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On sequences of large solutions for discrete anisotropic equations

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Abstract. In this paper, we determine a concrete interval of positive parameters λ, for which we prove the existence of infinitely many solutions for an anisotropic discrete Dirichlet problem

$$-\Delta\left(\alpha\left(k\right)\left|\Delta u(k-1)\right|^{p(k-1)-2}\Delta u(k-1)\right)=\lambda f(k,u(k)),\qquad k\in\mathbb{Z}[1,T],$$

where the nonlinear term $f\colon \mathbb{Z}[1,T]\times\mathbb{R}\to\mathbb{R}$ has an appropriate behavior at infinity, without any symmetry assumptions. The approach is based on critical point theory.

Keywords: discrete nonlinear boundary value problem, variational methods, anisotropic problem, infinitely many solutions.

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1 Introduction

Difference equations serve as mathematical models in diverse areas, such as economy, biology, physics, mechanics, computer science, finance – see for example [1, 9, 20]. Some of these models are of independent interest since their mathematical structure allows for obtaining new abstract tools. One of the models arising in the study of elastic mechanics is the $p(x)$-Laplacian. We consider the problem

$$\begin{cases}-\Delta\left(\alpha\left(k\right)\left|\Delta u(k-1)\right|^{p(k-1)-2}\Delta u(k-1)\right)=\lambda\ f(k,u(k)), & k\in\mathbb{Z}[1,T],\\ u(0)=u(T+1)=0,\end{cases}\tag{P_λ^f}$$

where λ is a positive parameter, $T\geq 2$ is an integer; $\mathbb{Z}[1,T]$ is a discrete interval $\{1,2,\dots,T\}$; $\Delta u(k-1)=u(k)-u(k-1)$ is the forward difference operator; $u(k)\in\mathbb{R}$ for all $k\in\mathbb{Z}[1,T]$; $\alpha\colon\mathbb{Z}[1,T+1]\to(0,+\infty)$ and $p\colon\mathbb{Z}[0,T]\to(1,+\infty)$ are some fixed functions; $f\colon\mathbb{Z}[1,T]\times\mathbb{R}\to\mathbb{R}$ is a continuous function, i.e. for any fixed $k\in\mathbb{Z}[1,T]$ a function $f(k,\cdot)$ is continuous. Let $p^-=\min_{k\in\mathbb{Z}[0,T]}p(k)$, $p^+=\max_{k\in\mathbb{Z}[0,T]}p(k)$, $\alpha^-=\min_{k\in\mathbb{Z}[1,T+1]}\alpha(k)$, $\alpha^+=\max_{k\in\mathbb{Z}[1,T+1]}\alpha(k)$.

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Several authors have investigated discrete BVPs with Dirichlet, periodic and Neumann boundary conditions by the critical point theory. They applied classical variational tools such as direct methods, the mountain geometry, linking arguments, the degree theory. We refer to the following works far from being exhaustive: [3,4,14,16,21,25,26]. Inspiration to our investigations in this note lies in [22], where a concrete interval of positive parameters for which the anisotropic problem (P_λ^f) admits infinitely many nonzero solutions which converges to zero is obtained. The main purpose of this paper is to investigate the existence of an unbounded sequence of solutions for problem (P_λ^f), by using the critical point theorem obtained in [23]. Our idea here is to transfer the problem of existence of solutions for problem (P_λ^f) into the problem of existence of critical points for a suitable associated energy functional. For the case of constant exponents see [5,7]. For some other approach towards discrete anisotropic problems we refer to [11–13]

Continuous versions of problems like (P_λ^f)are known to be mathematical models of various phenomena arising in the study of elastic mechanics, see [27], electrorheological fluids, see [24], or image restoration, see [8]. Variational continuous anisotropic problems were started by Fan and Zhang in [10] and later considered by many authors and the use of many methods, see [15] for an extensive survey of such boundary value problems. Finally, we cite the recent monograph by Kristály, Rădulescu and Varga [19] as general reference on variational methods adopted here.

We note that most multiplicity results for discrete problems assume that the nonlinearities are odd functions. Only a few papers deal with nonlinearities for which this property does not hold; see, for instance, the papers [17] and [18].

In our approach we do not require any symmetry hypothesis. A special case of our contributions reads as follows.

Theorem 1.1. *Let $g\colon \mathbb{R} \to \mathbb{R}$ be a nonnegative and continuous function. Assume that*

$$\liminf_{t\to+\infty} \frac{\int_0^t g(s)\,ds}{t^{p^-}} = 0 \quad \textit{and} \quad \limsup_{t\to+\infty} \frac{\int_0^t g(s)\,ds}{t^{p^+}} = +\infty.$$

Then for each $\lambda > 0$, the problem

$$\begin{cases} -\Delta\Big(\alpha(k)\,|\Delta u(k-1)|^{p(k-1)-2}\Delta u(k-1)\Big) = \lambda\, g(u(k)), & k \in \mathbb{Z}[1,T], \\ u(0) = u(T+1) = 0, \end{cases} \tag{P_λ^g}$$

admits an unbounded sequence of solutions.

The structure of the paper is the following: Section 2 is devoted to our abstract framework, while Section 3 is dedicated to main results. Concrete examples of application of the attained abstract results are presented in Section 4.

2 Auxiliary results

Solutions to (P_λ^f) will be investigated in the function space

$$X = \{u\colon \mathbb{Z}[0,T+1] \to \mathbb{R};\ u(0) = u(T+1) = 0\}$$

considered with the inner product

$$\langle u, v\rangle = \sum_{k=1}^{T+1} \Delta u(k-1)\Delta v(k-1), \qquad \forall\, u, v \in X,$$

with which X becomes a T-dimensional Hilbert space (see [2]) with a corresponding norm

$$\|u\| = \left(\sum_{k=1}^{T+1} |\Delta u(k-1)|^2\right)^{1/2}.$$

Let $J_\lambda \colon X \to \mathbb{R}$ be the functional associated to problem (P^f_λ) defined by

$$J_\lambda(u) = \Phi(u) - \lambda\Psi(u),$$

where

$$\Phi(u) := \sum_{k=1}^{T+1} \frac{\alpha(k)}{p(k-1)}|\Delta u(k-1)|^{p(k-1)} \quad \text{and} \quad \Psi(u) := \sum_{k=1}^{T} F(k, u(k)),$$

and $F(k,s) = \int_0^s f(k,t)\,dt$ for $s \in \mathbb{R}$ and $k \in \mathbb{Z}[1,T]$. The functional J_λ is continuously Gâteaux differentiable and its Gâteaux derivative J'_λ at u reads

$$J'_\lambda(u)(v) = \sum_{k=1}^{T+1} \alpha\,(k)\,|\Delta u(k-1)|^{p(k-1)-2}\Delta u(k-1)\Delta v(k-1) - \lambda \sum_{k=1}^{T} f(k, u(k))v(k),$$

for all $v \in X$. Summing by parts and taking boundary values into account, we have

$$J'_\lambda(u)(v) = -\sum_{k=1}^{T} \Delta(\alpha\,(k)\,|\Delta u(k-1)|^{p(k-1)-2}\Delta u(k-1))v(k) - \lambda \sum_{k=1}^{T} f(k, u(k))v(k),$$

for all $v \in X$. Hence, an element $u \in X$ is a solution for (P^f_λ) iff $J'_\lambda(u)(v) = 0$ for every $v \in X$, i.e. u is a critical point of J_λ.

Our main tool is a general critical points theorem due to Bonanno and Molica Bisci (see [6]) that is generalization of a result of Ricceri [23]. Here we state it in a smooth version for the reader's convenience.

Theorem 2.1. *Let $(X, \|\cdot\|)$ be a reflexive real Banach space, let $\Phi, \Psi \colon X \to \mathbb{R}$ be two functions of class C^1 on X with Φ coercive, i.e. $\lim_{\|u\|\to\infty} \Phi(u) = +\infty$. For every $r > \inf_X \Phi$, let us put*

$$\varphi(r) := \inf_{u\in\Phi^{-1}((-\infty,r))} \frac{\left(\sup_{v\in\Phi^{-1}((-\infty,r))} \Psi(v)\right) - \Psi(u)}{r - \Phi(u)}$$

and

$$\gamma := \liminf_{r\to+\infty} \varphi(r).$$

Let $J_\lambda := \Phi(u) - \lambda\Psi(u)$ for all $u \in X$. If $\gamma < +\infty$ then, for each $\lambda \in \left(0, \frac{1}{\gamma}\right)$, the following alternative holds:
either
(a) J_λ possesses a global minimum,
or
(b) there is a sequence $\{u_n\}$ of critical points (local minima) of J_λ such that $\lim_{n\to+\infty} \Phi(u_n) = +\infty$.

We will also need the following lemma.

Lemma 2.2. *The functional* $\Phi\colon X \to \mathbb{R}$ *is coercive, i.e.*

$$\lim_{\|u\|\to+\infty} \sum_{k=1}^{T+1} \frac{\alpha(k)}{p(k-1)} |\Delta u(k-1)|^{p(k-1)} = +\infty.$$

Proof. By [21, Lemma 1, part (a)], there exist two positive constants C_1 and C_2 such that

$$\sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)} \geq C_1 \|u\| - C_2,$$

for every $u \in X$ with $\|u\| > 1$. Hence we have

$$\begin{aligned}\Phi(u) &= \sum_{k=1}^{T+1} \frac{\alpha(k)}{p(k-1)} |\Delta u(k-1)|^{p(k-1)} \geq \frac{\alpha^-}{p^+} \left(\sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)} \right) \\ &\geq \frac{\alpha^-}{p^+} (C_1 \|u\| - C_2) \to +\infty,\end{aligned}$$

as $\|u\| \to \infty$. □

3 Main results

We state our main result. Let

$$A := \liminf_{t\to+\infty} \frac{\sum_{k=1}^{T} \max_{|\xi|\leq t} F(k,\xi)}{t^{p^-}}$$

and

$$B_+ := \limsup_{t\to+\infty} \frac{\sum_{k=1}^{T} F(k,t)}{|t|^{p^+}}, \qquad B_- := \limsup_{t\to-\infty} \frac{\sum_{k=1}^{T} F(k,t)}{|t|^{p^+}}.$$

Let $B := \max\{B_+, B_-\}$. For convenience we put $\frac{1}{0^+} = +\infty$ and $\frac{1}{+\infty} = 0$.

Theorem 3.1. *Assume that the following inequality holds:* $A < \frac{p^-\alpha^-}{2p^+\alpha^+T^{p^-}} \cdot B$. *Then, for each* $\lambda \in \left(\frac{2\alpha^+}{Bp^-}, \frac{\alpha^-}{AT^{p^-}p^+}\right)$, *problem* (P_λ^f) *admits an unbounded sequence of solutions.*

Proof. It is clear that $A \geq 0$. Put $\lambda \in \left(\frac{2\alpha^+}{Bp^-}, \frac{\alpha^-}{AT^{p^-}p^+}\right)$ and put Φ, Ψ, J_λ as in the previous section. Our aim is to apply Theorem 2.1 to function J_λ. By Lemma 2.2, the functional Φ is coercive. Therefore, our conclusion follows provided that $\gamma < +\infty$ as well as that J_λ does not possess a global minimum. To this end, let $\{c_m\} \subset (0, +\infty)$ be a sequence such that $\lim_{m\to\infty} c_m = +\infty$ and

$$\lim_{m\to+\infty} \frac{\sum_{k=1}^{T} \max_{|\xi|\leq c_m} F(k,\xi)}{c_m^{p^-}} = A.$$

Set

$$r_m := \frac{\alpha^-}{T^{p^-}p^+} c_m^{p^-}$$

for every $m \in \mathbb{N}$.

Let $m_0 \in \mathbb{N}$ be such that $\frac{p^+}{\alpha^-} r_m > 1$ for all $m > m_0$. We claim that

$$\Phi^{-1}\left((-\infty, r_m)\right) \subset \{v \in X : |v(k)| \leq c_m \quad \text{for all} \quad k \in \mathbb{Z}[0, T+1]\}. \tag{3.1}$$

Indeed, if $v \in X$ and $\Phi(v) < r_m$, one has

$$\sum_{k=1}^{T+1} \frac{\alpha(k)}{p(k-1)} |\Delta v(k-1)|^{p(k-1)} < r_m.$$

Then

$$|\Delta v(k-1)| < \left(\frac{p(k-1)}{\alpha(k)} r_m\right)^{1/p(k-1)} \leq \left(\frac{p^+}{\alpha^-} r_m\right)^{1/p^-}$$

for every $k \in \mathbb{Z}[1, T+1]$ and $m > m_0$. From this and since $v \in X$ we deduce by easy induction that

$$\begin{aligned} |v(k)| &\leq |\Delta v(k-1)| + |v(k-1)| < \left(\frac{p^+}{\alpha^-} r_m\right)^{1/p^-} + |v(k-1)| \\ &\leq k \cdot \left(\frac{p^+}{\alpha^-} r_m\right)^{1/p^-} \leq T \cdot \left(\frac{p^+}{\alpha^-} r_m\right)^{1/p^-} = c_m \end{aligned}$$

for every $k \in \mathbb{Z}[1, T]$ and this gives (3.1). From this and $\Phi(0) = \Psi(0) = 0$ we have

$$\begin{aligned} \varphi(r_m) &\leq \frac{\sup_{\Phi(v) < r_m} \sum_{k=1}^{T} F(k, v(k))}{r_m} \leq \frac{\sum_{k=1}^{T} \max_{|t| \leq c_m} F(k, t)}{r_m} \\ &= \frac{T^{p^-} p^+}{\alpha^-} \cdot \frac{\sum_{k=1}^{T} \max_{|t| \leq c_m} F(k, t)}{c_m^{p^-}} \end{aligned}$$

for every $m > m_0$. Hence, it follows that

$$\gamma \leq \lim_{m \to +\infty} \varphi(r_m) \leq \frac{T^{p^-} p^+}{\alpha^-} \cdot A < \frac{1}{\lambda} < +\infty.$$

Next we show that J_λ does not possess a global minimum. First, we assume that $B = B_-$. We begin with $B = +\infty$. Accordingly, let M be such that $M > \frac{2\alpha^+}{\lambda p^-}$ and let $\{b_m\}$ be a sequence of positive numbers, with $\lim_{m \to +\infty} b_m = +\infty$, such that $b_m > 1$ and

$$\sum_{k=1}^{T} F(k, -b_m) > M b_m^{p^+}$$

for all $m \in \mathbb{N}$. Thus, take in X a sequence $\{s_m\}$ such that, for every $m \in \mathbb{N}$, $s_m(k) := -b_m$ for every $k \in \mathbb{Z}[1, T]$. Then, one has

$$\begin{aligned} J_\lambda(s_m) &= \sum_{k=1}^{T+1} \frac{\alpha(k)}{p(k-1)} |\Delta s_m(k-1)|^{p(k-1)} - \lambda \sum_{k=1}^{T} F(k, s_m(k)) \\ &< \frac{2\alpha^+ b_m^{p^+}}{p^-} - \lambda M b_m^{p^+} = \left(\frac{2\alpha^+}{p^-} - \lambda M\right) b_m^{p^+} \end{aligned}$$

which gives $\lim_{m \to +\infty} J_\lambda(s_m) = -\infty$.

Next, assume that $B < +\infty$. Since $\lambda > \frac{2\alpha^+}{Bp^-}$, we can fix $\varepsilon > 0$ such that $\varepsilon < B - \frac{2\alpha^+}{\lambda p^-}$. Therefore, also taking $\{b_m\}$ a sequence of positive numbers, with $\lim_{m\to+\infty} b_m = +\infty$, such that $b_m > 1$ and

$$(B-\varepsilon)\, b_m^{p^+} < \sum_{k=1}^{T} F(k,-b_m) < (B+\varepsilon)\, b_m^{p^+}$$

for all $m \in \mathbb{N}$, choosing $\{s_m\}$ in X as above, one has

$$J_\lambda(s_m) < \left(\frac{2\alpha^+}{p^-} - \lambda(B-\varepsilon)\right) b_m^{p^+}.$$

So, also in this case, $\lim_{m\to+\infty} J_\lambda(s_m) = -\infty$. The same reasoning applies to the case $B = B_+$. Finally, the above facts mean that J_λ does not possess a global minimum. Hence, by Theorem 2.1, we obtain a sequence $\{u_m\}$ of critical points (local minima) of J_λ such that $\lim_{m\to+\infty} \Phi(u_m) = +\infty$. Since Φ is continuous on the finite dimensional space X, we have $\lim_{m\to+\infty} \|u_m\| = +\infty$. The proof is complete. □

Remark 3.2. We note that, if $f(k,\cdot)$ is a nonnegative continuous function for each $k \in \mathbb{Z}[1,T]$, then $\max_{|\xi|\le t} F(k,\xi) = F(k,t)$. Consequently, Theorem 1.1 immediately follows from Theorem 3.1.

As the immediate consequence of Theorem 3.1 we infer the existence of solutions to boundary value problems for finite difference equations with p-Laplacian operator. In this setting, set $p > 1$ and consider the real map $\phi_p\colon \mathbb{R} \to \mathbb{R}$ given by $\phi_p(s) := |s|^{p-2}s$, for every $s \in \mathbb{R}$. Denote

$$\tilde{A} := \liminf_{t\to+\infty} \frac{\sum_{k=1}^{T} \max_{|\xi|\le t} F(k,\xi)}{t^p}$$

and

$$\tilde{B}_+ := \limsup_{t\to+\infty} \frac{\sum_{k=1}^{T} F(k,t)}{|t|^p}, \qquad \tilde{B}_- := \limsup_{t\to-\infty} \frac{\sum_{k=1}^{T} F(k,t)}{|t|^p}$$

and put $\tilde{B} = \max\{\tilde{B}_+, \tilde{B}_-\}$

With the previous notations, taking maps $p\colon \mathbb{Z}[0,T] \to \mathbb{R}$ and $\alpha\colon \mathbb{Z}[1,T+1] \to (0,+\infty)$ such that $p(k) = p$ for every $k \in \mathbb{Z}[0,T]$ and $\alpha(k) = 1$ for every $k \in \mathbb{Z}[1,T+1]$ we have the following corollary.

Corollary 3.3. *Assume that* $\tilde{A} < \frac{\tilde{B}}{2T^p}$. *Then, for each* $\lambda \in \left(\frac{2}{\tilde{B}}, \frac{1}{\tilde{A}T^p}\right)$, *the problem*

$$\begin{cases} -\Delta\left(\phi_p(\Delta u(k-1))\right) = \lambda f(k,u(k)), & k \in \mathbb{Z}[1,T], \\ u(0) = u(T+1) = 0, \end{cases} \tag{D_λ^f}$$

admits an unbounded sequence of solutions.

A more technical version of Theorem 3.1 can be written as follows.

Theorem 3.4. *Assume that there exist real sequences* $\{a_m\}$ *and* $\{b_m\}$, *with* $\lim_{m\to+\infty} a_m = +\infty$ *and* $a_m \ge 1$ *for each* $m \in \mathbb{N}$, *such that*

(a) $a_m^{p^+} < \left(\frac{p^-\alpha^-}{2p^+\alpha^+T^{p^-}}\right) b_m^{p^-}$, *for each* $m \in \mathbb{N}$;

(b) $C < \frac{B}{2p^+\alpha^+T^{p^-}}$, *where*

$$C := \lim_{m\to+\infty} \frac{\sum_{k=1}^{T}\max_{|t|\le b_m} F(k,t) - \sum_{k=1}^{T} F(k,a_m)}{p^-\alpha^- b_m^{p^-} - 2p^+\alpha^+T^{p^-}a_m^{p^+}}.$$

Then, for each $\lambda \in \left(\frac{2\alpha^+}{Bp^-}, \frac{1}{Cp^-p^+T^{p^-}}\right)$, *problem* ($P_\lambda^f$) *admits an unbounded sequence of solutions.*

Proof. We will keep the above notations. Putting $r_m := \frac{\alpha^-}{T^{p^-}p^+} b_m^{p^-}$, we have

$$\varphi(r_m) \le \inf_{w\in\Phi^{-1}((-\infty,r_m))} \frac{\sum_{k=1}^{T}\max_{|t|\le b_m} F(k,t) - \sum_{k=1}^{T} F(k,w(k))}{r_m - \Phi(w)}. \tag{3.2}$$

Let $w_m \in X$ be defined by $w_m(k) := a_m$ for every $k \in \mathbb{Z}[1,T]$. Then $\|w_m\| \to +\infty$ and

$$\Phi(w_m) = \sum_{k=1}^{T+1} \frac{\alpha(k)}{p(k-1)} |\Delta w_m(k-1)|^{p(k-1)} \le \frac{2\alpha^+}{p^-} a_m^{p^+},$$

since $a_m \ge 1$. This and condition (a) gives

$$r_m - \Phi(w_m) \ge \frac{\alpha^-}{T^{p^-}p^+} b_m^{p^-} - \frac{2\alpha^+}{p^-} a_m^{p^+} > 0.$$

We also have $w_m \in \Phi^{-1}((-\infty, r_m))$, so inequality (3.2) yields

$$\varphi(r_m) \le \left(T^{p^-}p^+p^-\right) \frac{\sum_{k=1}^{T}\max_{|t|\le b_m} F(k,t) - \sum_{k=1}^{T} F(k,a_m)}{p^-\alpha^- b_m^{p^-} - 2p^+\alpha^+T^{p^-}a_m^{p^+}},$$

for every $m \in \mathbb{N}$. Further, by hypothesis (b), we obtain

$$\gamma \le \lim_{m\to+\infty} \varphi(r_m) \le T^{p^-}p^+p^- \cdot C < \frac{1}{\lambda} < +\infty.$$

From now on, arguing exactly as in the proof of Theorem 3.1 we obtain the assertion. □

4 Examples

Now, we will show the example of a function for which we can apply Theorem 3.1.

Example 4.1. Let $\hat{A}$, $\hat{B}$ be some positive real numbers. Let p^+, p^- be real numbers, such that $1 < p^- < p^+ < +\infty$. Choose a real number a such that $\left(\frac{\hat{B}}{\hat{A}} \cdot a^{p^+}\right)^{1/p^-} > a$. Let $\{a_m\}$ be a sequence defined by recursion

$$\begin{cases} a_1 := a \\ a_{m+1} := 1 + \left(\frac{\hat{B}}{\hat{A}} \cdot a_m^{p^+}\right)^{\frac{1}{p^-}} \quad \text{for } \ m \ge 2 \end{cases}$$

Then $a_{m+1} - 1 > a_m$ for every $m \in \mathbb{N}$. Let $\{h_m\}$ be a sequence such that $h_1 = \hat{B}a^{p^+}$ and

$$h_m := \hat{B}\left(a_m^{p^+} - a_{m-1}^{p^+}\right)$$

for $m \geq 2$. Let $\hat{f}\colon \mathbb{R} \to \mathbb{R}$ be the continuous nonnegative function given by

$$\hat{f}(s) := \sum_{m\in\mathbb{N}} 2h_m \left(1 - 2\left|s - a_m + \tfrac{1}{2}\right|\right) \mathbf{1}_{[a_m-1,a_m]}(s)$$

where the symbol $\mathbf{1}_{[\alpha,\beta]}$ denotes the characteristic function of the interval $[\alpha,\beta]$. It is easy to verify that, for every $m \in \mathbb{N}$,

$$\int_{a_m-1}^{a_m} \hat{f}(t)\,dt = h_m.$$

Set $F(t) := \int_0^t \hat{f}(s)\,ds$ for every $t \in \mathbb{R}$. Then $F(a_m) = \sum_{k=1}^m h_k = \hat{B}a_m^{p^+}$. It is easy to check that

$$\liminf_{t\to+\infty} \frac{F(t)}{t^{p^-}} = \lim_{m\to+\infty} \frac{F(a_{m+1}-1)}{(a_{m+1}-1)^{p^-}}$$

and

$$\limsup_{t\to+\infty} \frac{F(t)}{t^{p^+}} = \lim_{m\to+\infty} \frac{F(a_m)}{a_m^{p^+}}.$$

Therefore

$$\liminf_{t\to+\infty} \frac{F(t)}{t^{p^-}} = \lim_{m\to+\infty} \frac{F(a_{m+1}-1)}{(a_{m+1}-1)^{p^-}} = \lim_{m\to+\infty} \frac{F(a_m)}{\frac{\hat{B}}{\hat{A}}\cdot a_m^{p^+}} = \hat{A}$$

and

$$\limsup_{t\to+\infty} \frac{F(t)}{t^{p^+}} = \lim_{m\to+\infty} \frac{F(a_m)}{a_m^{p^+}} = \hat{B}.$$

Now, if we put $f(k,s) = \hat{f}(s)$ for every $k \in \mathbb{Z}[1,T]$ and assume that $\hat{A} < \frac{p^-\alpha^-}{2p^+\alpha^+T^{p^-}} \cdot \hat{B}$, then Theorem 3.1 applies.

And now, we will show the example of a function for which we can apply Theorem 1.1.

Example 4.2. Let p^+, p^- be real numbers, such that $1 < p^- \leq p^+ < +\infty$. Let $\{b_m\}$ be a sequence defined by recursion

$$\begin{cases} b_1 := 1 \\ b_{m+1} := 1 + (m^2 b_m^{p^+})^{\frac{1}{p^-}} \quad \text{for } \ m \geq 2 \end{cases}$$

and let $\{h_m\}$ be a sequence such that $h_1 = 1$ and

$$h_m := m b_m^{p^+} - (m-1) b_{m-1}^{p^+}$$

for $m \geq 2$. Let $g\colon \mathbb{R} \to \mathbb{R}$ be the continuous nonnegative function given by

$$g(s) := \sum_{m\in\mathbb{N}} 2h_m \left(1 - 2\left|s - b_m + \tfrac{1}{2}\right|\right) \mathbf{1}_{[b_m-1,b_m]}(s).$$

It is easy to verify that, for every $m \in \mathbb{N}$,

$$\int_{b_m-1}^{b_m} f(t)\,dt = h_m.$$

Set $G(t) := \int_0^t g(s)\,ds$ for every $t \in \mathbb{R}$. Then $G(b_m) = \sum_{k=1}^m h_k = m b_m^{p^+}$. We have

$$\liminf_{t\to+\infty} \frac{G(t)}{t^{p^-}} \leq \lim_{m\to+\infty} \frac{G(b_{m+1}-1)}{(b_{m+1}-1)^{p^-}} = \lim_{m\to+\infty} \frac{G(b_m)}{m^2 b_m^{p^+}} = \lim_{m\to+\infty} \frac{1}{m} = 0$$

and

$$\limsup_{t\to+\infty} \frac{G(t)}{t^{p^+}} \geq \lim_{m\to+\infty} \frac{G(b_m)}{b_m^{p^+}} = \lim_{m\to+\infty} m = +\infty.$$

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