TRANSVERSAL HOMOCLINICS IN NONLINEAR SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. Bifurcation of transversal homoclinics is studied for a pair of ordinary differential equations with periodic perturbations when the first unperturbed equation has a manifold of homoclinic solutions and the second unperturbed equation is vanishing. Such ordinary differential equations often arise in perturbed autonomous Hamiltonian systems.

1. Introduction

Let us consider the system of ordinary differential equations given by

$$\begin{aligned} \dot{x} &= f(x,y) + \epsilon h(x,y,t,\epsilon), \\ \dot{y} &= \epsilon\Big(Ay + g(y) + p(x,y,t,\epsilon) + \epsilon q(y,t,\epsilon)\Big), \end{aligned} \tag{1.1}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $\epsilon \neq 0$ is sufficiently small, A is an $m \times m$ matrix, and all mappings are smooth, 1-periodic in the time variable $t \in \mathbb{R}$ and such that

(i) $f(0,\cdot) = 0$, $g(0) = 0$, $g_y(0) = 0$, $p(0,\cdot,\cdot,\cdot) = 0$. Here g_y means the derivative of g with respect to y. Similar notations are used below.

(ii) The eigenvalues of A and $f_x(0,\cdot)$ lie off the imaginary axis.

(iii) There exists a smooth mapping $\gamma(\theta,y,t) \neq 0$, where $\theta \in \mathbb{R}^{d-1}$, $d \geq 1$ and y is small, such that

$$\begin{aligned} &\dot{\gamma}(\theta,y,t) = f(\gamma(\theta,y,t),y), \quad \gamma(\theta,y,t) = O\big(\mathrm{e}^{-c_1|t|}\big) \\ &\gamma_y(\theta,y,t) = O\big(\mathrm{e}^{-c_1|t|}\big), \quad \gamma_{yy}(\theta,y,t) = O\big(\mathrm{e}^{-c_1|t|}\big) \end{aligned}$$

for a constant $c_1 > 0$, and uniformly for θ, y. Moreover, we suppose

$$d = \dim W^s(y) \cap W^u(y) = \dim T_{\gamma(\theta,y,t)} W^s(y) \cap T_{\gamma(\theta,y,t)} W^u(y)\,.$$

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Here $W^{s(u)}(y)$ is the stable (unstable) manifold to $x=0$ of $\dot{x}=f(x,y)$, respectively, and $T_zW^{s(u)}(y)$ is the tangent bundle of $W^{s(u)}(y)$ at $z\in W^{s(u)}(y)$, respectively.

Consequently, assumption (iii) means that equation $\dot{x}=f(x,y)$ has a nondegenerate homoclinic manifold [5,7,10]

$$W_h(y)=W^s(y)\cap W^u(y)=\Big\{\gamma(\theta,y,t)\mid \theta\in\mathbb{R}^{d-1},\,t\in\mathbb{R}\Big\}\,.$$

We suppose that $\overline{W}_h(y)$ are compact. We are interested in homoclinic solutions of (1.1) near the family $W_h(y)$. Moreover, we search for transversal such solutions to show chaos for (1.1) [2,5,10].

Systems like (1.1) are investigated in [3], where the existence of chaos is proved, but the situation of this note is not included in [3]. Usually such systems occur in perturbed Hamiltonian systems [7,10], but in this note, equation $\dot{x}=f(x,y)$ has not to be necessary Hamiltonian in x uniformly for y small. For proving our results, we follow [3]. Related results are studied also in the papers [1,8,11,12].

2. Transversal Homoclinics

We take in (1.1) the following change of variables

$$\begin{aligned}&x(t)=\gamma(\theta,\epsilon y(t),t)+\epsilon z(t)\,,\\&y\leftrightarrow\epsilon y,\quad t\leftrightarrow t+\alpha\,,\end{aligned}$$

then by (iii) we get

$$\begin{aligned}\dot{z}&=f_x(\gamma(\theta,0,t),0)z+h(\gamma(\theta,0,t),0,t+\alpha,0)\\&\quad-\gamma_y(\theta,0,t)p(\gamma(\theta,0,t),0,t+\alpha,0)+O(\epsilon),\\\dot{y}&=\epsilon\Big(\big(A+p_y(\gamma(\theta,0,t),0,t+\alpha,0)\big)y+p_\epsilon(\gamma(\theta,0,t),0,t+\alpha,0)\\&\qquad+q(0,t+\alpha,0)+p_x(\gamma(\theta,0,t),0,t+\alpha,0)z+O\big(\mathrm{e}^{-c_1|t|}\big)+O(\epsilon)\Big)\\&\quad+p(\gamma(\theta,0,t),0,t+\alpha,0)\,.\end{aligned}\tag{2.1}$$

Now we consider the variational equation given by

$$\dot{u}=f_x(\gamma(\theta,0,t),0)u\,.\tag{2.2}$$

According to (iii), we note that the system

$$\Big\{\frac{\partial}{\partial\theta_i}\gamma(\theta,0,t)\Big\}_{i=1}^{d-1}\cup\dot{\gamma}(\theta,0,t)$$

is a family of bounded solutions of (2.2), where $\theta=\big(\theta_1,\theta_2,\cdots,\theta_{d-1}\big)$. We can assume that these vectors are linearly independent. Then this family represents a basis of bounded solutions of (2.2). Let $U_\theta(t)$ denote a fundamental solution of (2.2) with $u_{\theta j}(t)$ the jth column of $U_\theta(t)$ and define $U_\theta^\perp(t)=\big(U_\theta(t)^{-1}\big)^*$, where $*$ is a transposition with respect to a scalar product $\langle\cdot,\cdot\rangle$ on $\mathbb{R}^n$. We can suppose that $u_{\theta j}(t)$ and $u_{\theta j+d}^\perp(t)$, $j=1,2,\cdots,d$ form bases of the bounded solutions of (2.2) and of the adjoint equation

$$\dot{u}=-f_x(\gamma(\theta,0,t),0)^*u\,,\tag{2.3}$$

respectively, where $u_{\theta j}^\perp(t)$ is the jth column of $U_\theta^\perp(t)$. Moreover, we can assume the smoothness of $U_\theta(t)$ on both θ and t. We note that $U_\theta^\perp(t)$ is a fundamental solution of (2.3).

Now by following [3], we get the following result.

Theorem 2.1. *Let us define a mapping*

$$M\colon \mathbb{R}^{d-1}\times\mathbb{R}\to\mathbb{R}^d,\quad M=(M_1,M_2,\cdots,M_d)\,,$$

by

$$M_l(\theta,\alpha)=\int_{-\infty}^{\infty}\big\langle u^{\perp}_{\theta l+d}(t),h(\gamma(\theta,0,t),0,t+\alpha,0)\big\rangle\,dt \\ -\int_{-\infty}^{\infty}\big\langle u^{\perp}_{\theta l+d}(t),\gamma_y(\theta,0,t)p(\gamma(\theta,0,t),0,t+\alpha,0)\big\rangle\,dt\,. \tag{2.4}$$

If there is a simple root (θ_0,α_0) of $M(\theta,\alpha)=0$, i.e. $M(\theta_0,\alpha_0)=0$ and the matrix $M_{(\theta,\alpha)}(\theta_0,\alpha_0)$ is nonsingular, then (1.1) has for any $\epsilon\neq 0$ sufficiently small a transversal homoclinic solution near $\gamma(\theta_0,0,\cdot+\alpha_0)\times 0$.

Proof. Since the proof is very similar as of Theorem 2.10 of [3], so we only sketch it here. Let us define the following Banach spaces

$$Z=\Big\{z\in C(\mathbb{R},\mathbb{R}^n)\Big|\,|z|=\sup_t|z(t)|<\infty\Big\}\,,$$

$$Y_\theta=\Big\{h\in Z\Big|\int_{-\infty}^{\infty}\langle h(t),u^{\perp}_{\theta i+d}(t)\rangle\,dt=0\quad\text{for any}\quad i=1,2,\ldots,d\Big\}\,,$$

$$X=\Big\{v\in C(\mathbb{R},\mathbb{R}^m)\Big|\,|v|=\sup_t|v(t)|<\infty\Big\}\,.$$

We need the following two results.

Claim 1. ([3]) *The nonhomogeneous equation*

$$\dot z=f_x(\gamma(\theta,0,t),0)z+h(t),\quad h\in Z$$

has a solution $z\in Z$ if and only if $h\in Y_\theta$. The solution is unique if it satisfies $\int_{-\infty}^{\infty}\langle z(t),u_{\theta i}(t)\rangle\,dt=0$ for any $i=1,2,\ldots,d$. This solution is smooth in θ and h.

Claim 2. ([3]) *For $\epsilon\neq 0$ sufficiently small, the nonhomogeneous equation*

$$\dot y=\epsilon\Big(\big(A+p_y(\gamma(\theta,0,t),0,t+\alpha,0)\big)y+w\Big),\quad w\in X$$

has a unique solution in X which we denote $t\to y(t,\alpha,\theta,\epsilon)$. This solution satisfies $|y|\le c_2|w|$ for a constant $c_2>0$, and $\left|\frac{\partial y}{\partial\alpha}\right|=O(\epsilon|w|)$. If in addition $\int_{-\infty}^{\infty}|w(s)|\,ds<\infty$ then $|y|\le c_3|\epsilon|\int_{-\infty}^{\infty}|w(s)|\,ds$ for a constant $c_3>0$.

Now by using the standard way of Lyapunov-Schmidt like in [3], we can solve (2.1) to get the statement of the theorem. ☐

We note that usually we start with a system of the form

$$\begin{aligned} \dot{x} &= f_1(x,y) + \epsilon h_1(x,y,t,\epsilon)\,, \\ \dot{y} &= \epsilon g_1(x,y,t,\epsilon)\,. \end{aligned} \tag{2.5}$$

Then we suppose that $f_1(x,y)=0$ has a smooth solution $x=\psi(y)$ and by changing the variables, we can suppose that $f_1(0,y)=0$. Then we consider the equation $\dot{y}=\epsilon g_1(0,y,t,\epsilon)$ and we take its averaged equation $\dot{y}=\epsilon\int_0^1 g_1(0,y,t,0)\,dt$ (see [9]). Let $y=0$ be a hyperbolic root of $\int_0^1 g_1(0,y,t,0)\,dt=0$, i.e. $\int_0^1 g_1(0,0,t,0)\,dt=0$ and the matrix $\int_0^1 g_{1y}(0,0,t,0)\,dt$ has no eigenvalues on the imaginary axis. By taking in (2.5) the usual averaging change of variables of the form $y\leftrightarrow y+\epsilon H(y,t)$, where H is smooth and 1-periodic in t, we arrive at the system like (1.1). So let us take $y(t)=v(t)+\epsilon H(v(t),t)$ in (1.1). Then we get

$$\begin{aligned} \dot{x} &= f(x,v) + \epsilon\big(f_y(x,v)H(v,t) + h(x,v,t,0)\big) + O(\epsilon^2) \\ &= f_1(x,v) + \epsilon h_1(x,v,t,\epsilon)\,, \\ \dot{v} &= \epsilon\big(I+\epsilon H_v(v,t)\big)^{-1}\Big(Av + g(v+\epsilon H(v,t)) - H_t(v,t) \\ &\quad + \epsilon AH(v,t) + p(x,v+\epsilon H(v,t),t,\epsilon) + \epsilon q(v+\epsilon H(v,t),t,\epsilon)\Big) \\ &= \epsilon g_1(x,v,t,\epsilon)\,. \end{aligned} \tag{2.6}$$

The unperturbed equation of (2.6) has the same form as for (1.1). For the mapping $M=(M_1,M_2,\cdots,M_d)$ of (2.4) in terms of (2.6), we have

$$\begin{aligned} M_l(\theta,\alpha) = &-\int_{-\infty}^{\infty}\big\langle u^{\perp}_{\theta l+d}(t), f_y(\gamma(\theta,0,t),0)H(0,t+\alpha) + \gamma_y(\theta,0,t)H_t(0,t+\alpha)\big\rangle\,dt \\ &+\int_{-\infty}^{\infty}\big\langle u^{\perp}_{\theta l+d}(t), h_1(\gamma(\theta,0,t),0,t+\alpha,0) - \gamma_y(\theta,0,t)g_1(\gamma(\theta,0,t),0,t+\alpha,0)\big\rangle\,dt\,. \end{aligned}$$

Assumption (iii) for $\omega(t)=\gamma_y(\theta,0,t)H(0,t+\alpha)$ gives

$$\begin{aligned} \dot{\omega}(t) &= f_x(\gamma(\theta,0,t),0)\omega(t) \\ &\quad + f_y(\gamma(\theta,0,t),0)H(0,t+\alpha) + \gamma_y(\theta,0,t)H_t(0,t+\alpha)\,. \end{aligned} \tag{2.7}$$

Since $\omega\in Z$, equation (2.7) and Claim 1 imply

$$f_y(\gamma(\theta,0,t),0)H(0,t+\alpha) + \gamma_y(\theta,0,t)H_t(0,t+\alpha) \in Y_\theta\,.$$

Hence we get

$$\begin{aligned} M_l(\theta,\alpha) = &\int_{-\infty}^{\infty}\big\langle u^{\perp}_{\theta l+d}(t), h_1(\gamma(\theta,0,t),0,t+\alpha,0)\big\rangle\,dt \\ &-\int_{-\infty}^{\infty}\big\langle u^{\perp}_{\theta l+d}(t), \gamma_y(\theta,0,t)g_1(\gamma(\theta,0,t),0,t+\alpha,0)\big\rangle\,dt\,. \end{aligned} \tag{2.8}$$

When $f_1(0,\cdot)=0$ in (2.5), then (2.8) expresses the mapping M in terms of (2.5) without using its averaged form (1.1).

Generally, when $f_1(\psi(y),y)=0$ and $\gamma(\theta,y,t)$ are homoclinics to the hyperbolic fixed points $x=\psi(y)$ of $\dot{x}=f_1(x,y)$, and $y=y_0$ is a hyperbolic root of the equation $\int_0^1 g_1(\psi(y),y,t)\,dt=0$, then the mapping $M=(M_1,M_2,\cdots,M_d)$ has the form

$$
\begin{aligned}
M_l(\theta,\alpha) &= \int_{-\infty}^{\infty} \left\langle u^{\perp}_{\theta l+d}(t), h_1(\gamma(\theta,y_0,t),y_0,t+\alpha,0)\right\rangle dt \\
&\quad - \int_{-\infty}^{\infty} \left\langle u^{\perp}_{\theta l+d}(t), \gamma_y(\theta,y_0,t) g_1(\gamma(\theta,y_0,t),y_0,t+\alpha,0)\right\rangle dt\,,
\end{aligned}
\tag{2.9}
$$

where (2.2) has to be replaced by

$$\dot{u}=f_x(\gamma(\theta,y_0,t),y_0)u\,.$$

3. An Example

Let us consider the system

$$
\begin{aligned}
\ddot{z} &= z-(v^2+\dot{v}^2)z(z^2+w^2+u)+\epsilon\delta\dot{v}\,,\\
\ddot{w} &= w-(v^2+\dot{v}^2)w(z^2+w^2+u)\,,\\
\dot{u} &= (1+v^2+\dot{v}^2)u+\epsilon w^2\,,\\
\ddot{v}+v &= \epsilon\big((1-v^2)\dot{v}+w\big)\,,
\end{aligned}
\tag{3.1}
$$

where δ is a constant and ϵ is a small parameter. By taking the polar coordinates

$$v=y\sin\phi,\quad \dot{v}=y\cos\phi\,,$$

(3.1) possesses the form

$$
\begin{aligned}
x_1' &= x_2/g_2(y,\phi,x,\epsilon)\,,\\
x_2' &= \big(x_1-y^2x_1(x_1^2+x_3^2+x_5)+\epsilon\delta y\cos\phi\big)/g_2(y,\phi,x,\epsilon)\,,\\
x_3' &= x_4/g_2(y,\phi,x,\epsilon)\,,\\
x_4' &= \big(x_3-y^2x_3(x_1^2+x_3^2+x_5)\big)/g_2(y,\phi,x,\epsilon)\,,\\
x_5' &= \big((1+y^2)x_5+\epsilon x_3^2\big)/g_2(y,\phi,x,\epsilon)\,,\\
y' &= \epsilon\big((1-y^2\sin^2\phi)y\cos^2\phi+x_3\cos\phi\big)/g_2(y,\phi,x,\epsilon)\,,
\end{aligned}
\tag{3.2}
$$

where $'=\frac{d}{d\phi}$, $x=(x_1,x_2,x_3,x_4,x_5)$ and

$$g_2(y,\phi,x,\epsilon)=1-\epsilon\big((1-y^2\sin^2\phi)\cos\phi\sin\phi+\frac{x_3}{y}\sin\phi\big)\,.$$

Of course, we suppose that $y \neq 0$. The unperturbed equation of (3.2) has the form

$$
\begin{aligned}
x_1' &= x_2\,,\\
x_2' &= x_1 - y^2 x_1(x_1^2 + x_3^2 + x_5)\,,\\
x_3' &= x_4\,,\\
x_4' &= x_3 - y^2 x_3(x_1^2 + x_3^2 + x_5)\,,\\
x_5' &= (1+y^2)x_5\,.
\end{aligned} \tag{3.3}
$$

By putting $r(t) = \operatorname{sech} t$, for (3.3) we have [5,6]

$$
\begin{aligned}
\gamma(\theta, y, t) &= \frac{\sqrt{2}}{y}\Big(\sin\theta r(t), \sin\theta \dot r(t), \cos\theta r(t), \cos\theta \dot r(t), 0\Big)\,,\\
u_{\theta 3}^{\perp}(y, t) &= \Big(-\sin\theta \ddot r(t), \sin\theta \dot r(t), -\cos\theta \ddot r(t), \cos\theta \dot r(t), 0\Big)\,,\\
u_{\theta 4}^{\perp}(y, t) &= \Big(-\cos\theta \dot r(t), \cos\theta r(t), \sin\theta \dot r(t), -\sin\theta r(t), 0\Big)\,.
\end{aligned} \tag{3.4}
$$

Now we consider the equation

$$
\begin{aligned}
y' &= \epsilon \frac{(1 - y^2\sin^2\phi) y \cos^2\phi}{1 - \epsilon(1 - y^2 \sin^2\phi)\cos\phi\sin\phi}\\
&= \epsilon\big((1 - y^2\sin^2\phi) y\cos^2\phi + O(\epsilon)\big)
\end{aligned}
$$

and its first-order averaging is given by

$$
y' = \epsilon y\Big(\frac{1}{2} - \frac{y^2}{8}\Big)\,.
$$

$y_0 = 2$ is a simple root of $\frac{1}{2} - \frac{y^2}{8} = 0$. Hence we take $y = 2$ in the formulas (3.4).

In the notation of (2.5), we have

$$
\begin{aligned}
& h_1(x, 2, \phi, 0)\\
&= \Big(x_2, x_1 - 4x_1(x_1^2 + x_3^2 + x_5), x_4, x_3 - 4x_1(x_1^2 + x_3^2 + x_5), 5x_5\Big) g_3(x, \phi)\\
&+ 2\delta\big(0, \cos\phi, 0, 0, 0\big) + \big(0, 0, 0, 0, x_3^2\big)\,,\\
& g_3(x, \phi) = (1 - 4\sin^2\phi)\sin\phi\cos\phi + \frac{x_3}{2}\sin\phi\,,\\
& g_1(x, 2, \phi, 0) = 2(1 - 4\sin^2\phi)\cos^2\phi + x_3\cos\phi\,.
\end{aligned}
$$

We see that

$$
\gamma_y(\theta, 2, t) = -\gamma(\theta, 2, t)/2\,.
$$

Since

$$
\begin{aligned}
& h_1(\gamma(\theta, 2, t), 2, t + \alpha, 0) = \dot\gamma(\theta, 2, t) g_3(\gamma(\theta, 2, t), t + \alpha)\\
& + 2\delta\big(0, \cos(t + \alpha), 0, 0, 0\big) + \frac{1}{2}\big(0, 0, 0, 0, \cos^2\theta r(t)^2\big)\,,\\
& u_{\theta 2}(t) = \dot\gamma(\theta, 2, t), \quad \langle u_{\theta 2}(t), u_{\theta i+2}^{\perp}(t)\rangle = 0, \quad i = 1,\, 2\\
& \langle \gamma_y(\theta, 2, t), u_{\theta 4}^{\perp}(t)\rangle = 0\,,
\end{aligned}
$$

the formula (2.9) has after several calculations [4] now the form

$$
\begin{aligned}
M_1(\theta,\alpha) &= 2\delta\pi \operatorname{sech}\frac{\pi}{2}\sin\theta\sin\alpha + \frac{2\pi\sqrt{2}}{3}\operatorname{cosech}\pi\cos 2\alpha \\
&\qquad + \frac{10\pi\sqrt{2}}{3}\operatorname{cosech}2\pi\cos 4\alpha + \frac{5\pi}{24}\operatorname{sech}\frac{\pi}{2}\cos\theta\cos\alpha\,, \\
M_2(\theta,\alpha) &= 2\delta\pi \operatorname{sech}\frac{\pi}{2}\cos\theta\cos\alpha\,.
\end{aligned}
$$

For finding a simple root of $M(\theta,\alpha)=0$, we suppose $\delta\neq 0$ and take $\theta=-\pi/2$ while $\alpha\neq\pm\pi/2$ must be a simple zero of the equation

$$
\delta = \sqrt{2}\,\frac{\operatorname{cosech}\pi\cos 2\alpha + 5\operatorname{cosech}2\pi\cos 4\alpha}{3\operatorname{sech}(\pi/2)\sin\alpha} = \Omega(\alpha)\,. \tag{3.5}
$$

Function $\Omega(\alpha)$ is odd and it is satisfying

$$
\Omega(\alpha)=\Omega(\pi-\alpha),\quad \Omega(\alpha)=-\Omega(\pi+\alpha),\quad \lim_{\alpha\to 0_+}\Omega(\alpha)=+\infty\,.
$$

Furthermore, Ω has on $(0,\pi)$ only three critical points α_1, $\alpha_2=\pi-\alpha_1$, $\alpha_3=\pi/2$ for some $\alpha_1\simeq 1.378$. Moreover, Ω attains on $(0,\pi)$ its global minimum at α_1, α_2 and a local maximum at α_3. We note that $\Omega(\alpha_1)=\Omega(\alpha_2)$. Consequently as $\Omega(\pi/2)<0$, (3.5) has a simple zero for any δ.

Summarizing, by applying Theorem 2.1 and results of the papers [2,5], we arrive at the following result.

Theorem 3.1. *Let $\delta\neq 0$ be fixed. Equation (3.1) has chaos for any $\epsilon\neq 0$ sufficiently small.*

We note that for any compact interval $[a_1,a_2]\subset\mathbb{R}$, $0\notin[a_1,a_2]$, there is an $\epsilon_0>0$ such that (3.1) has chaos for any $\delta\in[a_1,a_2]$ and $0<|\epsilon|<\epsilon_0$. On the other hand, the function $M_2(\theta,\alpha)$ is vanishing for $\delta=0$, and we should derive higher-degenerate Melnikov mapping to get a reasonable bifurcation result as δ is crossing 0. We do not follow this line in this paper.

When $w=u=0$ in (3.1), we get the simpler system

$$
\begin{aligned}
\ddot z &= z-(v^2+\dot v^2)z^3+\epsilon\delta\dot v\,, \\
\ddot v + v &= \epsilon(1-v^2)\dot v\,.
\end{aligned} \tag{3.6}
$$

Then (3.3) has the form

$$
x_1' = x_2,\quad x_2' = x_1 - y^2x_1^3\,. \tag{3.7}
$$

(3.7) has a homoclinic $\gamma(y,t)=\frac{\sqrt{2}}{y}\big(r(t),\dot r(t)\big)$. So now we have $d=1$ and $u_2^\perp(y,t)=\big(-\ddot r(t),\dot r(t)\big)$. The Melnikov function has now the form

$$
M(\alpha) = 2\delta\int_{-\infty}^{\infty}\cos(t+\alpha)\dot r(t)\,dt = 2\delta\pi\operatorname{sech}\frac{\pi}{2}\sin\alpha\,.
$$

We see that $\alpha_0=0$ is a simple root of $M(\alpha)=0$ for $\delta\neq 0$. Consequently, (3.6) is chaotic for $\delta\neq 0$ fixed and $\epsilon\neq 0$ sufficiently small. Hence (3.1) has, in addition to Theorem 3.1, also “trivial” chaos of (3.6) with $w=u=0$.

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