Electronic Journal of Qualitative Theory of Differential Equations
2015, No. **5**, 1–13; http://www.math.u-szeged.hu/ejqtde/

On a cyclic system of m difference equations having exponential terms

Garyfalos Papaschinopoulos✉[1], **Nikos Psarros**[1] and
Kyriakos B. Papadopoulos[2]

[1]School of Engineering, Democritus University of Thrace, Xanthi, 67100, Greece
[2]American University of the Middle East, Kuwait

Received 8 January 2015, appeared 15 February 2015
Communicated by Stevo Stević

Abstract. In this paper we study the asymptotic behavior of the positive solutions of a cyclic system of the following m difference equations:

$$x_{n+1}^{(i)} = a_i x_n^{(i+1)} + b_i x_{n-1}^{(i)} e^{-x_n^{(i+1)}}, \qquad i = 1, 2, \dots, m-1,$$
$$x_{n+1}^{(m)} = a_m x_n^{(1)} + b_m x_{n-1}^{(m)} e^{-x_n^{(1)}},$$

where $n = 0, 1, \dots$, and a_i, b_i, $i = 1, 2, \dots, m$ are positive constants and the initial values $x_{-1}^{(i)}, x_0^{(i)}$, $i = 1, 2, \dots, m$ are positive numbers.

Keywords: difference equations, boundedness, persistence, asymptotic behavior.

2010 Mathematics Subject Classification: 39A10.

1 Introduction

In [14] the authors obtained results concerning the global behavior of the positive solutions for the difference equation:

$$x_{n+1} = a x_n + b x_{n-1} e^{-x_n}, \qquad n = 0, 1, \dots \tag{1.1}$$

where a, b are positive constants and the initial values x_{-1}, x_0 are positive numbers. This equation can be considered as a biological model, since it arises from models studying the amount of litter in a perennial grassland.

In addition, in [23] the authors studied analogous results for the system of difference equations:

$$x_{n+1} = a y_n + b x_{n-1} e^{-y_n}, \qquad y_{n+1} = c x_n + d y_{n-1} e^{-x_n} \tag{1.2}$$

where a, b, c, d are positive constants and the initial values x_{-1}, x_0, y_{-1}, y_0 are also positive numbers. For $a = c$ and $b = d$ the system is symmetric, so it is a close to symmetric system.

✉Corresponding author. Email: gpapas@env.duth.gr

Studying symmetric and close to symmetric systems of difference equations is an area of a considerable recent interest (see, for example, [5,7,8,10,11,22–24,31–35,38–40]).

In this paper we obtain results concerning the behavior of the positive solutions for the following cyclic system of difference equations:

$$\begin{aligned} x_{n+1}^{(i)} &= a_i x_n^{(i+1)} + b_i x_{n-1}^{(i)} e^{-x_n^{(i+1)}}, \qquad i = 1, 2, \ldots, m-1, \\ x_{n+1}^{(m)} &= a_m x_n^{(1)} + b_m x_{n-1}^{(m)} e^{-x_n^{(1)}}, \end{aligned} \tag{1.3}$$

$n = 0, 1, \ldots$, and a_i, b_i, $i = 1, 2, \ldots, m$ are positive constants and the initial values $x_{-1}^{(i)}$, $x_0^{(i)}$, $i = 1, 2, \ldots, m$ are positive numbers. More precisely, we study the existence of the unique nonnegative equilibrium of (1.3). In addition, we investigate the boundedness and the persistence of the positive solutions of system (1.3). Finally, we investigate the convergence of the positive solutions of (1.3) to the unique nonnegative equilibrium. We note that if $a_1 = a_2 = \cdots = a_m = a$, $b_1 = b_2 = \cdots = b_m = b$ and $(x_n^1, x_n^2, \ldots, x_n^m)$ is a solution of (1.3) and $x_{-1}^{(1)} = x_{-1}^{(2)} = \cdots = x_{-1}^{(m)}$, $x_0^{(1)} = x_0^{(2)} = \cdots = x_0^{(m)}$, then it is obvious that $x_n^{(1)} = x_n^{(2)} = \cdots = x_n^{(m)} = x_n$, $n = 0, 1, \ldots$, and so x_n is a solution of (1.1). Moreover, if $m = 2$ then system (1.3) reduces to system (1.2). Studying cyclic systems of difference equations has attracted some attention recently (see [16,36,37] and the related references therein).

Difference equations and systems of difference equations containing exponential terms have numerous potential applications in biology. A large number of papers dealing with such or related equations have been published. See for example, [2,14,18,20–23,26,29,41] and the references cited therein. We also note that since difference equations have many applications in applied sciences, there is a quite rich bibliography concerning theory and applications of difference equations (see, for example, [1–41] and the references cited therein).

2 Existence and uniqueness of a nonnegative equilibrium for (1.3).

In this section we study the existence and the uniqueness of the positive equilibrium of (1.3).

Proposition 2.1. *The following statements are true.*

I. *Suppose that*

$$a_i, b_i \in (0,1), \qquad a_i + b_i > 1, \qquad i = 1, 2, \ldots, m. \tag{2.1}$$

Then system (1.3) *has a unique positive equilibrium* $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_m)$.

II. *Consider that* a_i, b_i, $i = 1, 2, \ldots, m$ *are positive constants such that:*

$$a_i, b_i \in (0,1), \qquad a_i + b_i < 1. \tag{2.2}$$

Then, the zero equilibrium $(0, 0, \ldots, 0)$ *is the unique nonnegative equilibrium of system* (1.3).

Proof. **I**. We consider the functions $h_i \colon \mathbb{R}^+ \to \mathbb{R}^+$, $\mathbb{R}^+ = (0, \infty)$,

$$h_i(x) = \frac{a_i x}{1 - b_i e^{-x}}, \qquad i = 1, \ldots, m.$$

Then we define

$$\prod_{s=j}^{k} h_j = h_j \circ h_{j+1} \circ \cdots \circ h_k, \qquad k \geq j, \qquad \prod_{s=j+1}^{j} h_j = I,$$

where I is the identity function. If we set $x_{m+1} = x_1$ we consider the system of algebraic equations

$$x_i = h_i(x_{i+1}), \qquad i = 1, 2, \dots, m. \tag{2.3}$$

From (2.3) for a $j \in \{1, 2, \dots, m\}$ we get

$$\begin{aligned}
x_j &= h_j(x_{j+1}) = h_j \circ h_{j+1}(x_{j+2}) = \prod_{s=j}^{m} h_j \circ \prod_{s=1}^{j-1} h_s(x_j) = h_j \circ \prod_{s=j+1}^{m} h_j \circ \prod_{s=1}^{j-1} h_s(x_j) \\
&= \frac{a_j \prod_{s=j+1}^{m} h_s \circ \prod_{s=1}^{j-1} h_s(x_j)}{1 - b_j e^{-\prod_{k=j+1}^{m} h_k \circ \prod_{k=1}^{j-1} h_k(x_j)}} = \frac{a_j h_{j+1} \circ \prod_{s=j+2}^{m} h_s \circ \prod_{s=1}^{j-1} h_s(x_j)}{1 - b_j e^{-\prod_{k=j+1}^{m} h_k \circ \prod_{k=1}^{j-1} h_k(x_j)}} \\
&= \frac{a_j a_{j+1} \prod_{s=j+2}^{m} h_s \circ \prod_{s=1}^{j-1} h_s(x_j)}{\left(1 - b_j e^{-\prod_{k=j+1}^{m} h_k \circ \prod_{k=1}^{j-1} h_k(x_j)}\right)\left(1 - b_{j+1} e^{-\prod_{k=j+2}^{m} h_k \circ \prod_{k=1}^{j-1} h_k(x_j)}\right)} \\
&\vdots \\
&= \frac{\prod_{s=j}^{m-1} a_s h_m \circ \prod_{s=1}^{j-1} h_s(x_j)}{\prod_{s=j}^{m-1}\left(1 - b_s e^{-\prod_{k=j+1}^{m} h_k \circ \prod_{k=1}^{j-1} h_k(x_j)}\right)} \\
&= \frac{\prod_{s=j}^{m} a_s \prod_{s=1}^{j-1} h_s(x_j)}{\prod_{s=j}^{m-1}\left(1 - b_s e^{-\prod_{k=j+1}^{m} h_k \circ \prod_{k=1}^{j-1} h_k(x_j)}\right)\left(1 - b_m e^{-\prod_{k=1}^{j-1} h_k(x_j)}\right)} \\
&= \frac{\prod_{s=j}^{m} a_s h_1 \circ \prod_{s=2}^{j-1} h_s(x_j)}{\prod_{s=j}^{m-1}\left(1 - b_s e^{-\prod_{k=j+1}^{m} h_k \circ \prod_{k=1}^{j-1} h_k(x_j)}\right)\left(1 - b_m e^{-\prod_{k=1}^{j-1} h_k(x_j)}\right)} \\
&= \frac{a_1 \prod_{s=j}^{m} a_s \prod_{s=2}^{j-1} h_s(x_j)}{\prod_{s=j}^{m-1}\left(1 - b_s e^{-\prod_{k=j+1}^{m} h_k \circ \prod_{k=1}^{j-1} h_k(x_j)}\right)\left(1 - b_m e^{-\prod_{k=1}^{j-1} h_k(x_j)}\right)\left(1 - b_1 e^{-\prod_{k=2}^{j-1} h_k(x_j)}\right)} \\
&= \frac{a_1 \prod_{s=j}^{m} a_s \prod_{s=2}^{j-1} h_s(x_j)}{\prod_{s=j}^{m}\left(1 - b_s e^{-\prod_{k=j+1}^{m} h_k \circ \prod_{k=1}^{j-1} h_k(x_j)}\right)\left(1 - b_1 e^{-\prod_{k=2}^{j-1} h_k(x_j)}\right)}
\end{aligned}$$

$$= \frac{a_1 \prod_{s=j}^{m} a_s h_2 \circ \prod_{s=3}^{j-1} h_s(x_j)}{\prod_{s=j}^{m}\left(1 - b_s e^{-\prod_{k=j+1}^{m} h_k \circ \prod_{k=1}^{j-1} h_k(x_j)}\right)\left(1 - b_1 e^{-\prod_{k=2}^{j-1} h_k(x_j)}\right)}$$

$$= \frac{a_1 a_2 \prod_{s=j}^{m} a_s \prod_{s=3}^{j-1} h_s(x_j)}{\prod_{s=j}^{m}\left(1 - b_s e^{-\prod_{k=j+1}^{m} h_k \circ \prod_{k=1}^{j-1} h_k(x_j)}\right)\left(1 - b_1 e^{-\prod_{k=2}^{j-1} h_k(x_j)}\right)\left(1 - b_2 e^{-\prod_{k=3}^{j-1} h_k(x_j)}\right)}$$

$$\vdots$$

$$= \frac{x_j \prod_{s=1}^{m} a_s}{\prod_{s=j}^{m}\left(1 - b_s e^{-\prod_{k=s+1}^{m} h_k \circ \prod_{k=1}^{j-1} h_k(x_j)}\right)\prod_{s=1}^{j-1}\left(1 - b_s e^{-\prod_{k=s+1}^{j-1} h_k(x_j)}\right)}.$$

We consider the function

$$F_j(x) = \frac{\prod_{s=1}^{m} a_s}{\prod_{s=j}^{m}\left(1 - b_s e^{-\prod_{k=s+1}^{m} h_k \circ \prod_{k=1}^{j-1} h_k(x)}\right)\prod_{s=1}^{j-1}\left(1 - b_s e^{-\prod_{k=s+1}^{j-1} h_k(x)}\right)} - 1. \tag{2.4}$$

Since $h_k(0) = 0$ for $k = 1, 2, \ldots, m$, from (2.4) we can prove that

$$F_j(0) = \frac{\prod_{s=1}^{m} a_s}{\prod_{s=1}^{m}(1 - b_s)} - 1. \tag{2.5}$$

Then from (2.1) we have that $F_j(0) > 0$. Moreover, since $\lim_{x\to\infty} h_k(x) = \infty$, $k = 1, 2 \ldots, m$, from (2.1) and (2.4) we get

$$\lim_{x\to\infty} F_j(x) = \prod_{s=1}^{k} a_s - 1 < 0.$$

Therefore there exists an $\bar{x}_j$, $j = 1, 2, \ldots, m$ such that $F_j(\bar{x}_j) = 0$, $j = 1, 2, \ldots, m$. So, $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_m)$ is a positive equilibrium for (1.3). Moreover since $h'_k(x) = a_k \frac{1 - b_k e^{-x}(x+1)}{(1 - b_k e^{-x})^2}$ then from (2.1) and $e^{-x}(x+1) < 1$ we get $h'_k(x) > 0$, $k = 1, 2, \ldots, m$. Therefore for all $k = 1, 2, \ldots, m$ the functions h_k are increasing. Hence for all $j = 1, 2, \ldots, m$ the functions F_j are decreasing. This implies that $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_m)$ is the unique positive equilibrium for (1.3). This completes the proof of statement I.

II. From (2.2) and (2.5) we have $F_j(0) < 0$ for all $j = 1, 2, \ldots, m$. Since for all $j = 1, 2, \ldots, m$ F_j are decreasing functions it is obvious that the zero equilibrium is the only nonnegative equilibrium. This completes the proof of the proposition. □

3 Asymptotic behavior of the positive solutions of (1.3)

In this section we study the boundedness and persistence of the positive solutions of (1.3) and the convergence of the positive solutions of (1.3) to the unique nonnegative equilibrium.

Proposition 3.1.

I. *Suppose that*

$$a_i, b_i \in (0,1), \qquad i = 1,2,\dots,m. \tag{3.1}$$

Then every positive solution of (1.3) *is bounded.*

II. *Consider that a_i, b_i, $i = 1,2,\dots,m$ are positive constants such that* (2.1) *holds. Then, every positive solution of* (1.3) *persists.*

Proof. **I.** Let $\left(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(m)}\right)$ be an arbitrary solution of (1.3) and

$$M = \max\left\{x_j^{(i)}, \quad \ln\left(\frac{1}{1-a_i}\right), \quad i = 1,2,\dots,m, \; j = -1,0\right\}.$$

Then arguing as in Lemma 1 of [14] and Theorem 3.1 of [22] we can prove that

$$x_n^{(i)} \le M, \qquad i = 1,2,\dots,m, \;\; n = 1,2,\dots$$

and so the solution $\left(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(m)}\right)$ is bounded.

II. Let

$$r = \min\left\{x_j^{(i)}, \;\; z_i \;\; i = 1,2,\dots,m, \; j = -1,0\right\}$$

where $z_i = \ln(\frac{b_i}{1-a_i})$. Arguing as in the proof of Proposition 3.1 of [23] we have the following:

If for $i = 2,3,\dots,m$

$$x_0^{(i)} \le z_{i-1},$$

then

$$x_1^{(i-1)} \ge \min\left\{x_0^{(i)}, x_{-1}^{(i-1)}\right\}.$$

In addition, if

$$x_0^{(i)} > z_{i-1}, \qquad x_{-1}^{(i-1)} \le z_{i-1},$$

we take

$$x_1^{(i-1)} > x_{-1}^{(i-1)}.$$

Finally, if

$$x_0^{(i)} > z_{i-1}, \qquad x_{-1}^{(i-1)} > z_{i-1},$$

we get

$$x_1^{(i-1)} > z_{i-1}.$$

So, we have that

$$x_1^{(i-1)} \ge r.$$

We consider now the case

$$x_0^{(1)} \le z_m,$$

then

$$x_1^{(m)} \ge \min\left\{x_0^{(1)}, x_{-1}^{(m)}\right\}.$$

Furthermore, if

$$x_0^{(1)} > z_m, \qquad x_{-1}^{(m)} \leq z_m,$$

we take

$$x_1^{(m)} > x_{-1}^{(m)}.$$

Finally, if

$$x_0^{(1)} > z_m, \qquad x_{-1}^{(m)} > z_m,$$

we get

$$x_1^{(m)} > z_m.$$

So, we have that

$$x_1^{(m)} \geq r.$$

Arguing as above and using the method of induction we can prove that:

$$x_n^{(i)} \geq r, \qquad n = 1, 2, \ldots, \quad i = 1, 2 \ldots, m.$$

This completes the proof of the proposition. □

In the following proposition we study the convergence of the positive equilibrium of (1.3) to the unique positive equilibrium.

Proposition 3.2. *Consider system* (1.3) *such that relations* (2.1) *hold. Suppose also that there exists a* $v \in \{1, 2, \ldots, m\}$ *such that*

$$a_j \leq a_v, \qquad b_j \leq \prod_{s=1, s\neq v}^{m} a_s, \qquad j = 1, 2, \ldots, m. \tag{3.2}$$

Then every positive solution of (1.3) *tends to unique positive equilibrium.*

Proof. Let $\left(x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(m)}\right)$ be an arbitrary solution of (1.3). From Proposition 3.1 there exists numbers l_i, L_i, $i = 1, 2, \ldots, m$, $0 < l_i < L_i < \infty$ such that

$$\liminf_{n\to\infty} x_n^{(i)} = l_i, \qquad \limsup_{n\to\infty} x_n^{(i)} = L_i, \qquad i = 1, 2, \ldots, m. \tag{3.3}$$

Moreover, since relations (2.1) hold, from Proposition 2.1 System (1.3) has a unique positive equilibrium $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_m)$.

First of all we prove that

$$l_i \leq L_i \leq \bar{x}_i, \qquad i = 1, 2, \ldots, m. \tag{3.4}$$

From (3.3) for every ϵ there exists a n_0 such that for every $n \geq n_0$

$$l_i - \epsilon \leq x_n^{(i)} \leq L_i + \epsilon, \qquad i = 1, 2, \ldots, m. \tag{3.5}$$

So, relations (1.3) and (3.5) imply that for $i = 1, 2, \ldots, m-1$ and $n = 0, 1, \ldots,$

$$x_{n+1}^{(i)} = a_i x_n^{(i+1)} + b_i x_{n-1}^{(i)} e^{-x_n^{(i+1)}} \leq a_i x_n^{(i+1)} + b_i (L_i + \epsilon) e^{-x_n^{(i+1)}}. \tag{3.6}$$

We set $g_{L_i+\epsilon}(x) = a_i x + b_i (L_i + \epsilon) e^{-x}$. Since $g''_{L_i+\epsilon}(x) = b_i (L_i + \epsilon) e^{-x} > 0$ we have

$$g_{L_i+\epsilon}(x) \leq \max\left\{g_{L_i+\epsilon}(l_{i+1} - \epsilon),\ g_{L_i+\epsilon}(L_{i+1} + \epsilon)\right\}, \qquad x \in [l_{i+1} - \epsilon,\ L_{i+1} + \epsilon]. \tag{3.7}$$

Moreover, from (3.5), (3.6) and (3.7) it follows that

$$x_{n+1}^{(i)} \leq g_{L_i+\epsilon}(x_n^{(i+1)}) \leq \max\left\{g_{L_i+\epsilon}(l_{i+1}-\epsilon),\ g_{L_i+\epsilon}(L_{i+1}+\epsilon)\right\},$$

which implies that

$$L_i \leq \max\left\{g_{L_i+\epsilon}(l_{i+1}-\epsilon),\ g_{L_i+\epsilon}(L_{i+1}+\epsilon)\right\}.$$

Then for $\epsilon \to 0$ it holds

$$L_i \leq \max\{g_{L_i}(l_{i+1}),\ g_{L_i}(L_{i+1})\}, \qquad i = 1,2,\ldots,m-1. \tag{3.8}$$

We claim that

$$L_i \leq g_{L_i}(L_{i+1}). \tag{3.9}$$

Since $g'_{L_i}(x) = a_i - b_i L_i e^{-x}$, for $x \geq \ln(\frac{b_i L_i}{a_i})$ we have $g'_{L_i}(x) > 0$. So, from (3.8) for $l_{i+1} \geq \ln(\frac{b_i L_i}{a_i})$ we have that (3.9) is true. For $l_{i+1} < \ln(\frac{b_i L_i}{a_i})$ we take

$$g_{L_i}(l_{i+1}) \leq g_{L_i}(0) = b_i L_i < L_i. \tag{3.10}$$

Then using (3.8) and (3.10), for $l_{i+1} < \ln(\frac{b_i L_i}{a_i})$ relation (3.9) is satisfied. Hence, our claim (3.9) is true. Then using (3.9) we take

$$L_i \leq a_i L_{i+1} + b_i L_i e^{-L_{i+1}}, \qquad i = 1,2.\ldots,m-1$$

and so we get

$$L_i \leq \frac{a_i L_{i+1}}{1 - b_i e^{-L_{i+1}}}, \qquad i = 1,2.\ldots,m-1. \tag{3.11}$$

Similarly we can prove that

$$L_m \leq \frac{a_m L_1}{1 - b_m e^{-L_1}}. \tag{3.12}$$

Therefore since the functions h_i, $i = 1,2,\ldots,m$ defined in the proof of Proposition 2.1 are increasing, then from relations (3.11), (3.12) and arguing as in Proposition 2.1 we can prove that

$$F_i(L_i) \geq 0, \qquad i = 1,2,\ldots,m. \tag{3.13}$$

Then from (3.13) and since the functions F_i, $i = 1,2,\ldots,m$ are decreasing and from Proposition 2.1 $F(\bar{x}_i) = 0$, $i = 1,2,\ldots,m$ we take relations (3.4).

We prove now that

$$\bar{x}_j \leq l_j \leq L_j, \qquad j = 1,2,\ldots,m. \tag{3.14}$$

Since $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_m)$ is the unique positive equilibrium of (1.3) we have that $\bar{x}_i$, $i = 1,2,\ldots,m$ satisfy system (2.3). Then by setting $\bar{x}_{m+1} = \bar{x}_1$ we have

$$\bar{x}_{j+1} = \ln\left(\frac{b_j \bar{x}_j}{\bar{x}_j - a_j \bar{x}_{j+1}}\right), \qquad j = 1,2,\ldots,m.$$

Then since $b_j < 1$ we have,

$$\bar{x}_{j+1} \leq \frac{b_j \bar{x}_j}{\bar{x}_j - a_j \bar{x}_{j+1}} - 1 \leq \frac{a_j \bar{x}_{j+1}}{\bar{x}_j - a_j \bar{x}_{j+1}}, \qquad j = 1,2,\ldots,m,$$

and so

$$\bar{x}_j - a_j \bar{x}_{j+1} \leq a_j, \qquad j = 1,2,\ldots,m-1, \qquad \bar{x}_m - a_m \bar{x}_1 \leq a_m. \tag{3.15}$$

From (3.15) for a $j \in \{1, 2, \dots, m\}$ it holds

$$\begin{aligned} &\bar{x}_j - a_j \bar{x}_{j+1} \leq a_j, \quad \bar{x}_{j+1} - a_{j+1}\bar{x}_{j+2} \leq a_{j+1}, \dots, \quad \bar{x}_m - a_m \bar{x}_1 \leq a_m, \\ &\bar{x}_1 - a_1 \bar{x}_2 \leq a_1, \quad \bar{x}_2 - a_2 \bar{x}_3 \leq a_2, \dots, \quad \bar{x}_{j-1} - a_{j-1}\bar{x}_j \leq a_{j-1}. \end{aligned} \tag{3.16}$$

From the first two relations of (3.16) we get

$$\bar{x}_j - a_j a_{j+1} \bar{x}_{j+2} \leq a_j + a_j a_{j+1}$$

and working similarly to (3.16) we can prove that

$$\bar{x}_j \leq \frac{\sum_{s=j}^{m} \left(\prod_{i=j}^{s} a_i \right) + \prod_{s=j}^{m} a_s \left(\sum_{r=1}^{j-1} \left(\prod_{w=1}^{r} a_w \right) \right)}{1 - \prod_{i=1}^{m} a_i}. \tag{3.17}$$

Then from (3.2) and (3.17) we get

$$\bar{x}_j \leq \frac{a_v}{1 - a_v}, \qquad j = 1, 2, \dots, m. \tag{3.18}$$

From (1.3) and (3.5) for an $\epsilon > 0$ there exists a n_0 such that for $n \geq n_0$ we get

$$x_{n+1}^{(j)} \geq a_j x_n^{(j+1)} + b_j (l_j - \epsilon) e^{-x_n^{(j+1)}} \tag{3.19}$$

where $j \in \{1, 2, \dots, m-1\}$. We consider the function

$$k_{l_j - \epsilon}(y) = a_j y + b_j (l_j - \epsilon) e^{-y}.$$

Then since $k'_{l_j-\epsilon}(y) = a_j - b_j(l_j - \epsilon)e^{-y}$ we have that $k_{l_j-\epsilon}$ is increasing for $y \geq \ln(b_j(l_j - \epsilon)/a_j)$. We claim that

$$l_{j+1} - \epsilon \prod_{s=1, s \neq j}^{m} a_s > \ln \left(\frac{b_j (l_j - \epsilon)}{a_j} \right). \tag{3.20}$$

From (3.4) and (3.18) we get

$$l_j \leq \bar{x}_j \leq \frac{a_v}{1 - a_v}$$

and so $\frac{l_j}{a_v} - 1 \leq l_j$. Therefore since $a_v < 1$ for an $\epsilon > 0$ we get

$$\frac{l_j - \epsilon}{a_v} - 1 < \frac{l_j - \epsilon a_v}{a_v} - 1 \leq l_j - \epsilon. \tag{3.21}$$

Moreover, from (1.3) we take

$$\begin{aligned} &l_{j-1} \geq a_{j-1} l_j, \quad l_{j-2} \geq a_{j-2} l_{j-1}, \dots, \quad l_1 \geq a_1 l_2, \quad l_m \geq a_m l_1, \\ &l_{m-1} \geq a_{m-1} l_m, \quad l_{m-2} \geq a_{m-2} l_{m-1}, \dots, \quad l_{j+1} \geq a_{j+1} l_{j+2}. \end{aligned}$$

Then we have

$$l_j \leq \frac{l_{j-1}}{a_{j-1}} \leq \frac{l_{j-2}}{a_{j-1} a_{j-2}} \leq \dots \leq \frac{l_1}{\prod_{s=1}^{j-1} a_s} \leq \frac{l_m}{a_m \prod_{s=1}^{j-1} a_s} \leq \frac{l_{m-1}}{a_{m-1} a_m \prod_{s=1}^{j-1} a_s} \leq \dots \leq \frac{l_{j+1}}{\prod_{s=1, s \neq j}^{m} a_s}. \tag{3.22}$$

Then, from (3.21) and (3.22) we have,

$$\frac{l_j-\epsilon}{a_v}-1\leq\frac{l_{j+1}}{\prod_{s=1,s\neq j}^{m}a_s}-\epsilon$$

and so

$$\frac{l_j-\epsilon}{a_v}\prod_{s=1,s\neq j}^{m}a_s-\prod_{s=1,s\neq j}^{m}a_s\leq l_{j+1}-\epsilon\prod_{s=1,s\neq j}^{m}a_s.$$

Then from (2.1) and (3.2) it follows that

$$\frac{b_j}{a_j}(l_j-\epsilon)-1<l_{j+1}-\epsilon\prod_{s=1,s\neq j}^{m}a_s. \tag{3.23}$$

Therefore, from (3.23) and since $\ln x\leq x-1$ our claim (3.20) is true. Moreover, there exists a n_1 such that for $n\geq n_1$

$$x_n^{(j+1)}\geq l_{j+1}-\epsilon\prod_{s=1,s\neq j}^{m}a_s. \tag{3.24}$$

Since $k_{l_j-\epsilon}$ is an increasing function for $y\geq\ln(b_j(l_j-\epsilon)/a_j)$, then from (3.20) and (3.24) we take

$$k_{l_j-\epsilon}(x_n^{(j+1)})\geq k_{l_j-\epsilon}\left(l_{j+1}-\epsilon\prod_{s=1,s\neq j}^{m}a_s\right).$$

Then from (3.19) it follows that

$$x_{n+1}^{(j)}\geq a_j\left(l_{j+1}-\epsilon\prod_{s=1,s\neq j}^{m}a_s\right)+b_j(l_j-\epsilon)e^{-\left(l_{j+1}-\epsilon\prod_{s=1,s\neq j}^{m}a_s\right)}$$

and so

$$l_j\geq a_j\left(l_{j+1}-\epsilon\prod_{s=1,s\neq j}^{m}a_s\right)+b_j(l_j-\epsilon)e^{-\left(l_{j+1}-\epsilon\prod_{s=1,s\neq j}^{m}a_s\right)}. \tag{3.25}$$

For $\epsilon\to 0$ to (3.25) we have

$$l_j\geq a_jl_{j+1}+b_jl_je^{-l_{j+1}},\qquad j=1,2,\dots,m-1. \tag{3.26}$$

Similarly we can prove that

$$l_m\geq a_ml_1+b_ml_me^{-l_1}. \tag{3.27}$$

From (3.26) and (3.27) we can prove that

$$F_j(l_j)\leq 0,\qquad j=1,2,\dots,m. \tag{3.28}$$

But since from Proposition 2.1 $F_j(\bar{x}_j)=0$, $j=1,2,\dots,m$ we get

$$F_j(l_j)\leq F_j(\bar{x}_j),\qquad j=1,2,\dots,m.$$

Since F_j, $j=1,2,\dots,m$ are decreasing functions we take (3.14). Then from (3.4) and (3.14) we have $\bar{x}_j=l_j=L_j$, $j=1,2,\dots,m$. This completes the proof of the proposition. □

In the last proposition we study the convergence of the positive solutions of (1.3) to the zero equilibrium.

Proposition 3.3. *Consider system* (1.3) *such that the constants* a_i, b_i, $i = 1, 2, \ldots, m$ *satisfy* (2.2). *Then every positive solution of* (1.3) *tends to the zero equilibrium* $(0, 0, \ldots, 0)$.

Proof. Since (2.2) holds, from Proposition 2.1 the only nonnegative equilibrium is the zero equilibrium. Let $(x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(m)})$ be an arbitrary solution of (1.3). From (1.3) we take

$$\begin{aligned} x_{n+1}^{(i)} &\le a_i x_n^{(i+1)} + b_i x_{n-1}^{(i)}, \qquad i = 1, 2, \ldots, m-1, \\ x_{n+1}^{(m)} &\le a_m x_n^{(1)} + b_m x_{n-1}^{(m)}. \end{aligned} \tag{3.29}$$

We consider the system of difference equations

$$\begin{aligned} y_{n+1}^{(i)} &= a_i y_n^{(i+1)} + b_i y_{n-1}^{(i)}, \qquad i = 1, 2, \ldots, m-1, \\ y_{n+1}^{(m)} &= a_m y_n^{(1)} + b_m y_{n-1}^{(m)}. \end{aligned} \tag{3.30}$$

Let $(y_n^{(1)}, y_n^{(2)}, \ldots, y_n^{(m)})$ be a solution of (3.29) with initial values $y_{-1}^{(-i)} = x_{-1}^{(i)}$, $y_0^{(i)} = x_0^{(i)}$, $i = 1, 2, \ldots, m$. Then from (3.29) and (3.30), by induction we can easily prove that

$$x_n^{(i)} \le y_n^{(i)}, \qquad i = 1, 2, \ldots, m, \quad n = 0, 1, \ldots \tag{3.31}$$

We prove that every positive solution of (3.30) tends to the zero equilibrium $(0, 0, \ldots, 0)$. System (3.30) is equivalent to the system

$$\bar{y}_{n+1} = A\bar{y}_n, \tag{3.32}$$

where $\bar{y}_n = \operatorname{col}(y_n^{(1)}, y_n^{(2)}, \ldots, y_n^{(m)}, y_{n-1}^{(1)}, y_{n-1}^{(2)}, \ldots, y_{n-1}^{(m)})$ and A is a matrix where in the (i)th line $1 \le i \le m-1$ the only non zero elements are a_i which is the $(i+1)$th element and b_i which is the $(i+m)$th element, in the (m)th line the only non zero elements are a_m which is the first element and b_m which is the last element, and finally in the $(m+j)$th line, $1 \le j \le m$ the unique nonzero element is the (j)th element which is 1.

Let

$$T = \operatorname{diag}(1, \epsilon^{-1}, \epsilon^{-2}, \ldots, \epsilon^{-2m+1})$$

where ϵ is a positive number such that

$$a_i + b_i < \epsilon^m, \qquad i = 1, 2, \ldots, m. \tag{3.33}$$

We take the change of variables $\bar{y}_n = T\bar{z}_n$ and we get the system

$$\bar{z}_{n+1} = T^{-1}AT\bar{z}_n, \tag{3.34}$$

$T^{-1}AT$ is matrix where in the (i)th line $1 \le i \le m-1$ the only non zero elements are $\epsilon^{-1}a_i$ which is the $(i+1)$th element and $\epsilon^{-m}b_i$ which is the $(i+m)$th element, in the (m)th line the only non zero elements are $\epsilon^{m-1}a_m$ which is the first element and $\epsilon^{-m}b_m$ which is the last element, and finally in the $(m+j)$th line, $1 \le j \le m$ the unique nonzero element is the (j)th element which is ϵ^m.

If for a $2m \times 2m$ matrix $C = (c_{ij})$ we take the norm $|C| = \sup_{0 \le i \le 2m}\{\sum_{j=1}^{2m} |c_{ij}|\}$ then we take

$$\begin{aligned} |T^{-1}AT| &= \max\Big\{\epsilon^{-1}a_1 + \epsilon^{-m}b_1,\ \epsilon^{-1}a_2 + \epsilon^{-m}b_2, \ldots, \epsilon^{-1}a_{m-1} + \epsilon^{-m}b_{m-1}, \\ &\qquad \epsilon^{m-1}a_m + \epsilon^{-m}b_m, \epsilon^m\Big\} \\ &\le \max\{\epsilon^{-m}(a_i + b_i), 1 \le i \le m\}. \end{aligned} \tag{3.35}$$

So, from (3.33) and (3.35) we take $|T^{-1}AT| < 1$. Then since from a known result it holds $|\lambda_i| \leq |T^{-1}AT| < 1$ where λ_i are the eigenvalues of $T^{-1}AT$ we have that $\lambda_i < 1$, $i = 1, 2, \ldots, 2m$. So, every solution of system (3.34) tends to $(0, 0, \ldots, 0)$ as $n \to \infty$. This implies that every solution of (3.32) tends to $(0, 0, \ldots, 0)$ as $n \to \infty$. Therefore, from (3.31) the proof of the proposition is completed. □

Acknowledgements

The authors would like to thank the referee for his helpful suggestions.

References

[1] A. M. Amleh, E.Camouzis, G.Ladas, On the dynamics of a rational difference equation. II., *Int. J. Difference Equ.* **3** (2008), No. 2, 195–225. MR2548126

[2] J. R. Beddington, C. A. Free, J. H. Lawton, Dynamic complexity in predator–prey models framed in difference equations, *Nature* **255**(1975), 58–60. url

[3] D. Benest, C. Froeschlé (Eds.), *Analysis and modelling of discrete dynamical systems*, Advances in Discrete Mathematics and Applications, Vol. 1, Gordon and Breach Science Publishers, Amsterdam, 1998. MR1678615

[4] K. Berenhaut, J. Foley, S. Stević, The global attractivity of the rational difference equation $y_n = 1 + (y_{n-k}/y_{n-m})$, *Proc. Amer. Math. Soc.* **135**(2007), No. 4, 1133–1140. MR2262916

[5] L. Berg, S. Stević, Periodicity of some classes of holomorphic difference equations, *J. Difference Equ. Appl.* **12**(2006), No. 8, 827–835. MR2248789; url

[6] L. Berg, S. Stević, On the asymptotics of the difference equation $y_n(1 + y_{n-1} \cdots y_{n-k+1}) = y_{n-k}$, *J. Difference Equ. Appl.* **17**(2011), No. 4, 577–586. MR2783369; url

[7] L. Berg, S. Stević, On some systems of difference equations, *Appl. Math. Comput.* **218** (2011), No. 5, 1713–1718. MR2831394; url

[8] E. Camouzis, M. R. S. Kulenović, G. Ladas, O. Merino, Rational systems in the plane, *J. Difference Equ. Appl.* **15**(2009), No. 3, 303–323. MR2498776; url

[9] E. Camouzis, G. Ladas, *Dynamics of third-order rational difference equations with open problems and conjectures*, Chapman & Hall/CRC, Boca Raton, London, 2008. MR2363297

[10] D. Clark, M.R.S. Kulenovic, A coupled system of rational difference equations, *Comput. Math. Appl.* **43**(2002), No. 6–7, 849–867. MR1884331; url

[11] D. Clark, M. R. S. Kulenović, J. F. Selgrade, Global asymptotic behavior of a two-dimensional difference equation modelling competition, *Nonlinear Anal.* **52**(2003), No. 7, 1765–1776. MR1956175; url

[12] J. M. Cushing, Periodically forced nonlinear systems of difference equations, *J. Difference Equ. Appl.* **3**(1998), No. 5–6, 547–561. MR1624860; url

[13] L. Edelstein-Keshet, *Mathematical models in biology*, Birkhäuser Mathematical Series, New York, 1988. MR1010228

[14] E. A. Grove, G. Ladas, N. R. Prokup, R. Levins, On the global behavior of solutions of a biological model, *Comm. Appl. Nonlinear Anal.* **7**(2000), No. 2, 33–46. MR1756640

[15] E. A. Grove, G. Ladas, *Periodicities in nonlinear difference equations*, Chapman & Hall/CRC, 2005. MR2193366

[16] B. Iričanin, S. Stević, Some systems of nonlinear difference equations of higher order with periodic solutions, *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.* **13**(2006), No. 3–4, 499–507. MR2220850

[17] B. Iričanin, S. Stević, Eventually constant solutions of a rational difference equation, *Appl. Math. Comput.* **215**(2009), No. 2, 854–856. MR2561544; url

[18] V. L. Kocić, G. Ladas, *Global behavior of nonlinear difference equations of higher order with applications*, Kluwer Academic Publishers, Dordrecht, 1993. MR1247956

[19] M. R. S. Kulenovic, G. Ladas, *Dynamics of second order rational difference equations*, Chapman & Hall/CRC, 2002. MR1935074

[20] E. El-Metwally, E. A. Grove, G. Ladas, R. Levins, M. Radin, On the difference equation, $x_{n+1} = \alpha + \beta x_{n-1} e^{-x_n}$, *Nonlinear Anal.* **47**(2001) 4623–4634. MR1975856; url

[21] I. Ozturk, F. Bozkurt, S. Ozen, On the difference equation $y_{n+1} = \frac{\alpha+\beta e^{-y_n}}{\gamma+y_{n-1}}$, *Appl. Math. Comput.* **181**(2006), No. 2, 1387–1393. MR2270770; url

[22] G. Papaschinopoulos, G. Ellina, K. B. Papadopoulos, Asymptotic behavior of the positive solutions of an exponential type system of difference equations, *Appl. Math. Comput.* **245**(2014) 181–190. MR3260707; url

[23] G. Papaschinopoulos, N. Fotiades, C. J. Schinas, On a system of difference equations including negative exponential terms, *J. Difference Equ. Appl.* **20**(2014), No. 5–6, 717–732. MR3210311; url

[24] G. Papaschinopoulos, C. J. Schinas, Invariants and oscillation for systems of two nonlinear difference equations, *Nonlinear Anal.* **46**(2001), No. 7, 967–978. MR1866733; url

[25] Y. Saito, W. Ma, T. Hara, A necessary and sufficient condition for permanence of a Lotka–Volterra discrete system with delays, *J. Math. Anal. Appl.* **256**(2001), No. 1, 162–174. MR1820074; url

[26] S. Stević, Asymptotic behavior of a nonlinear difference equation, *Indian J. Pure Appl. Math.* **34**(2003), No. 12, 1681–1687. MR2030114

[27] S. Stević, On the recursive sequence $x_{n+1} = A / \prod_{i=0}^{k} x_{n-i} + 1 / \prod_{j=k+2}^{2(k+1)} x_{n-j}$, *Taiwanese J. Math.* **7** (2003), No. 2, 249–259. MR1978014

[28] S. Stević, A short proof of the Cushing–Henson conjecture, *Discrete Dyn. Nat. Soc.* **2006**, Art. ID 37264, 5 pp. MR2272408

[29] S. Stević, On a discrete epidemic model, *Discrete Dyn. Nat. Soc.* **2007**, Art. ID 87519, 10 pp. MR2272415

[30] S. Stević, On a nonlinear generalized max-type difference equation, *J. Math. Anal. Appl.* **376**(2011), 317–328. MR2745409; url

[31] S. Stević, On a system of difference equations, *Appl. Math. Comput.* **218**(2011), No. 7, 3372–3378. MR2851439; url

[32] S. Stević, On a system of difference equations with period two coefficients, *Appl. Math. Comput.* **218**(2011), No. 8, 4317–4324. MR2862101; url

[33] S. Stević, On a third-order system of difference equations, *Appl. Math. Comput.* **218**(2012), No. 14, 7649–7654. MR2892731; url

[34] S. Stević, On some periodic systems of max-type difference equations, *Appl. Math. Comput.* **218**(2012), No. 23, 11483–11487. MR2943993; url

[35] S. Stević, On some solvable systems of difference equations, *Appl. Math. Comput.* **218**(2012), No. 9, 5010–5018. MR2870025; url

[36] S. Stević, On a solvable system of difference equations of kth order, *Appl. Math. Comput.* **219**(2013), No. 14, 7765–7771. MR3032616; url

[37] S. Stević, On a cyclic system of difference equations, *J. Difference Equ. Appl.* **20** (2014), No. 5–6, 733–743. MR3210312; url

[38] S. Stević, Representation of solutions of bilinear difference equations in terms of generalized Fibonacci sequences, *Electron. J. Qual. Theory Differ. Equ.* **2014**, No. 67, 1–15.

[39] S. Stević, M. A. Alghamdi, A. Alotaibi, N. Shahzad, On a higher-order system of difference equations, *Electron. J. Qual. Theory Differ. Equ.* **2013**, No. 47, 1–18. MR3090580

[40] S. Stević, M. A. Alghamdi, A. Alotaibi, N. Shahzad, Boundedness character of a max-type system of difference equations of second order, *Electron. J. Qual. Theory Differ. Equ.* **2014**, No. 45, 1–12. MR3262799

[41] D. C. Zhang, B. Shi, Oscillation and global asymptotic stability in a discrete epidemic model, *J. Math. Anal. Appl.* **278**(2003), No. 1, 194–202. MR1963474; url