega_n''(t)|\le\Phi_n\int_{t_0}^{\infty}e^{-\beta(t-\tau)}|p(\lambda_2,\tau)|d\tau,\;\;\forall\,t\ge t_0, \tag{4.22}$$

$$\exists\,\Phi\in\mathbb{R}_+ \;:\; \Phi_n\to\Phi \text{ when } n\to\infty. \tag{4.23}$$

In the induction step for $n=1$ the estimate (4.22) is satisfied with $\Phi_1=A_2$, since by the definition of the operator T given on (4.21), the property $F(s,0,0,0)=0$, the estimates of type (4.19) for g_2 and the fact that $\beta\in[\lambda_3-\lambda_2,0[\,\subset[\lambda_3-\lambda_2,\lambda_1-\lambda_2]$, we deduce the following bound

$$\begin{aligned} &|\omega_1(t)|+|\omega_1'(t)|+|\omega_1''(t)| \\ &\quad= |T\omega_0(t)|+|T'\omega_0(t)|+|T''\omega_0(t)| \\ &\quad\le\frac{1}{|\Upsilon_2|}\Bigg\{|\lambda_3-\lambda_4|\Big(1+|\lambda_1-\lambda_2|+|\lambda_1-\lambda_2|^2\Big)\int_{t_0}^{t}e^{-(\lambda_1-\lambda_2)(t-s)}|p(\lambda_2,s)|\,ds \\ &\qquad + \Big[|\lambda_3-\lambda_1|\Big(1+|\lambda_4-\lambda_2|+|\lambda_4-\lambda_2|^2\Big)+|\lambda_4-\lambda_1|\Big(1+|\lambda_3-\lambda_2|+|\lambda_3-\lambda_2|^2\Big)\Big] \\ &\qquad \times\int_t^{\infty}e^{-\beta(t-s)}|p(\lambda_2,\tau)|\,ds\Bigg\} \\ &\quad\le A_2\left\{\int_{t_0}^{t}e^{-\beta(t-s)}|p(\lambda_2,\tau)|\,ds+\int_t^{\infty}e^{-\beta(t-s)}|p(\lambda_2,\tau)|\,ds\right\} \\ &\quad= A_2\int_{t_0}^{\infty}e^{-\beta(t-\tau)}|p(\lambda_2,\tau)|d\tau. \end{aligned}$$

Noticing that similar inequalities to (4.20), with λ_2 instead of λ_1 and integration on $[t_0,\infty[$ instead of $[t,\infty[$, we deduce that

$$\begin{aligned} J_1(t) &:= \int_{t_0}^{t}e^{-(\lambda_1-\lambda_2)(t-s)}|p(\lambda_2,s)|\,ds+\int_t^{\infty}e^{-(\lambda_3-\lambda_2)(t-s)}|p(\lambda_2,s)|\,ds \\ &\le\int_{t_0}^{t}e^{-\beta(t-s)}|p(\lambda_2,\tau)|\,ds+\int_t^{\infty}e^{-\beta(t-s)}|p(\lambda_2,\tau)|\,ds=\int_{t_0}^{\infty}e^{-\beta(t-\tau)}|p(\lambda_2,\tau)|\,d\tau, \\ J_2(t) &:= \int_{t_0}^{t}e^{-(\lambda_3-\lambda_2)(t-s)}\Big[\|(b,f,h)(s)\|_1+\Big(\|(q,f,h)(s)\|_1+\|\mathbf{C}\|_1\Big)\eta\Big]\Big\|\Big(\omega_k,\omega_k',\omega_k''\Big)(s)\Big\|_1\,ds \\ &\quad + \int_t^{\infty}e^{-(\lambda_1-\lambda_2)(t-s)}\Big[\|(b,f,h)(s)\|_1+\Big(\|(q,f,h)(s)\|_1+\|\mathbf{C}\|_1\Big)\eta\Big]\Big\|\Big(\omega_k,\omega_k',\omega_k''\Big)(s)\Big\|_1\,ds \end{aligned}$$

$$
\begin{aligned}
&\leq \int_{t_0}^{t} e^{-\beta(t-s)}\Big[\|(b,f,h)(s)\|_1 + \Big(\|(q,f,h)(s)\|_1 + \|\mathbf{C}\|_1\Big)\eta\Big]\Big\|\Big(\omega_k,\omega_k',\omega_k''\Big)(s)\Big\|_1 ds\\
&\quad + \int_{t}^{\infty} e^{-\beta(t-s)}\Big[\|(b,f,h)(s)\|_1 + \Big(\|(q,f,h)(s)\|_1 + \|\mathbf{C}\|_1\Big)\eta\Big]\Big\|\Big(\omega_k,\omega_k',\omega_k''\Big)(s)\Big\|_1 ds\\
&= \int_{t_0}^{\infty} e^{-\beta(t-s)}\Big[\|(b,f,h)(s)\|_1 + \Big(\|(q,f,h)(s)\|_1 + \|\mathbf{C}\|_1\Big)\eta\Big]\Big\|\Big(\omega_k,\omega_k',\omega_k''\Big)(s)\Big\|_1 ds\\
&\leq \Phi_k\int_{t_0}^{\infty} e^{-\beta(t-s)}\Big[\|(b,f,h)(s)\|_1 + \Big(\|(q,f,h)(s)\|_1 + \|\mathbf{C}\|_1\Big)\eta\Big]\int_{t_0}^{\infty} e^{-\beta(s-\tau)}|p(\lambda_2,\tau)|\,d\tau\,ds\\
&= \Phi_k\int_{t_0}^{\infty} e^{-\beta(t-s)}\Big[\|(b,f,h)(s)\|_1 + \Big(\|(q,f,h)(s)\|_1 + \|\mathbf{C}\|_1\Big)\eta\Big]\int_{t_0}^{\infty} e^{-\beta(t-\tau)}|p(\lambda_2,\tau)|\,d\tau\,ds\\
&\leq \Phi_k\,\rho_2\,\sigma_2\int_{t_0}^{\infty} e^{-\beta(t-\tau)}|p(\lambda_2,\tau)|\,d\tau.
\end{aligned}
$$

Then, the general induction step can be proved as follows

$$
\begin{aligned}
&|\omega_{k+1}(t)| + |\omega_{k+1}'(t)| + |\omega_{k+1}''(t)|\\
&\quad = |T\omega_k(t)| + |T'\omega_k(t)| + |T''\omega_k(t)|\\
&\quad \leq A_2\int_{t_0}^{t} e^{-(\lambda_1-\lambda_2)(t-s)}\Big|p(\lambda_2,s) + F\Big(s,z(s),z'(s),z''(s)\Big)\Big|\,ds\\
&\qquad + A_2\int_{t}^{\infty} e^{-(\lambda_3-\lambda_2)(t-s)}\Big|p(\lambda_2,s) + F\Big(s,z(s),z'(s),z''(s)\Big)\Big|\,ds\\
&\quad \leq A_2\Bigg[\int_{t_0}^{t} e^{-(\lambda_1-\lambda_2)(t-s)}\Big\{|p(\lambda_2,s)| + \Big[\|(b,f,h)(s)\|_1 + \Big(\|(q,f,h)(s)\|_1 + \|\mathbf{C}\|_1\Big)\eta\Big]\\
&\qquad\qquad \times\Big\|\Big(\omega_k,\omega_k^{(1)},\omega_k^{(2)}\Big)(s)\Big\|_1\Big\}\,ds\\
&\qquad + \int_{t}^{\infty} e^{-(\lambda_3-\lambda_2)(t-s)}\Big\{|p(\lambda_2,s)| + \Big[\|(b,f,h)(s)\|_1 + \Big(\|(q,f,h)(s)\|_1 + \|\mathbf{C}\|_1\Big)\eta\Big]\\
&\qquad\qquad \times\Big\|\Big(\omega_k,\omega_k',\omega_k''\Big)(s)\Big\|_1\Big\}\,ds\Bigg]\\
&\quad = A_2\Big[J_1(t) + J_2(t)\Big]\\
&\quad \leq A_2\Big(1 + \Phi_k\,\rho_2\,\sigma_2\Big)\int_{t_0}^{\infty} e^{-\beta(t-\tau)}|p(\lambda_2,\tau)|\,d\tau.
\end{aligned}
$$

Hence the thesis of the inductive steps holds with $\Phi_n = A_2(1+\Phi_{n-1}\rho_2\,\sigma_2)$.

We proceed in an analogous way to the case $i=1$ and deduce that (4.23) is satisfied with $\Phi = A_2/(1-\rho_2\,\sigma_2 A_2) > 0$.

Therefore, the sequence $\{\Phi_n\}$ is convergent and z_2 (the limit of ω_n in the topology of $C_0^2([t_0,\infty]))$ satisfies (3.6).

4.3.3 Proof of (3.6) with $i = 3$ and $i = 4$

The proof of the cases $i=3$ and $i=4$ is completely analogous to cases $i=2$ and $i=1$, respectively.

4.4 Proof of Theorem 3.6

By Lemma 3.2, we have that the fundamental system of solutions for (1.1) is given by (3.4). Moreover, by (3.4) we deduce the identities

$$\frac{y_i'(t)}{y_i(t)} = [\lambda_i + z_i(t)] \tag{4.24}$$

$$\frac{y_i''(t)}{y_i(t)} = [\lambda_i + z_i(t)]^2 + z_i'(t), \tag{4.25}$$

$$\frac{y_i'''(t)}{y_i(t)} = [\lambda_i + z_i(t)]^3 + 3[\lambda_i + z_i(t)]z_i'(t) + z_i''(t), \tag{4.26}$$

$$\frac{y_i^{(\mathrm{iv})}(t)}{y_i(t)} = [\lambda_i + z_i(t)]^4 + 6[\lambda_i + z_i(t)]^2 z_i'(t) + 3[z_i'(t)]^2 + 4[\lambda_i + z_i(t)]z_i''(t) + z_i''(t), \tag{4.27}$$

Now, using the fact that $z_i \in C_0^2([t_0, \infty[)$ is a solution of (3.4) with $\mu = \lambda_i$, we deduce the proof of (3.7). Now, by the definition of the $W[y_1, \ldots, y_4]$, some algebraic rearrangements and (3.7), we deduce (3.8).

The proof of (3.9) follows from the identity

$$\begin{aligned}\int_{t_0}^{t} e^{-a\tau} \int_{\tau}^{\infty} e^{as} H(s)\, ds\, d\tau \\ = -\frac{1}{a}\left[\int_{t}^{\infty} e^{-a(t-s)} H(s)\, ds - \int_{t_0}^{\infty} e^{-a(t_0-s)} H(s)\, ds\right] + \frac{1}{a}\int_{t_0}^{t} H(\tau)\, d\tau\end{aligned} \tag{4.28}$$

and from (4.8)–(4.9). Now, we develop the proof for $i = 1$. Indeed, by (3.4) we have that

$$y_1(t) = \exp\Big(\int_{t_0}^{t} (\lambda_1 + z_1(\tau))\, d\tau\Big) = e^{\lambda_1(t-t_0)} \exp\Big(\int_{t_0}^{t} z_1(\tau)\, d\tau\Big). \tag{4.29}$$

By (4.8)–(4.9), (4.15), (4.16), (4.28), and the fact that $\pi_1 = (\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)(\lambda_4 - \lambda_1)$, we have that

$$\begin{aligned}
\int_{t_0}^{t} z_1(\tau)d\tau &= \frac{1}{\Upsilon_1}\int_{t_0}^{t}\int_{t_0}^{\infty} g_1(\tau, s)\Big(p(\lambda_1, s) + F(s, z_1(s), z_1'(s), z_1''(s)\Big)d\tau\, ds \\
&= \frac{1}{\Upsilon_1}\int_{t_0}^{t}\int_{\tau}^{\infty}\Big[(\lambda_4 - \lambda_3)e^{-(\lambda_2-\lambda_1)(\tau-s)} + (\lambda_2 - \lambda_4)e^{-(\lambda_3-\lambda_1)(\tau-s)} \\
&\qquad + (\lambda_3 - \lambda_2)e^{-(\lambda_4-\lambda_1)(\tau-s)}\Big]\Big(p(\lambda_1, s) + F(s, z_1(s), z_1'(s), z_1''(s))\Big)ds\, d\tau \\
&= \frac{1}{\Upsilon_1}\left[\frac{\lambda_4 - \lambda_3}{\lambda_2 - \lambda_1} + \frac{\lambda_2 - \lambda_4}{\lambda_3 - \lambda_1} + \frac{\lambda_3 - \lambda_2}{\lambda_4 - \lambda_1}\right]\int_{t_0}^{t}\Big(p(\lambda_1, s) + F(s, z_1(s), z_1'(s), z_1''(s))\Big)ds \\
&\quad + \frac{1}{\Upsilon_1}\left[\frac{\lambda_4 - \lambda_2}{\lambda_2 - \lambda_1}\left\{\int_{t}^{\infty} e^{-(\lambda_2-\lambda_1)(t-s)}\Big(p(\lambda_1, s) + F(s, z_1(s), z_1'(s), z_1''(s))\Big)ds \right.\right. \\
&\qquad \left.\left. - \int_{t_0}^{\infty} e^{-(\lambda_2-\lambda_1)(t_0-s)}\Big(p(\lambda_1, s) + F(s, z_1(s), z_1'(s), z_1''(s))\Big)ds\right\}\right] \\
&\quad + \frac{1}{\Upsilon_1}\left[\frac{\lambda_2 - \lambda_4}{\lambda_3 - \lambda_1}\left\{\int_{t}^{\infty} e^{-(\lambda_3-\lambda_1)(t-s)}\Big(p(\lambda_1, s) + F(s, z_1(s), z_1'(s), z_1''(s))\Big)ds \right.\right. \\
&\qquad \left.\left. - \int_{t_0}^{\infty} e^{-(\lambda_3-\lambda_1)(t_0-s)}\Big(p(\lambda_1, s) + F(s, z_1(s), z_1'(s), z_1''(s))\Big)ds\right\}\right]
\end{aligned}$$

$$+\frac{1}{\Upsilon_1}\left[\frac{\lambda_3-\lambda_2}{\lambda_4-\lambda_1}\left\{\int_t^\infty e^{-(\lambda_4-\lambda_1)(t-s)}\Big(p(\lambda_1,s)+F(s,z_1(s),z_1'(s),z_1''(s))\Big)ds\right.\right.$$
$$\left.\left.-\int_{t_0}^\infty e^{-(\lambda_4-\lambda_1)(t_0-s)}\Big(p(\lambda_1,s)+F(s,z_1(s),z_1'(s),z_1''(s))\Big)ds\right\}\right]$$
$$=\frac{1}{\pi_1}\int_{t_0}^t\Big(p(\lambda_1,s)+F(s,z_1(s),z_1'(s),z_1''(s))\Big)ds+o(1).$$

Then, (3.9) is valid for $i=1$. The proof of (3.9) for $i=2,3,4$ is analogous. Now the proof of (3.10) follows by (3.9) and (4.24)–(4.27).

5 Example

In this section we consider an example where some classical results cannot be applied. However, we can apply the Theorem 3.6.

Consider the differential equation

$$y^{(\mathrm{iv})}-5y''+[\sin(t^q)+4]y=0,\quad \text{with } q\in\,]2,\infty[, \tag{5.1}$$

which is of type (1.1) with $(a_0,a_1,a_2,a_3)=(4,0,-5,0)$ and $(r_0,r_1,r_2,r_3)(t)=(\sin(t^q),0,0,0)$. Introducing the change of variable

$$\begin{pmatrix} y(t)\\ y'(t)\\ y''(t)\\ y'''(t)\end{pmatrix}=\begin{pmatrix}1&1&1&1\\2&1&-1&-2\\4&1&-1&4\\8&1&-1&-8\end{pmatrix}\begin{pmatrix}x_1(t)\\x_2(t)\\x_3(t)\\x_4(t)\end{pmatrix},$$

we note that (5.1) is equivalent to the following system

$$\begin{pmatrix}x_1(t)\\x_2(t)\\x_3(t)\\x_4(t)\end{pmatrix}'=\left\{\begin{pmatrix}2&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-2\end{pmatrix}+\frac{\sin(t^q)}{12}\begin{pmatrix}-1&-1&-1&-1\\2&2&2&2\\-2&-2&-2&-2\\1&1&1&1\end{pmatrix}\right\}\begin{pmatrix}x_1(t)\\x_2(t)\\x_3(t)\\x_4(t)\end{pmatrix}.$$

We note that

$$\sin(t^q)\not\to 0\quad\text{when } t\to\infty,\qquad \sin(t^q)\notin L^p([t_0,\infty[)\quad\text{for any } p\geq 1.$$

Then, the classical generalizations of Poincaré type theorems [31], the Levinson theorem [10, Theorem 1.3.1], the Hartman–Wintner [10, Theorem 1.5.1] or the Eastham theorem [10, Theorem 1.6.1] can not be applied to obtain the asymptotic behavior of (5.1).

Now, in order to apply the Theorem 3.6, we have the following.

(a) The set of characteristic roots of the linear no perturbed equation associated to (5.1), i.e. $y^{(\mathrm{iv})}-5y''+4y=0$, is given by $\{2,1,-1,-2\}$. Then (H_1) is satisfied.

(b) By integration by parts we note that

$$\int_{t_0}^\infty r_0(t)\,dt=\int_{t_0}^\infty\sin(t^q)\,dt=-\frac{\cos(t_0^{q-1})}{q\,t_0^{q-1}}+\frac{1-q}{q}\int_{t_0}^\infty\frac{\cos(t^q)}{t^q}\,dt.$$

Then $|\int_{t_0}^{\infty}\sin(t^q)\,dt|$ is bounded and r_0 is conditionally integrable. Now, we prove that

$$\lim_{t\to\infty}\left|\int_t^{\infty} e^{c(t-s)}\sin(s^q)\,ds\right| = 0, \qquad c>0,\ q>2, \tag{5.2}$$

$$\lim_{t\to\infty}\left|\int_{t_0}^{t} e^{c(s-t)}\sin(s^q)\,ds\right| = 0, \qquad c>0,\ q>2, \tag{5.3}$$

$$\lim_{t\to\infty}\int_t^{\infty} e^{c(t-s)}|\sin(s^q)|\,ds = 0, \qquad c>0,\ q>2, \tag{5.4}$$

$$\lim_{t\to\infty}\int_{t_0}^{t} e^{c(s-t)}|\sin(s^q)|\,ds = 0, \qquad c>0,\ q>2. \tag{5.5}$$

Indeed, by integration by parts, we deduce that

$$\begin{aligned}
0 &\le \left|\int_t^{\infty} e^{c(t-s)}\sin(s^q)\,ds\right| \\
&= \left|\frac{\cos(t^q)}{qt^{q-1}} - \frac{1}{q}\left[c\int_t^{\infty}\frac{e^{c(t-s)}\cos(s^q)}{s^{q-1}}\,dt + (q-1)\int_t^{\infty}\frac{e^{c(t-s)}\cos(s^q)}{s^q}\,dt\right]\right| \\
&\le \left|\frac{\cos(t^q)}{qt^{q-1}}\right| + \frac{c}{q}\int_t^{\infty}\left|\frac{e^{c(t-s)}\cos(s^q)}{s^{q-1}}\right|dt + \frac{q-1}{q}\int_t^{\infty}\left|\frac{e^{c(t-s)}\cos(s^q)}{s^q}\right|dt \\
&\le \frac{1}{qt^{q-1}} + \frac{c}{q(q-2)t^{q-2}} + \frac{1}{qt^{q-1}},
\end{aligned}$$

which implies (5.2). Similarly, we get the following bound

$$\begin{aligned}
0 &\le \left|\int_{t_0}^{t} e^{c(t-s)}\sin(s^q)\,ds\right| \\
&\le \frac{1}{qt^{q-1}} + \frac{1}{t_0e^{c(t-t_0)}} + \frac{\int_{t_0}^{t} e^{cs}s^{1-q}\,ds}{qe^{ct}} + \frac{(q-1)\int_{t_0}^{t} e^{cs}s^{-q}\,ds}{qe^{ct}},
\end{aligned}$$

which implies (5.3). To prove (5.4) and (5.5), we can apply similar arguments or more directly the change of variable $m = s^q$.

We note that $p(\lambda_i, s) = r_0(s)$. Then, by application of (4.2), (5.2) and (5.3) we deduce that $\mathcal{G}(p(\lambda_i,\cdot))(t) \to 0$ when $t\to\infty$. Moreover, $\mathcal{L}(r_0)(t) \to 0$ when $t\to\infty$ by application of (4.2), (5.4) and (5.5) and $\mathcal{L}(r_1)(t) = \mathcal{L}(r_2)(t) = \mathcal{L}(r_3)(t) = 0$ since $r_1 = r_2 = r_3 = 0$. Thus, (H_2) is also satisfied.

(c) We note that

$$\begin{gathered}
\mathbb{F}_1(1)(t) = 1,\quad \mathbb{F}_2(1)(t) = 1 - e^{-t_0} + \frac{1}{2},\quad \mathbb{F}_3(1)(t) = \frac{1-e^{-2t_0}}{2} + 1,\quad \mathbb{F}_4(1)(t) = 1, \\
\sigma_1 = 19 + 76\eta,\quad \sigma_2 = 11 + 39\eta,\quad \sigma_3 = 11 + 39\eta,\quad \sigma_4 = 19 + 76\eta, \\
A_1 = \frac{32}{3},\quad A_2 = \frac{35}{6},\quad A_3 = \frac{5}{2},\quad A_4 = \frac{32}{3}.
\end{gathered}$$

Then, the sets $\mathcal{F}_i([t_0,\infty[)$ given in (2.4) are well defined. We note that, naturally, $r_1 = r_2 = r_3 = 0 \in \mathcal{F}_i$. Moreover, from (4.2), (5.4) and (5.5) we can prove that $\mathbb{F}_i(r_0)(t) \to 0$ when $t\to\infty$. Then, we have that $r_0 \in \mathcal{F}_i([t_0,\infty[)$. Hence, (H_3) is satisfied.

Thus, from (a)–(c), we can apply the Theorem 3.6 and the asymptotic formulas are given by

$$\begin{aligned} y_1(t) &= e^{2t}\exp\Big(\frac{1}{12}\int_{t_0}^{t}\Big\{\sin(s^q)-f_1(s)\Big\}\,ds\Big),\\ y_2(t) &= e^{t}\exp\Big(-\frac{1}{6}\int_{t_0}^{t}\Big\{\sin(s^q)-f_2(s)\Big\}\,ds\Big),\\ y_3(t) &= e^{-t}\exp\Big(\frac{1}{6}\int_{t_0}^{t}\Big\{\sin(s^q)-f_3(s)\Big\}\,ds\Big),\\ y_4(t) &= e^{-2t}\exp\Big(-\frac{1}{12}\int_{t_0}^{t}\Big\{\sin(s^q)-f_4(s)\Big\}\,ds\Big), \end{aligned}$$

where

$$\begin{aligned} f_1(t) &= 3(z_1'(s))^2+24z_1''(s)+4z_1(s)z_1''(s)+6[z_1(s)]^2z_1'(s)+8[z_1(s)]^3+[z_1(s)]^4\\ f_2(t) &= 3(z_2'(s))^2+12z_2^2(s)+4z_2(s)z_2''(s)+12[z_2(s)]^2z_2'(s)+4[z_2(s)]^3+[z_2(s)]^4\\ f_3(t) &= 3(z_3'(s))^2-6z_3^2(s)+4z_3(s)z_3''(s)+6[z_3(s)]^2z_3'(s)-4[z_3(s)]^3+[z_3(s)]^4\\ f_4(t) &= 3(z_4'(s))^2-12z_4^2(s)+4z_4(s)z_4''(s)+6[z_4(s)]^2z_4'(s)-8[z_4(s)]^3+[z_4(s)]^4 \end{aligned}$$

and $z_i(t)$ satisfies the following asymptotic behavior

$$z_i(t),z_i'(t),z_i''(t)=\begin{cases} O\left(\displaystyle\int_{t}^{\infty}e^{-\beta(t-s)}|\sin(s^p)|\,ds\right), & i=1, \quad \beta\in[-1,0[,\\ O\left(\displaystyle\int_{t_0}^{\infty}e^{-\beta(t-s)}|\sin(s^p)|\,ds\right), & i=2, \quad \beta\in[-2,0[,\\ O\left(\displaystyle\int_{t_0}^{\infty}e^{-\beta(t-s)}|\sin(s^p)|\,ds\right), & i=3, \quad \beta\in[-1,0[,\\ O\left(\displaystyle\int_{t_0}^{t}e^{-\beta(t-s)}|\sin(s^p)|\,ds\right), & i=4, \quad \beta\in]0,1]. \end{cases}$$

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References

[1] A. R. Aftabizadeh, Existence and uniqueness theorems for fourth-order boundary value problems, *J. Math. Anal. Appl.* **116**(1986), No. 2, 415–426. MR842808; url

[2] R. Bellman, *A survey of the theory of the boundedness, stability, and asymptotic behavior of solutions of linear and non-linear differential and difference equations*, Office of Naval Research, Washington, D. C., 1949. MR0030662

[3] R. Bellman, On the asymptotic behavior of solutions of $u''-(1+f(t))u=0$, *Ann. Mat. Pura Appl. (4)* **31**(1950), 83–91. MR0042579

[4] R. Bellman, *Stability theory of differential equations*, McGraw-Hill Book Company, Inc., New York–Toronto–London, 1953. MR0061235

[5] E. A. Coddington, N. Levinson, *Theory of ordinary differential equations*, McGraw-Hill Book Company, Inc., New York–Toronto–London, 1955. MR0069338

[6] W. A. Coppel, *Stability and asymptotic behavior of differential equations*, D. C. Heath and Co., Boston, Mass., 1965. MR0190463

[7] A. R. Davies, A. Karageorghis, T. N. Phillips, Spectral Galerkin methods for the primary two-point boundary value problem in modelling viscoelastic flows, *Int. J. Numer. Methods Engng.* **26**(1988), 647–662

[8] T. A. Dzhangveladze, G. B. Lobzhanidze, On a nonlocal boundary value problem for a fourth-order ordinary differential equation (in Russian), *Differ. Uravn.* **47**(2011), No. 2, 181–188; translation in *Differ. Equ.* **47**(2011), No. 2, 179–186 MR2895673; url

[9] M. S. P. Eastham, The asymptotic solution of higher-order differential equations with small final coefficient, *Portugal. Math.*, **45**(1988), No. 4, 351–362. MR982904

[10] M. S. P. Eastham, *The asymptotic solution of linear differential systems. Applications of the Levinson theorem*, London Mathematical Society Monographs, Vol. 4, Oxford University Press, New York, 1989. MR1006434

[11] U. Elias, H. Gingold, A framework for asymptotic integration of differential systems, *Asymptot. Anal.* **35**(2003), No. 3–4, 281–300. MR2011791

[12] M. V. Fedoryuk, *Asymptotic analysis. Linear ordinary differential equations* (Translated from the Russian by Andrew Rodick), Springer-Verlag, Berlin, 1993. MR1295032; url

[13] P. Figueroa, M. Pinto, Asymptotic expansion of the variable eigenvalue associated to second order differential equations, *Nonlinear Stud.* **13**(2006), No. 3, 261–272. MR2261608

[14] P. Figueroa, M. Pinto, Riccati equations and nonoscillatory solutions of third order differential equations, *Dynam. Systems Appl.* **17**(2008), No. 3–4, 459–475. MR2569513

[15] P. Figueroa, M. Pinto, L^p-solutions of Riccati-type differential equations and asymptotics of third order linear differential equations, *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.* **17**(2010), No. 4, 555–571. MR2682803

[16] F. Gazzola, H. C. Grunau,Radial entire solutions for supercritical biharmonic equations, *Math. Ann.* **334**(2006), No. 4, 905–936. MR2209261; url

[17] W. A. Harris Jr., D. A. Lutz, On the asymptotic integration of linear differential systems, *J. Math. Anal. Appl.* **48**(1974), 1–16. MR0355222

[18] W. A. Harris Jr., D. A. Lutz, A unified theory of asymptotic integration, *J. Math. Anal. Appl.* **57**(1977), No. 3, 571–586. MR0430436

[19] P. Hartman, Unrestricted solution fields of almost-separable differential equations, *Trans. Amer. Math. Soc.* **63**(1948), 560–580. MR0025052

[20] P. Hartman, A. Wintner, Asymptotic integrations of linear differential equations, *Amer. J. Math.* **77**(1955), 45–86. MR0066520

[21] P. Karageorgis, Asymptotic expansion of radial solutions for supercritical biharmonic equations, *NoDEA Nonlinear Differential Equations Appl.* **19**(2012), No. 4, 401–415. MR2949625; url

[22] A. Lamnii, O. El-khayyari, J. Dabounou, Solving linear fourth order boundary value problem by using a hyperbolic splines of order 4, *Int. Electron. J. Pure Appl. Math.* **7**(2014), No. 2, 85–98. MR3171533; url

[23] N. Levinson, The asymptotic nature of solutions of linear systems of differential equations, *Duke Math. J.* **15**(1948), No. 1, 111–126. MR0024538

[24] F. W. J. Olver, *Asymptotics and special functions* (Reprint of the 1974 original Academic Press, New York) AKP Classics, A. K. Peters, Ltd., Wellesley, MA, 1997. MR0435697

[25] O. Perron, Über einen Satz des Herrn Poincaré (in German), *J. Reine Angew. Math.*, **136**(1909), 17–38. MR1580774; url

[26] G. W. Pfeiffer *Asymptotic solutions of the equation* $y''' + qy' + ry = 0$, ProQuest LLC, Ann Arbor, MI, PhD thesis, University of Georgia, 1970.

[27] G. W. Pfeiffer, Asymptotic solutions of $y''' + qy' + ry = 0$, *J. Differential Equations* **11**(1972), 145–155. MR0296438

[28] B. Pietruczuk, Resonance phenomenon for potentials of Wigner–von Neumann type, in: *Geometric Methods in Physics* (P. Kielanowski, S. T. Ali, A. Odesskii, A. Odzijewicz, M. Schlichenmaier, T. Voronov (Eds.), Trends in Mathematics, Springer Basel, pp. 203–207, 2013. url

[29] M. Pinto, Null solutions of difference systems under vanishing perturbation, *J. Difference Equ. Appl.* **9**(2003), No. 1, 1–13. MR1958299; url

[30] H. Poincaré, Sur les équations linéaires aux différentielles ordinaires et aux différences finies (in French), *Amer. J. Math.* **7**(1885), No. 3, 203–258. MR1505385; url

[31] J. Šimša, An extension of a theorem of Perron, *SIAM J. Math. Anal.* **19**(1988), No. 2, 460–472. MR930038; url

[32] S. A. Stepin, The WKB method and dichotomy for ordinary differential equations (in Russian), *Dokl. Akad. Nauk* **404**(2005), 749–752. MR2258499

[33] S. A. Stepin, Asymptotic integration of nonoscillatory second-order differential equations (in Russian), *Dokl. Akad. Nauk* **434**(2010), No. 3, 315–318; translation in *Dokl. Math.* **82**(2010), No. 2, 751–754. MR2757854; url

[34] S. P. Timoshenko, *Theory of elastic stability*, McGraw-Hill Book, New York, NY, USA, 2nd edition, 1961. MR0134026

[35] Y.-X. Yao, Y.-T. Shen, Z.-H. Chen, Biharmonic equation and an improved Hardy inequality, *Acta Math. Appl. Sin. Engl. Ser.* **20**(2004), No. 3, 433–440. MR2086765; url