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Solvability for second-order nonlocal boundary value problems with a p-Laplacian at resonance on a half-line*

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Abstract: This paper investigates the solvability of the second-order boundary value problems with the one-dimensional p-Laplacian at resonance on a half-line

$$\begin{cases} (c(t)\phi_p(x'(t)))' = f(t, x(t), x'(t)), & 0 < t < \infty, \\ x(0) = \sum\limits_{i=1}^{n} \mu_i x(\xi_i), \quad \lim\limits_{t\to+\infty} c(t)\phi_p(x'(t)) = 0 \end{cases}$$

and

$$\begin{cases} (c(t)\phi_p(x'(t)))' + g(t)h(t, x(t), x'(t)) = 0, & 0 < t < \infty, \\ x(0) = \int_0^\infty g(s)x(s)ds, \quad \lim\limits_{t\to+\infty} c(t)\phi_p(x'(t)) = 0 \end{cases}$$

with multi-point and integral boundary conditions, respectively, where $\phi_p(s) = |s|^{p-2}s$, $p > 1$. The arguments are based upon an extension of Mawhin's continuation theorem due to Ge. And examples are given to illustrate our results.

Keywords: Boundary value problem; Multi-point boundary condition; Integral boundary condition; Resonance; Half-line; p-Laplacian

MSC: 34B10; 34B15; 34B40

1. Introduction

In this paper, we consider the second-order boundary value problems with a p-Laplacian on a half line

$$(c(t)\phi_p(x'(t)))' = f(t, x(t), x'(t)), \quad 0 < t < \infty, \tag{1.1}$$

$$x(0) = \sum_{i=1}^{n} \mu_i x(\xi_i), \quad \lim_{t\to+\infty} c(t)\phi_p(x'(t)) = 0, \tag{1.2}$$

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with $0 \leq \xi_i < \infty$, $\mu_i \in \mathbb{R}$, $i = 1, 2, \cdots, n$,

$$\sum_{i=1}^{n} \mu_i = 1 \tag{1.3}$$

and

$$(c(t)\phi_p(x'(t)))' + g(t)h(t, x(t), x'(t)) = 0, \quad 0 < t < \infty, \tag{1.4}$$

$$x(0) = \int_0^\infty g(s)x(s)ds, \quad \lim_{t\to+\infty} c(t)\phi_p(x'(t)) = 0 \tag{1.5}$$

with $g \in L^1[0, \infty)$, $g(t) > 0$ on $[0, \infty)$ and

$$\int_0^\infty g(s)ds = 1. \tag{1.6}$$

Throughout this paper, we assume

(A1) $c \in C[0, \infty) \cap C^1(0, \infty)$ and $c(t) > 0$ on $[0, \infty)$, $\phi_q(\frac{1}{c}) \in L^1[0, \infty)$.

(A2) $\sum\limits_{i=1}^{n} \mu_i \int_0^{\xi_i} \phi_q(\frac{e^{-s}}{c(s)})ds \neq 0$.

Due to the conditions (1.3) and (1.6), the differential operator $\frac{d}{dt}(c\phi_p(\frac{d}{dt}\cdot))$ in (1.1) and (1.4) is not invertible under the boundary conditions (1.2) and (1.5), respectively. In the literature, boundary value problems of this type are referred to problems at resonance.

The theory of boundary value problems (in short: BVPs) with multi-point and integral boundary conditions arises in a variety of different areas of applied mathematics and physics. For example, bridges of small size are often designed with two supported points, which leads to a standard two-point boundary condition and bridges of large size are sometimes contrived with multi-point supports, which corresponds to a multi-point boundary condition [1]. Heat conduction, chemical engineering, underground water flow, thermo-elasticity and plasma physics can be reduced to the nonlocal problems with integral boundary conditions [2,3]. The study of multi-point BVPs for linear second-order ordinary differential equations was initiated by Il'in and Moiseev [4] in 1987. Since then many authors have studied more nonlinear multi-point BVPs [7-14]. Recently, BVPs with integral boundary conditions have received much attention. To identify a few, we refer the readers to [17-22] and references therein.

Second-order BVPs on infinite intervals arising from the study of radially symmetric solutions of nonlinear elliptic equation and models of gas pressure in a semi-infinite porous medium, have received much attention. For an extensive collection of results on BVPs on unbounded domains, we refer the readers to a monograph by Agarwal and O'Regan [16]. Other recent results and methods for BVPs on a half-line can be found in [14,15] and the references therein.

From the existed results, we can see a fact: for the resonance case, only BVPs with linear differential operator on half-line were considered. The BVPs with multi-point and integral boundary conditions on a half-line have not investigated till now. Although some authors (see [5,9,10,12,17]) have studied BVPs with nonlinear differential operator, for example, with a p-Laplacian operator, the domains are bounded.

Motivated by the above works, we intend to discuss the BVPs (1.1)-(1.2) and (1.4)-(1.5) at resonance on a half-line. Due to the fact that the classical Mawhin's continuation theorem can't be directly used to discuss the BVP with nonlinear differential operator, in this paper, we investigate the BVPs (1.1)-(1.2) and (1.4)-(1.5) by applying an extension of Mawhin's continuation theorem due to Ge [5]. Furthermore, examples are given to illustrate the results.

2. Preliminaries

For the convenience of readers, we present here some definitions and lemmas.

Definition 2.1. We say that a mapping $f : [0, \infty)\times\mathbb{R}^2 \to \mathbb{R}$ satisfies the Carathéodory conditions, if the following two conditions are satisfied:

(B1) for each $(u, v) \in \mathbb{R}^2$, the mapping $t \mapsto f(t, u, v)$ is Lebesgue measurable;

(B2) for a.e. $t \in [0, \infty)$, the mapping $(u, v) \mapsto f(t, u, v)$ is continuous on $\mathbb{R}^2$.

In addition, f is called a L^1-Carathéodory function if (B1), (B2) and (B3) hold, f is called a g-Carathéodory function if (B1), (B2) and (B4) are satisfied.

(B3) for each $r > 0$, there exists $\alpha_r \in L^1[0, \infty)$ such that for a.e. $t \in [0, \infty)$ and every (u, v) such that $\max\{||u||_\infty, ||v||_\infty\} \leq r$, we have $|f(t, u, v)| \leq \alpha_r(t)$;

(B4) for each $l > 0$ and $g \in L^1[0, \infty)$, there exists a function $\psi_l : [0, \infty) \to [0, \infty)$ satisfying $\int_0^\infty g(s)\psi_l(s)ds < \infty$ such that

$$\max\{|u|, |v|\} \leq l \quad \text{implies} \quad |f(t, u, v)| \leq \psi_l(t) \;\text{ for a.e. } t \in [0, \infty).$$

Definition 2.2[5]**.** Let X and Z be two Banach spaces with norms $||\cdot||_X$ and $||\cdot||_Z$, respectively. A continuous operator $M : X\cap \operatorname{dom}M \to Z$ is said to be quasi-linear if

(C1) $\operatorname{Im}M = M(X\cap \operatorname{dom}M)$ is a closed subset of Z;

(C2) $\ker M = \{x \in X\cap \operatorname{dom}M : Mx = 0\}$ is linearly homeomorphic to $\mathbb{R}^n$, $n < \infty$.

Definition 2.3[6]**.** Let X be a Banach spaces and $X_1 \subset X$ a subspace. The operator $P : X \to X_1$ is said to be a projector provided $P^2 = P$, $P(\lambda_1x_1 + \lambda_2x_2) = \lambda_1Px_1 + \lambda_2Px_2$ for $x_1, x_2 \in X$, $\lambda_1, \lambda_2 \in \mathbb{R}$. The operator $Q : X \to X_1$ is said to be a semi-projector provided $Q^2 = Q$ and $Q(\lambda x) = \lambda Qx$ for $x \in X,\ \lambda \in \mathbb{R}$.

Let $X_1 = \ker M$ and X_2 be the complement space of X_1 in X, then $X = X_1 \oplus X_2$. On the other hand, suppose Z_1 is a subspace of Z and Z_2 is the complement of Z_1 in Z, then $Z = Z_1\oplus Z_2$. Let $P : X \to X_1$ be a projector and $Q : Z \to Z_1$ be a semi-projector, and $\Omega \subset X$ an open and bounded set with the origin $\theta \in \Omega$, where θ is the origin of a linear space. Suppose $N_\lambda : \overline{\Omega} \to Z$, $\lambda \in [0, 1]$ is a continuous operator. Denote N_1 by N. Let $\sum_\lambda = \{x \in \overline{\Omega} : Mx = N_\lambda x\}$.

Definition 2.4[5]**.** N_λ is said to be M-compact in $\overline{\Omega}$ if there is a vector subspace Z_1 of Z with $\dim Z_1 = \dim X_1$ and an operator $R : \overline{\Omega} \times [0,1] \to X_2$ being continuous and compact such that for $\lambda \in [0,1]$,

$$(I - Q)N_\lambda(\overline{\Omega}) \subset \mathrm{Im}M \subset (I - Q)Z, \tag{2.1}$$

$$QN_\lambda x = 0, \quad \lambda \in (0,1) \iff QNx = 0, \tag{2.2}$$

$$R(\cdot, 0) \text{ is the zero operator and } R(\cdot,\lambda)|_{\sum_\lambda} = (I - P)|_{\sum_\lambda}, \tag{2.3}$$

$$M[P + R(\cdot,\lambda)] = (I - Q)N_\lambda. \tag{2.4}$$

Theorem 2.1[5]**.** *Let X and Z be two Banach spaces with norms $||\cdot||_X$ and $||\cdot||_Z$, respectively, and $\Omega \subset X$ an open and bounded set. Suppose $M : X \cap \mathrm{dom}M \to Z$ is a quasi-linear operator and $N_\lambda : \overline{\Omega} \to Z$, $\lambda \in [0,1]$ is M-compact. In addition, if*

(D1) *$Mx \neq N_\lambda x$, for $\lambda \in (0,1)$, $x \in \mathrm{dom}M \cap \partial\Omega$;*

(D2) *$\deg\{JQN, \Omega \cap \ker M, 0\} \neq 0$, where $J : Z_1 \to X_1$ is a homeomorphism with $J(\theta) = \theta$.*

Then the abstract equation $Mx = Nx$ has at least one solution in $\overline{\Omega}$.

Proposition 2.1[6]**.** *ϕ_p has the following properties*

(E1) *ϕ_p is continuous, monotonically increasing and invertible. Moreover, $\phi_p^{-1} = \phi_q$ with $q > 1$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$;*

(E2) $\forall u, v \geq 0,$
$$\begin{aligned} &\phi_p(u+v) \leq \phi_p(u) + \phi_p(v), && \text{if } \ 1 < p < 2,\\ &\phi_p(u+v) \leq 2^{p-2}(\phi_p(u) + \phi_p(v)), && \text{if } \ p \geq 2. \end{aligned}$$

3. Related Lemmas

Let $AC[0,\infty)$ denote the space of absolutely continuous functions on the interval $[0,\infty)$. In this paper, we work in the following spaces

$$X = \{x : [0,\infty) \to \mathbb{R} \mid \ x, c\phi_p(x') \in AC[0,\infty), \lim_{t\to\infty} x(t) \text{ and } \lim_{t\to\infty} x'(t) \text{ exist and } (c\phi_p(x'))' \in L^1[0,\infty)\},$$

$$Y = L^1[0,\infty) \ \text{ and } \ Z = \{z : [0,\infty) \to \mathbb{R} : \int_0^\infty g(t)|z(t)|dt < \infty\}$$

with norms $||x||_X = \max\{||x||_\infty, ||x'||_\infty\}$, where $||x||_\infty = \sup\limits_{t\in[0,\infty)} |x(t)|$, $||y||_1 = \int_0^\infty |y(t)|dt$ and $||z||_Z = \int_0^\infty g(t)|z(t)|dt$ for $x \in X$, $y \in Y$ and $z \in Z$. By the standard arguments, we can prove that $(X, ||\cdot||_X)$, $(Y, ||\cdot||_1)$ and $(Z, ||\cdot||_Z)$ are all Banach spaces.

Define $M_1 : \mathrm{dom}M_1 \to Y$ and $N_\lambda^1 : X \to Y$ with

$$\mathrm{dom}M_1 = \{x \in X : x(0) = \sum_{i=1}^n \mu_i x(\xi_i), \ \lim_{t\to+\infty} c(t)\phi_p(x'(t)) = 0\}$$

by $M_1x(t)=(c(t)\phi_p(x'(t)))'$ and $N_\lambda^1 x(t)=\lambda f(t,x(t),x'(t)),\ t\in[0,\infty)$.

Let $M_2:\mathrm{dom}M_2\to Z$ and $N_\lambda^2:X\to Z$ with

$$\mathrm{dom}M_2=\{x\in X: gx\in L^1[0,\infty),\ x(0)=\int_0^\infty x(s)g(s)ds,\ \lim_{t\to\infty}c(t)\phi_p(x'(t))=0\}$$

be defined by $M_2x(t)=-\frac{1}{g(t)}(c(t)\phi_p(x'(t)))'$ and $N_\lambda^2 x(t)=\lambda h(t,x(t),x'(t)),\ t\in[0,\infty)$. Then the BVPs (1.1)-(1.2) and (1.4)-(1.5) can be written as $M_1x=N^1x$ and $M_2x=N^2x$, respectively, here denote $N_1^i=N^i$, $i=1,2$.

Lemma 3.1. *The operators $M_1:\mathrm{dom}M_1\to Y$ and $M_2:\mathrm{dom}M_2\to Z$ are quasi-linear.*

Proof. It is clear that $X_1=\ker M_1=\{x\in\mathrm{dom}M_1: x(t)\equiv a\in\mathbb{R}\text{ on }[0,\infty)\}$.

Let $x\in\mathrm{dom}M_1$ and consider the equation $(c(t)\phi_p(x'(t)))'=y(t)$. It follows from (1.2) that

$$c(t)\phi_p(x'(t))=-\int_t^\infty y(s)ds,$$

so that

$$x'(t)=-\phi_q(\frac{1}{c(t)})\phi_q(\int_t^\infty y(s)ds), \tag{3.1}$$

and

$$x(t)=\int_t^\infty \phi_q(\frac{1}{c(s)})\phi_q(\int_s^\infty y(\tau)d\tau)ds+C, \tag{3.2}$$

where C is a constant. In view of (1.2) and (1.3), we have

$$\sum_{i=1}^n \mu_i\int_0^{\xi_i}\phi_q(\frac{1}{c(s)})\phi_q(\int_s^\infty y(\tau)d\tau)ds=0. \tag{3.3}$$

Thus,

$$\mathrm{Im}M_1\subset\{y\in Y:\sum_{i=1}^n \mu_i\int_0^{\xi_i}\phi_q(\frac{1}{c(s)})\phi_q(\int_s^\infty y(\tau)d\tau)ds=0\}.$$

Conversely, if (3.3) holds for $y\in Y$, we take $x\in\mathrm{dom}\ M_1$ as given by (3.2), then $(c(t)\phi_p(x'(t)))'=y(t)$ for $t\in[0,\infty)$ and (1.2) is satisfied. Hence, we have

$$\mathrm{Im}M_1=\{y\in Y:\sum_{i=1}^n \mu_i\int_0^{\xi_i}\phi_q(\frac{1}{c(s)})\phi_q(\int_s^\infty y(\tau)d\tau)ds=0\}. \tag{3.4}$$

So we have $\dim\ker M_1=1<\infty$, $\mathrm{Im}M_1\subset Y$ is closed. Therefore, M_1 is a quasi-linear operator.

Similarly, we can calculate that

$$\ker M_2=\{x\in\mathrm{dom}M_2: x(t)\equiv a\in\mathbb{R}\text{ on }[0,\infty)\}$$

and prove that

$$\mathrm{Im}M_2=\{z\in Z:\int_0^\infty g(t)\int_0^t\phi_q(\frac{1}{c(s)})\phi_q(\int_s^\infty g(\tau)z(\tau)d\tau)dsdt=0\}. \tag{3.5}$$

Hence, M_2 is also a quasi-linear operator. $\square$

In order to apply Theorem 2.1, we have to prove that R is completely continuous, and then to prove that N is M-compact. Because the Arzelà-Ascoli theorem fails to the noncompact interval case, we will use the following criterion.

Lemma 3.2[14]**.** *Let X be the space of all bounded continuous vector-valued functions on $[0,\infty)$ and $S \subset X$. Then S is relatively compact if the following conditions hold:*

(F1) *S is bounded in X;*

(F2) *all functions from S are equicontinuous on any compact subinterval of $[0,\infty)$;*

(F3) *all functions from S are equiconvergent at infinity, that is, for any given $\varepsilon > 0$, there exists a $T = T(\varepsilon) > 0$ such that $||\chi(t) - \chi(\infty)||_{\mathbb{R}^n} < \varepsilon$ for all $t > T$ and $\chi \in S$.*

Lemma 3.3. *If f is a L^1-Carathéodory function, then the operator $N_\lambda^1 : \overline{U} \to Y$ is M_1-compact in $\overline{U}$, where $U \subset X$ is an open and bounded subset with $\theta \in U$.*

Proof. We recall the condition (A2) and define the continuous operator $Q_1 : Y \to Y_1$ by

$$Q_1 y(t) = \omega_1(t)\phi_p(\sum_{i=1}^{n}\mu_i \int_0^{\xi_i} \phi_q(\frac{1}{c(s)})\phi_q(\int_s^\infty y(\tau)d\tau)ds), \tag{3.6}$$

where $\omega_1(t) = e^{-t}/\phi_p(\sum\limits_{i=1}^{n}\mu_i \int_0^{\xi_i}\phi_q(\frac{e^{-s}}{c(s)})ds)$. It is easy to check that $Q_1^2 y = Q_1 y$ and $Q_1(\lambda y) = \lambda Q_1 y$ for $y \in Y$, $\lambda \in \mathbb{R}$, that is, Q_1 is a semi-projector and $\dim X_1$=1=$\dim Y_1$. Moreover, (3.4) and (3.6) imply that $\mathrm{Im} M_1$=$\ker Q_1$.

It is easy to see that $Q_1[(I - Q_1)N_\lambda^1(x)] = 0$, $\forall x \in \overline{U}$. So $(I - Q_1)N_\lambda^1(x) \in \ker Q_1 = \mathrm{Im} M_1$. For $y \in \mathrm{Im} M_1$, we have $Q_1 y = 0$. Thus, $y = y - Q_1 y = (I - Q_1)y \in (I - Q_1)Y$. Therefore, (2.1) is satisfied. Obviously, (2.2) holds.

Define $R_1 : \overline{U} \times [0,1] \to X_2$ by

$$R_1(x,\lambda)(t) = \int_t^\infty \phi_q(\frac{1}{c(s)})\phi_q(\int_s^\infty \lambda(f(\tau, x(\tau), x'(\tau)) - (Q_1 f)(\tau))d\tau)ds, \tag{3.7}$$

where X_2 is the complement space of $X_1 = \ker M_1$ in X. Clearly, $R_1(\cdot, 0) = \theta$. Now we prove that $R_1 : \overline{U} \times [0,1] \to X_2$ is compact and continuous.

We first assert that R_1 is relatively compact for any $\lambda \in [0,1]$. In fact, since $U \subset X$ is a bounded set, there exists $r > 0$ such that $\overline{U} \subset \{x \in X : ||x||_X \le r\}$. Because the function f is L^1-Carathéodory, there exists $\alpha_r \in L^1[0,\infty)$ such that for a.e. $t \in [0,\infty)$, $|f(t, x(t), x'(t))| \le \alpha_r(t)$ for $x \in \overline{U}$. Then for any $x \in \overline{U}$, $\lambda \in [0,1]$, we have

$$\begin{aligned} |R_1(x,\lambda)(t)| &\le \int_t^\infty |\phi_q(\frac{1}{c(s)})|\phi_q(\int_s^\infty \lambda|f(\tau, x(\tau), x'(\tau)) - (Q_1 f)(\tau)|d\tau)ds \\ &\le \int_0^\infty |\phi_q(\frac{1}{c(s)})|ds\phi_q[\int_0^\infty |\alpha_r(s)|ds + \int_0^\infty |(Q_1 f)(s)|ds] \end{aligned}$$

$$= \quad ||\phi_q(\frac{1}{c})||_1 \cdot \phi_q[||\alpha_r||_1 + ||Q_1 f||_1] =: L_1 < \infty.$$

From (A1), we can see that $\phi_q(\frac{1}{c})$ is bounded. Hence,

$$\begin{aligned} |R_1'(x,\lambda)(t)| &\leq |\phi_q(\frac{1}{c(t)})|\phi_q(\int_t^\infty \lambda|f(s,x(s),x'(s)) - (Q_1 f)(s)|ds \\ &\leq ||\phi_q(\frac{1}{c})||_\infty \cdot \phi_q[||\alpha_r||_1 + ||Q_1 f||_1] =: L_2 < \infty, \end{aligned}$$

that is, $R_1(\cdot,\lambda)\overline{U}$ is uniformly bounded. Meanwhile, for any $t_1, t_2 \in [0,T]$ with T a positive constant, one gets

$$|R_1(x,\lambda)(t_2) - R_1(x,\lambda)(t_1)| = |\int_{t_1}^{t_2} R_1'(x,\lambda)(s)ds| \leq L_2|t_2 - t_1| \to 0, \quad \text{as} \quad |t_2 - t_1| \to 0$$

and

$$\begin{aligned} & |\phi_p(R_1'(x,\lambda)(t_2)) - \phi_p(R_1'(x,\lambda)(t_1))| \\ = \; & |\frac{1}{c(t_2)}\int_{t_2}^\infty \lambda[f(s,x(s),x'(s)) - (Q_1 f)(s)]ds - \frac{1}{c(t_1)}\int_{t_1}^\infty \lambda[f(s,x(s),x'(s)) - (Q_1 f)(s)]ds| \\ \leq \; & |\frac{1}{c(t_2)}| \cdot |\int_{t_2}^{t_1} \lambda[f(s,x(s),x'(s)) - (Q_1 f)(s)]ds| \\ & + |[\frac{1}{c(t_2)} - \frac{1}{c(t_1)}]\int_{t_1}^\infty \lambda[f(s,x(s),x'(s)) - (Q_1 f)(s)]ds| \\ \leq \; & ||\frac{1}{c}||_\infty \cdot |\int_{t_1}^{t_2} [\alpha_r(s) + |(Q_1 f)(s)|]ds| + [||\alpha_r||_1 + ||Q_1 f||_1]|\frac{1}{c(t_2)} - \frac{1}{c(t_1)}| \to 0, \\ & \text{as} \quad |t_2 - t_1| \to 0. \end{aligned}$$

Then $|R_1'(x,\lambda)(t_2) - R_1'(x,\lambda)(t_1)| \to 0$, as $|t_2 - t_1| \to 0$. So, $R_1(\cdot,\lambda)\overline{U}$ is equicontinuous on [0,T].

In additional, we claim that $R_1(\cdot,\lambda)\overline{U}$ is equiconvergent at infinity. In fact,

$$\begin{aligned} |R_1(x,\lambda)(t) - R_1(x,\lambda)(+\infty)| &\leq \int_t^\infty |\phi_q(\frac{1}{c(s)})|\phi_q(\int_s^\infty \lambda|f(\tau,x(\tau),x'(\tau)) - (Q_1 f)(\tau)|d\tau)ds \\ &\leq \int_t^\infty L_2 ds \to 0 \quad \text{uniformly as} \quad t \to +\infty. \end{aligned}$$

$$\begin{aligned} |R_1'(x,\lambda)(t) - R_1'(x,\lambda)(+\infty)| &\leq |\phi_q(\frac{1}{c(t)})|\phi_q(\int_t^\infty \lambda|f(s,x(s),x'(s)) - (Q_1 f)(s)|ds) \\ &\leq ||\phi_q(\frac{1}{c})||_\infty \cdot \phi_q[\int_t^\infty (\alpha_r(s) + |(Q_1 f)(s)|)ds] \to 0 \\ &\qquad \text{uniformly as} \quad t \to +\infty. \end{aligned}$$

Thus, Lemma 3.2 implies that $R_1(\cdot,\lambda)\overline{U}$ is relatively compact. Since f is L^1-Carathéodory, the continuity of R_1 on $\overline{U}$ follows from the Lebesgue dominated convergence theorem.

Define a projector $P_1 : X \to X_1$ by $P_1x(t) = \lim_{t\to+\infty} x(t)$. For any $x \in \sum_\lambda^1 = \{x \in \overline{U} : M_1x = N_\lambda^1 x\}$, we have $\lambda f(t, x(t), x'(t)) = (c(t)\phi_p(x'(t)))' \in \mathrm{Im}M_1 = \ker Q_1$. Hence

$$\begin{aligned} R_1(x,\lambda)(t) &= \int_t^\infty \phi_q(\frac{1}{c(s)})\phi_q(\int_s^\infty \lambda(f(\tau, x(\tau), x'(\tau)) - (Q_1f)(\tau))d\tau)ds \\ &= \int_t^\infty \phi_q(\frac{1}{c(s)})\phi_q(\int_s^\infty (c(\tau)\phi_p(x'(\tau)))'d\tau)ds \\ &= -\int_t^\infty x'(s)ds = x(t) - \lim_{t\to+\infty} x(t) = [(I - P_1)x](t), \end{aligned}$$

which implies (2.3). For any $x \in \overline{U}$, we have

$$\begin{aligned} & M_1[P_1x + R_1(x,\lambda)](t) \\ = \ & M_1[\lim_{t\to+\infty} x(t) + \int_t^\infty \phi_q(\frac{1}{c(s)})\phi_q(\int_s^\infty \lambda(f(\tau, x(\tau), x'(\tau)) - (Q_1f)(\tau))d\tau)ds] \\ = \ & \lambda(f(t, x(t), x'(t)) - Q_1f(t, x(t), x'(t))) \\ = \ & [((I - Q_1)N_\lambda^1)(x)](t), \end{aligned}$$

which yields (2.4). As a result, N_λ^1 is M_1-compact in $\overline{U}$. □

Lemma 3.4. *If h is a g-Carathéodory function, then the operator $N_\lambda^2 : \overline{\Omega} \to Z$ is M_2-compact, where $\Omega \subset X$ is an open and bounded subset with $\theta \in \Omega$.*

Proof. As in the proof of Lemma 3.3, we first define the semi-projection $Q_2 : Z \to Z_1$ by

$$Q_2z(t) = \phi_p(\frac{1}{\omega_2}\int_0^\infty g(s)\int_0^s \phi_q(\frac{1}{c(\tau)})\phi_q(\int_\tau^\infty g(r)z(r)dr)d\tau ds), \tag{3.8}$$

where $\omega_2 = \int_0^\infty g(s)\int_0^s \phi_q(\frac{1}{c(\tau)})\phi_q(\int_\tau^\infty g(r)dr)d\tau ds$. (3.5) and (3.8) imply that $\mathrm{Im}M_2$=$\ker Q_2$. It is easy to check that the conditions (2.1) and (2.2) hold.

Let $R_2 : \overline{\Omega} \times [0, 1] \to X_2'$ be defined by

$$R_2(x,\lambda)(t) = \int_0^t \phi_q(\frac{1}{c(s)})\phi_q(\int_s^\infty \lambda g(\tau)(h(\tau, x(\tau), x'(\tau)) - (Q_2f)(\tau))d\tau)ds, \tag{3.9}$$

where X_2' is the complement space of $X_1' = \ker M_2$ in X. Clearly, $R_2(\cdot, 0) = \theta$.

Now we prove that $R_2 : \overline{\Omega} \times [0, 1] \to X_2'$ is compact and continuous. We first assert that R_2 is relatively compact for $\lambda \in [0, 1]$. In fact, there exists $l > 0$ such that $\overline{\Omega} \subset \{x \in X : ||x||_X \le l\}$. Again, since h is a g-Carathéodory function, there exists nonnegative function ψ_l satisfying $\int_0^\infty g(s)\psi_l(s)ds < \infty$ such that for a.e. $t \in [0,\infty)$, $|h(t, x(t), x'(t))| \le \psi_l(t)$ for $x \in \overline{\Omega}$. Then for any $x \in \overline{\Omega}$, $\lambda \in [0, 1]$, we have

$$\begin{aligned} |R_2(x,\lambda)(t)| &= |\int_0^t \phi_q(\frac{1}{c(s)})\phi_q[\int_s^\infty \lambda g(\tau)(h(\tau, x(\tau), x'(\tau)) - (Q_2f)(\tau))d\tau]ds| \\ &\le \int_0^\infty \phi_q(\frac{1}{c(s)})ds \cdot \phi_q(\int_0^\infty g(s)|\psi_l(s)|ds + \int_0^\infty g(s)|(Q_2f)(s)|ds) \end{aligned}$$

$$= \quad ||\phi_q(\frac{1}{c})||_1 \cdot \phi_q(||\psi_l||_Z + ||Q_2 f||_Z) =: L_3 < \infty$$

and

$$\begin{aligned} |R_2'(x,\lambda)(t)| &= |\phi_q(\frac{1}{c(t)})\phi_q[\int_t^\infty \lambda g(s)(h(s,x(s),x'(s)) - (Q_2 f)(s))]ds| \\ &\le ||\phi_q(\frac{1}{c})||_\infty \cdot \phi_q(||\psi_l||_Z + ||Q_2 f||_Z) =: L_4 < \infty, \end{aligned}$$

that is, $R_2(\cdot,\lambda)\overline{\Omega}$ is uniformly bounded. Meanwhile, for any $t_1, t_2 \in [0,T]$ with T a positive constant, as in the proof of Lemma 3.3, we can also show that $R_2(\cdot,\lambda)\overline{\Omega}$ is equicontinuous on [0,T] and equiconvergent at infinity. Thus, Lemma 3.2 yields that $R_2(\cdot,\lambda)\overline{\Omega}$ is relatively compact. Since f is a g-Carathéodory function, the continuity of R_1 on $\overline{\Omega}$ follows from the Lebesgue dominated convergence theorem.

Define $P_2 : X \to X_1'$ by $(P_2 x)(t) = x(0)$. Similar to the proof of Lemma 3.3, we can check that the conditions (2.3) and (2.4) are satisfied. Therefore, N_λ^2 is M_2-compact in $\overline{\Omega}$. $\square$

4. Existence result for (1.1)-(1.2)

Theorem 4.1. *If f is a L^1-Carathéodory function and suppose that*

(G1) *there exists a constant $A > 0$ such that*

$$\sum_{i=1}^n \mu_i \int_0^{\xi_i} \phi_q(\frac{1}{c(s)})\phi_q(\int_s^\infty f(\tau, x(\tau), x'(\tau))d\tau)ds \neq 0 \tag{4.1}$$

for $x \in \text{dom}M_1 \setminus \ker M_1$ with $|x(t)| > A$ on $t \in [0,\infty)$;

(G2) *there exist functions $\alpha, \beta, \gamma \in L^1[0,\infty)$ such that*

$$|f(t,x,y)| \le \alpha(t)|x|^{p-1} + \beta(t)|y|^{p-1} + \gamma(t), \quad \forall (x,y) \in \mathbb{R}^2, \quad a.e.\ t \in [0,\infty), \tag{4.2}$$

here denote $\alpha_1 = ||\alpha||_1$, $\beta_1 = ||\beta||_1$, $\gamma_1 = ||\gamma||_1$;

(G3) *there exist a constant $B > 0$ such that either*

$$b \cdot \sum_{i=1}^n \mu_i \int_0^{\xi_i} \phi_q(\frac{1}{c(s)})\phi_q(\int_s^\infty f(\tau, b, 0)d\tau)ds < 0 \tag{4.3}$$

or

$$b \cdot \sum_{i=1}^n \mu_i \int_0^{\xi_i} \phi_q(\frac{1}{c(s)})\phi_q(\int_s^\infty f(\tau, b, 0)d\tau)ds > 0 \tag{4.4}$$

for all $b \in \mathbb{R}$ with $|b| > B$.

Then the BVP(1.1)-(1.2) has at least one solution provided

$$2^{q-2}\beta_1^{q-1}||\phi_q(\frac{1}{c})||_\infty + 2^{2(q-2)}\alpha_1^{q-1}||\phi_q(\frac{1}{c})||_1 < 1 \quad \textit{for} \ \ p < 2, \tag{4.5}$$

$$\beta_1^{q-1}||\phi_q(\frac{1}{c})||_\infty + \alpha_1^{q-1}||\phi_q(\frac{1}{c})||_1 < 1 \quad for \;\; p \geq 2. \tag{4.6}$$

Before the proof of the main result, we first prove two lemmas.

Lemma 4.1. $U_1 = \{x \in \mathrm{dom} M_1 : M_1 x = N_\lambda^1 x$ *for some* $\lambda \in (0,1)\}$ *is bounded.*

Proof. Since $N_\lambda^1 x \in \mathrm{Im} M_1 = \ker Q_1$ for $x \in U_1$, $Q_1 N^1 x = 0$. It follows from (G1) that there exists $t_0 \in [0,\infty)$ such that $|x(t_0)| \leq A$. Now, $|x(t)| = |x(t_0) + \int_{t_0}^t x'(s)ds| \leq A + ||x'||_1$, that is,

$$||x||_\infty \leq A + ||x'||_1. \tag{4.7}$$

Also,

$$x'(t) = -\phi_q(\frac{1}{c(t)})\phi_q(\int_t^\infty \lambda f(s, x(s), x'(s))ds).$$

In the case $1 < p < 2$, by (G2) and Proposition 2.1, one gets

$$\begin{aligned}
||x'||_\infty &= \sup_{t\in[0,\infty)} |\phi_q(\frac{1}{c(t)})\phi_q(\int_t^\infty \lambda f(s, x(s), x'(s))ds)| \\
&\leq ||\phi_q(\frac{1}{c})||_\infty \cdot \phi_q[\alpha_1||x||_\infty^{p-1} + \beta_1||x'||_\infty^{p-1} + \gamma_1] \\
&\leq ||\phi_q(\frac{1}{c})||_\infty \cdot 2^{q-2}[\phi_q(\alpha_1||x||_\infty^{p-1} + \gamma_1) + \beta_1^{q-1}||x'||_\infty] \\
&\leq ||\phi_q(\frac{1}{c})||_\infty \cdot 2^{q-2}[2^{q-2}(\alpha_1^{q-1}||x||_\infty + \gamma_1^{q-1}) + \beta_1^{q-1}||x'||_\infty].
\end{aligned}$$

Noticing (4.5), one arrives at

$$||x'||_\infty \leq \frac{2^{2(q-2)}(\alpha_1^{q-1}||x||_\infty + \gamma_1^{q-1})||\phi_q(\frac{1}{c})||_\infty}{1 - 2^{q-2}\beta_1^{q-1}||\phi_q(\frac{1}{c})||_\infty} =: W_1 + W_2||x||_\infty, \tag{4.8}$$

where $W_1 = \frac{2^{2(q-2)}\gamma_1^{q-1}||\phi_q(\frac{1}{c})||_\infty}{1-2^{q-2}\beta_1^{q-1}||\phi_q(\frac{1}{c})||_\infty}$, $W_2 = \frac{2^{2(q-2)}\alpha_1^{q-1}||\phi_q(\frac{1}{c})||_\infty}{1-2^{q-2}\beta_1^{q-1}||\phi_q(\frac{1}{c})||_\infty}$.

$$\begin{aligned}
||x'||_1 &= \int_0^\infty |\phi_q(\frac{1}{c(t)})\phi_q(\int_t^\infty \lambda f(s, x(s), x'(s))ds)|dt \\
&\leq ||\phi_q(\frac{1}{c})||_1 \cdot \phi_q[\alpha_1||x||_\infty^{p-1} + \beta_1||x'||_\infty^{p-1} + \gamma_1] \\
&\leq ||\phi_q(\frac{1}{c})||_1 \cdot 2^{q-2}[\phi_q(\alpha_1||x||_\infty^{p-1} + \gamma_1) + \beta_1^{q-1}||x'||_\infty] \\
&\leq ||\phi_q(\frac{1}{c})||_1 \cdot 2^{q-2}[2^{q-2}(\alpha_1^{q-1}||x||_\infty + \gamma_1^{q-1}) + \beta_1^{q-1}(W_1 + W_2||x||_\infty)] \\
&\leq ||\phi_q(\frac{1}{c})||_1 \cdot 2^{q-2}[(2^{q-2}\alpha_1^{q-1} + W_2\beta_1^{q-1})||x||_\infty + (2^{q-2}\gamma_1^{q-1} + W_1\beta_1^{q-1})] \\
&=: W_3 + W_4||x||_\infty, \qquad (4.9)
\end{aligned}$$

where $W_3 = 2^{q-2}(2^{q-2}\gamma_1^{q-1} + W_1\beta_1^{q-1})||\phi_q(\frac{1}{c})||_1$, $W_4 = 2^{q-2}(2^{q-2}\alpha_1^{q-1} + W_2\beta_1^{q-1})||\phi_q(\frac{1}{c})||_1$.
Thus, from (4.7) and (4.9), we have $\quad ||x||_\infty \leq A + W_3 + W_4||x||_\infty$.

In view of (4.5), we can see $W_4 = \frac{2^{2(q-2)}\alpha_1^{q-1}||\phi_q(\frac{1}{c})||_1}{1-2^{q-2}\beta_1^{q-1}||\phi_q(\frac{1}{c})||_\infty} < 1$, then $||x||_\infty \leq \frac{A+W_3}{1-W_4} =: W_5$ and $||x'||_\infty \leq W_1 + W_2W_5 =: W_6$.

Similarly, in the case $p \geq 2$, it follows that

$$||x'||_\infty \leq ||\phi_q(\frac{1}{c})||_\infty \cdot [\alpha_1^{q-1}||x||_\infty + \beta_1^{q-1}||x'||_\infty + \gamma_1^{q-1}].$$

Again,

$$||x'||_\infty \leq \frac{(\gamma_1^{q-1} + \alpha_1^{q-1}||x||_\infty)||\phi_q(\frac{1}{c})||_\infty}{1-\beta_1^{q-1}||\phi_q(\frac{1}{c})||_\infty} =: V_1 + V_2||x||_\infty,$$

where $V_1 = \frac{\gamma_1^{q-1}||\phi_q(\frac{1}{c})||_\infty}{1-\beta_1^{q-1}||\phi_q(\frac{1}{c})||_\infty}$, $V_2 = \frac{\alpha_1^{q-1}||\phi_q(\frac{1}{c})||_\infty}{1-\beta_1^{q-1}||\phi_q(\frac{1}{c})||_\infty}$.

$$\begin{aligned} ||x'||_1 &\leq ||\phi_q(\frac{1}{c})||_1 \cdot [\alpha_1^{q-1}||x||_\infty + \beta_1^{q-1}||x'||_\infty + \gamma_1^{q-1}]. \\ &\leq ||\phi_q(\frac{1}{c})||_1 \cdot [(\alpha_1^{q-1} + V_2\beta_1^{q-1})||x||_\infty + (\gamma_1^{q-1} + V_1\beta_1^{q-1})] =: V_3 + V_4||x||_\infty, \end{aligned}$$

where $V_3 = (\gamma_1^{q-1} + V_1\beta_1^{q-1})||\phi_q(\frac{1}{c})||_1$, $V_4 = (\alpha_1^{q-1} + V_2\beta_1^{q-1})||\phi_q(\frac{1}{c})||_1$.

Thus, $||x||_\infty \leq A + V_3 + V_4||x||_\infty$, then $||x||_\infty \leq \frac{A+V_3}{1-V_4} := V_5$ and $||x'||_\infty \leq V_1 + V_2V_5 =: V_6$.

Therefore, U_1 is bounded. $\square$

Lemma 4.2. *If $U_2 = \{x \in \ker M_1 : -\lambda x + (1-\lambda)JQ_1N^1x = 0, \lambda \in [0,1]\}$, where $J : \mathrm{Im}Q_1 \to \ker M_1$ is a homomorphism, then U_2 is bounded.*

Proof. Define $J : \mathrm{Im}Q_1 \to \ker M_1$ by $J(b\omega_1(t)) = b$. Then for all $b \in U_2$,

$$\lambda b = (1-\lambda)\phi_p(\sum_{i=1}^n \mu_i \int_0^{\xi_i} \phi_q(\frac{1}{c(s)})\phi_q(\int_s^\infty f(\tau,b,0)d\tau)ds).$$

If $\lambda = 1$, then $b = 0$. In the case $\lambda \in [0,1)$, if $|b| > B$, then by (4.3), we have

$$0 \leq \lambda b^2 = (1-\lambda)b\phi_p(\sum_{i=1}^n \mu_i \int_0^{\xi_i} \phi_q(\frac{1}{c(s)})\phi_q(\int_s^\infty f(\tau,b,0)d\tau)ds) < 0,$$

which is a contradiction. Thus, $||x||_X = |b| \leq B$, $\forall x \in U_2$, that is, U_2 is bounded. $\square$

Proof of Theorem 4.1. Let $U \supset \overline{U}_1 \cup \overline{U}_2$ be a bounded and open set, then from Lemmas 4.1 and 4.2, we can obtain

(i) $M_1x \neq N_\lambda^1 x$ for all $(x,\lambda) \in [\mathrm{dom}M_1 \cap \partial U] \times (0,1)$;

(ii) Let $H(x,\lambda) = -\lambda x + (1-\lambda)JQ_1N^1x$, J is defined as in Lemma 4.2, we can see that $H(x,\lambda) \neq 0$, $\forall x \in \mathrm{dom}M \cap \partial U$. As a result, the homotopy invariance of Brouwer degree implies

$$\begin{aligned} \deg\{JQ_1N^1\mid_{\overline{U}\cap\ker M_1}, U\cap\ker M_1, 0\} &= \deg\{H(\cdot,0), U\cap\ker M_1, 0\} \\ &= \deg\{H(\cdot,1), U\cap\ker M_1, 0\} \\ &= \deg\{-I, U\cap\ker M_1, 0\} \neq 0. \end{aligned}$$

Theorem 2.1 yields that $M_1x = N^1x$ has at least one solution. The proof is completed. $\square$

Remark 4.1. When the second part of condition (G3) holds, we choose $\tilde{U}_2 = \{x \in \ker M_1 : \lambda x + (1-\lambda)JQ_1N^1x = 0, \lambda \in [0,1]\}$ and take homotopy $\tilde{H}(x,\lambda) = \lambda x + (1-\lambda)JQ_1N^1x$. By a similar argument, we can also complete the proof.

Example 4.1. Consider

$$\begin{cases} (e^{t+1}\phi_3(x'(t)))' = f(t, x(t), x'(t)), \qquad t \in (0,\infty), \\ x(0) = 2ex(\frac{1}{4}) + (1-2e)x(3), \quad \lim\limits_{t\to+\infty} e^{t+1}\phi_p(x'(t)) = 0. \end{cases} \tag{4.10}$$

Corresponding to the BVP (1.1)-(1.2), we have $p = 3$, $q = \frac{3}{2}$, $c(t) = e^{t+1}$, $\mu_1 = 2e$, $\mu_2 = 1-2e$, $\xi_1 = \frac{1}{4}$, $\xi_2 = 3$ and

$$f(t,u,v) = \frac{1}{1+t}e^{-t-1}u^2 + e^{-t-2}\sin t \cdot v^2 + \frac{1}{t^2+1}.$$

It is easy to verify that (A1)-(A2) hold. Let $\alpha(t) = e^{-t-1}$, $\beta(t) = e^{-t-2}$, $\gamma(t) = \frac{1}{t^2+1}$, then $\alpha_1 = \frac{1}{e}$, $\beta_1 = \frac{1}{e^2}$, $||\phi_q(\frac{1}{c})||_\infty = \frac{1}{\sqrt{e}}$, $||\phi_q(\frac{1}{c})||_1 = \frac{2}{\sqrt{e}}$. Also, we can check that (G1)-(G3) and (4.6) are all satisfied. Thus, the BVP (4.10) has at least one solution, by using Theorem 4.1.

5. Existence result for (1.4)-(1.5)

Theorem 5.1. *If h is a g-Carathéodory function and suppose that*

(H1) *there exists a constant $A' > 0$ such that*

$$\int_0^\infty g(s)\int_0^s \phi_q(\frac{1}{c(\tau)})\phi_q(\int_\tau^\infty g(r)h(r,x(r),x'(r))dr)d\tau ds \neq 0 \tag{5.1}$$

for $x \in \mathrm{dom}M_2 \backslash \ker M_2$ with $|x(t)| > A'$ on $t \in [0,\infty)$;

(H2) *there exist nonnegative functions $\delta, \zeta, \eta \in Z$ such that*

$$|h(t,u,v)| \leq \delta(t)|u|^{p-1} + \zeta(t)|v|^{p-1} + \eta(t), \quad \forall (u,v) \in \mathbb{R}^2, \quad a.e.\ t \in [0,\infty), \tag{5.2}$$

here denote $\delta_1 = ||\delta||_Z$, $\zeta_1 = ||\zeta||_Z$, $\eta_1 = ||\eta||_Z$;

(H3) *there exists a constant $B' > 0$ such that either*

$$d \cdot \int_0^\infty g(s)\int_0^s \phi_q(\frac{1}{c(\tau)})\phi_q(\int_\tau^\infty g(r)h(r,d,0)dr)d\tau ds < 0 \tag{5.3}$$

or

$$d \cdot \int_0^\infty g(s)\int_0^s \phi_q(\frac{1}{c(\tau)})\phi_q(\int_\tau^\infty g(r)h(r,d,0)dr)d\tau ds > 0 \tag{5.4}$$

for all $d \in \mathbb{R}$ with $|d| > B'$.

Then the BVP(1.4)-(1.5) has at least one solution on $[0,\infty)$ provided

$$\max\{2^{q-2}\zeta_1^{q-1}||\phi_q(\frac{1}{c})||_\infty,\ \frac{2^{2(q-2)}\delta_1^{q-1}||\phi_q(\frac{1}{c})||_1}{1-2^{q-2}\zeta_1^{q-1}||\phi_q(\frac{1}{c})||_\infty}\} < 1 \quad for\ \ p < 2, \tag{5.5}$$

$$\max\{\zeta_1^{q-1}||\phi_q(\frac{1}{c})||_\infty,\ \frac{\delta_1^{q-1}||\phi_q(\frac{1}{c})||_1}{1-\zeta_1^{q-1}||\phi_q(\frac{1}{c})||_\infty}\}<1 \quad for \quad p\geq 2. \tag{5.6}$$

Proof. Let $\Omega_1=\{x\in \mathrm{dom}M_2: M_2x=N_\lambda^2x$ for some $\lambda\in(0,1)\}$. As in the proof of Lemma 4.1, for $x\in\Omega_1$, $N_\lambda^2x\in \mathrm{Im}M_2=\ker Q_2$, then $Q_2N^2x=0$, i.e.,

$$\int_0^\infty g(s)\int_0^s \phi_q(\frac{1}{c(\tau)})\phi_q(\int_\tau^\infty g(r)h(r,x(r),x'(r))dr)d\tau ds=0.$$

It follows from (H1) that there exists $t_0\in[0,\infty)$ such that $|x(t_0)|\leq A'$. Thus, we can obtain

$$||x||_\infty\leq A'+||x'||_1. \tag{5.7}$$

Also,

$$x'(t)=\phi_q(\frac{1}{c(t)})\phi_q(\int_t^\infty \lambda g(s)h(s,x(s),x'(s))ds).$$

In the case $1<p<2$, by (H2), Proposition 2.1 and (5.5), one gets

$$||x'||_\infty\leq\frac{2^{2(q-2)}(\delta_1^{q-1}||x||_\infty+\eta_1^{q-1})||\phi_q(\frac{1}{c})||_\infty}{1-2^{q-2}\zeta_1^{q-1}||\phi_q(\frac{1}{c})||_\infty}=:W_1'+W_2'||x||_\infty, \tag{5.8}$$

where $W_1'=\frac{2^{2(q-2)}\eta_1^{q-1}||\phi_q(\frac{1}{c})||_\infty}{1-2^{q-2}\zeta_1^{q-1}||\phi_q(\frac{1}{c})||_\infty}$, $W_2'=\frac{2^{2(q-2)}\delta_1^{q-1}||\phi_q(\frac{1}{c})||_\infty}{1-2^{q-2}\zeta_1^{q-1}||\phi_q(\frac{1}{c})||_\infty}$.

$$\begin{aligned}
||x'||_1 &= \int_0^\infty |\phi_q(\frac{1}{c(t)})\phi_q(\int_t^\infty \lambda g(s)h(s,x(s),x'(s))ds|dt\\
&\leq ||\phi_q(\frac{1}{c})||_1\cdot\phi_q[\delta_1||x||_\infty^{p-1}+\zeta_1||x'||_\infty^{p-1}+\eta_1]\\
&\leq ||\phi_q(\frac{1}{c})||_1\cdot 2^{q-2}[(2^{q-2}\delta_1^{q-1}+W_2'\zeta_1^{q-1})||x||_\infty+(2^{q-2}\eta_1^{q-1}+W_1'\zeta_1^{q-1})]\\
&=: W_3'+W_4'||x||_\infty, \qquad (5.9)
\end{aligned}$$

where $W_3'=2^{q-2}(2^{q-2}\eta_1^{q-1}+W_1'\zeta_1^{q-1})||\phi_q(\frac{1}{c})||_1$, $W_4'=2^{q-2}(2^{q-2}\delta_1^{q-1}+W_2'\zeta_1^{q-1})||\phi_q(\frac{1}{c})||_1$.
Thus, from (5.7) and (5.9), we have $||x||_\infty\leq\frac{A'+W_3'}{1-W_4'}=:W_5'$. Then, $||x'||_\infty\leq W_1'+W_2'W_5'=:W_6'$.

Similarly, for $p\geq2$, it follows that

$$||x'||_\infty\leq\frac{(\eta_1^{q-1}+\delta_1^{q-1}||x||_\infty)||\phi_q(\frac{1}{c})||_\infty}{1-\zeta_1^{q-1}||\phi_q(\frac{1}{c})||_\infty}=:V_1'+V_2'||x||_\infty,$$

where $V_1'=\frac{\eta_1^{q-1}||\phi_q(\frac{1}{c})||_\infty}{1-\zeta_1^{q-1}||\phi_q(\frac{1}{c})||_\infty}$, $V_2'=\frac{\delta_1^{q-1}||\phi_q(\frac{1}{c})||_\infty}{1-\zeta_1^{q-1}||\phi_q(\frac{1}{c})||_\infty}$.

$$\begin{aligned}
||x'||_1 &\leq ||\phi_q(\frac{1}{c})||_1\cdot[\delta_1^{q-1}||x||_\infty+\zeta_1^{q-1}||x'||_\infty+\eta_1^{q-1}].\\
&\leq ||\phi_q(\frac{1}{c})||_1\cdot[(\delta_1^{q-1}+V_2'\zeta_1^{q-1})||x||_\infty+(\eta_1^{q-1}+V_1'\zeta_1^{q-1})]=:V_3'+V_4'||x||_\infty,
\end{aligned}$$

where $V_3'=(\eta_1^{q-1}+V_1'\zeta_1^{q-1})||\phi_q(\frac{1}{c})||_1$, $V_4'=(\delta_1^{q-1}+V_2'\zeta_1^{q-1})||\phi_q(\frac{1}{c})||_1$.
Thus, $||x||_\infty\leq\frac{A'+V_3'}{1-V_4'}=:V_5'$ and $||x'||_\infty\leq V_1'+V_2'V_5'=:V_6'$. As a result, Ω_1 is bounded.

Define $\Omega_2 = \{x \in \ker M_2 : -\mu x + (1-\mu)JQ_2N^2x = 0, \mu \in [0,1]\}$, where $J : \mathrm{Im}Q_2 \to \ker M_2$ is a homomorphism defined by $J(d) = d$. As in Lemma 4.2, we can prove that Ω_2 is bounded.

Let $\Omega \supset \overline{\Omega}_1 \cup \overline{\Omega}_2$ be a bounded and open set. Then $M_2x \neq N^2_\lambda x$, $\forall (x,\lambda) \in (\mathrm{dom}M_2 \cap \partial\Omega) \times (0,1)$. Define a homotopy operator

$$T(x,\mu) = -\mu x + (1-\mu)JQ_2N^2x.$$

We can see that $T(x,\mu) \neq 0$, $\forall x \in \mathrm{dom}M_2 \cap \partial\Omega$. Therefore,

$$\begin{aligned} \deg\{JQ_2N^2 \mid_{\overline{\Omega}\cap \ker M_2}, \Omega \cap \ker M_2, 0\} &= \deg\{T(\cdot,0), \Omega \cap \ker M_2, 0\} \\ &= \deg\{T(\cdot,1), \Omega \cap \ker M_2, 0\} \\ &= \deg\{-I, \Omega \cap \ker M_2, 0\} \neq 0. \end{aligned}$$

Theorem 2.1 implies that $M_2x = N^2x$ has at least one solution. The proof is completed. $\square$

Remark 5.1. When the second part of condition (H3) holds, we may choose $\tilde{\Omega}_2 = \{x \in \ker M_2 : \mu x + (1-\mu)JQ_2N^2x = 0, \mu \in [0,1]\}$ and take homotopy $\tilde{T}(x,\mu) = \mu x + (1-\mu)JQ_2N^2x$.

Remark 5.2. Under the multi-point boundary conditions, we can obtain the existence of solutions on a half-line by assume the nonlinear function f is L^1-Carathéodory. When the boundary conditions involved in the integral condition, however, this assumption on the nonlinear term is invalid if the domain is unbounded. In this paper, we overcome this difficulty by introducing the definition of g-Carathéodory function and multiplying the g-Carathéodory function h by the function $g \in L^1[0,\infty)$ in the equation (1.4).

Example 5.1. Consider

$$\begin{cases} 3e^t(e^t\phi_3(x'(t)))' + h(t,x(t),x'(t)) = 0, \quad t \in (0,\infty), \\ x(0) = \int_0^\infty e^{-t}x(t)dt, \quad \lim\limits_{t\to+\infty} 3e^t\phi_3(x'(t)) = 0. \end{cases} \tag{5.10}$$

Corresponding to the BVP (1.4)-(1.5), we have $p = 3$, $c(t) = 3e^t$, $g(t) = e^{-t}$ and $h(t,u,v) = te^{-2t}u^2 + e^{-t}v^2$. It is easy to verify that (A1) holds. Let $\delta(t) = te^{-2t}$, $\zeta(t) = e^{-t}$, then $\delta_1 = \frac{1}{9}$, $\zeta_1 = \frac{1}{2}$, $||\phi_q(\frac{1}{c})||_\infty = \frac{1}{\sqrt{3}}$, $||\phi_q(\frac{1}{c})||_1 = \frac{2}{\sqrt{3}}$. Also, we can check that (H1)-(H3) and (5.6) are all satisfied. Thus, thanks to Theorem 5.1, the BVP (5.10) has at least one solution.

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