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# Upper and lower solutions method and a fractional differential equation boundary value problem.

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**Abstract.** The method of lower and upper solutions for fractional differential equation $D^\delta u(t)+g(t,u(t))=0, t\in(0,1), 1<\delta\leq 2$, with Dirichlet boundary condition $u(0)=a, u(1)=b$ is used to give sufficient conditions for the existence of at least one solution.

**Keywords:** Fractional differential equation; Boundary value problem; Upper and lower solutions; Existence.

**MR(2000) Subject Classification 26A33, 34B12.**

## 1 Introduction

In this paper, we consider the two-point boundary value problem

$$\begin{cases} D^\delta u(t)+g(t,u)=0, t\in(0,1), 1<\delta\leq 2 \\ \\ u(0)=a, u(1)=b, \end{cases} \tag{1.1}$$

where $g:[0,1]\times R\to R$, $a,b\in R$, and $D^\delta$ is Caputo fractional derivative of order $1<\delta\leq 2$ defined by (see [1])

$$D^\delta u(t)=I^{2-\delta}u''(t)=\frac{1}{\Gamma(2-\delta)}\int_0^t (t-s)^{1-\delta}u''(s)ds,$$

$I^{2-\delta}$ is the Riemann-Liouville fractional integral of order $2-\delta$, see [1].

Differential equations of fractional order occur more frequently in different research areas and engineering, such as physics, chemistry, etc. Recently, many people pay attention to the

*Corresponding author, E-mail address:zsqjk@163.com existence of solution to boundary value problem for fractional differential equations, such as [2] − [7], by means of some fixed point theorems. However, as far as we know, there are no papers dealing with the existence of solution to boundary value problem for fractional differential equations, by means of the lower and upper solutions method. The lower and upper solutions method plays very important role in investigating the existence of solutions to ordinary differential equation problems of integer orders, for example, [8] − [11].

In this paper, by generalizing the concept of lower and upper solutions to boundary value problem for fractional differential equation (1.1), we shall present sufficient conditions for the existence of at least one solution satisfying (1.1)

## 2 Extremum principle for the Caputo derivative

In order to apply the upper and lower solutions method to fractional differential equation two-point boundary value problem (1.1), we need the following results about Caputo derivative.

**Theorem 2.1** Let a function $f \in C^2(0,1) \cap C[0,1]$, attain its maximum over the interval $[0,1]$ at the point $t_0, t_0 \in (0,1]$. Then the Caputo derivative of the function $f$ is non-positive at the point $t_0$ for any $\alpha$, $D^\alpha f(t_0) \leq 0, 1 < \alpha \leq 2$.

**Proof** For given function $f \in C^2(0,1) \cap C[0,1]$, Since $f^{''} \in L_1(0,t_0)$, hence,

$$\forall \delta > 0, \exists \quad 0 < \varepsilon < t_0 \quad \text{such that} \quad \left|\frac{1}{\Gamma(2-\alpha)} \int_0^\varepsilon (t_0 - s)^{1-\alpha} f^{''}(s) ds\right| \leq \delta. \tag{2.1}$$

For $\varepsilon >$ obtained in (2.1), let us consider the following two cases: Case $(i)$: $f^{'}(\varepsilon) \geq 0$; Case $(ii)$: $f^{'}(\varepsilon) < 0$.

For case $(i)$, we consider an auxiliary function

$$h(t) = f(t_0) - f(t), t \in [0,1].$$

Because the function $f$ attains its maximum over the interval $[0,1]$ at the point $t_0, t_0 \in (0,1]$, the Caputo derivative is a linear operator and $D^\alpha c \equiv 0$($c$ being a constant), hence, function $h$ possesses the following properties:

$$\begin{cases} h(t) \geq 0, t \in [0,1]; h(t_0) = 0; h^{'}(t_0) = -f^{'}(t_0) = 0; \\ \\ D^\alpha h(t) = -D^\alpha f(t), t \in (0,1]. \end{cases} \tag{2.2}$$

Obviously, $h^{'}(\varepsilon) = -f^{'}(\varepsilon) \leq 0$. Since

$$D^\alpha h(t_0) = \frac{1}{\Gamma(2-\alpha)} \int_0^{t_0} (t-s)^{1-\alpha} h^{''}(s) ds$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(2-\alpha)}\int_0^{\varepsilon}(t_0-s)^{1-\alpha}h^{''}(s)ds+\frac{1}{\Gamma(2-\alpha)}\int_{\varepsilon}^{t_0}(t_0-s)^{1-\alpha}h^{''}(s)ds\\
&= I_1+I_2
\end{aligned}
$$

is valid for $\varepsilon$ in (2.1); since $h\in C^2(0,1)$, $h(t_0)=h^{'}(t_0)=0$, there are

$$
|h(t)|=|h(t)-h(t_0)|\le|h^{'}(\overline{t})|(t_0-t)=|h^{'}(\overline{t})-h^{'}(t_0)|(t_0-t)\le c_1(t_0-t)^2,
$$

$$
|h^{'}(t)|=|h^{'}(t)-h^{'}(t_0)|\le c_2(t_0-t), t\in[\varepsilon,t_0],
$$

where $c_1>0, c_2>0$ are positive constants, and $\overline{t}\in(t,t_0)$. Hence, we have

$$
\begin{aligned}
I_2 = &\ \frac{1}{\Gamma(2-\alpha)}\int_{\varepsilon}^{t_0}(t_0-s)^{1-\alpha}h^{''}(s)ds\\
= &\ \frac{1-\alpha}{\Gamma(2-\alpha)}\int_{\varepsilon}^{t_0}(t_0-s)^{-\alpha}h^{'}(s)ds-\frac{(t_0-\varepsilon)^{1-\alpha}h^{'}(\varepsilon)}{\Gamma(2-\alpha)}\\
= &\ -\frac{(1-\alpha)(t_0-\varepsilon)^{-\alpha}h(\varepsilon)}{\Gamma(2-\alpha)}-\frac{\alpha(1-\alpha)}{\Gamma(2-\alpha)}\int_{\varepsilon}^{t_0}(t_0-s)^{-\alpha-1}h(s)ds\\
&\ -\frac{(t_0-\varepsilon)^{1-\alpha}h^{'}(\varepsilon)}{\Gamma(2-\alpha)},
\end{aligned}
$$

which leads to the relation

$$
I_2\ge 0
$$

that together with (2.1) complete the proof of the theorem.

We consider case $(ii)$ in the remaining part of the proof. Here, we consider the following auxiliary function

$$
h(t)=f(t_0)-f(t)+\varphi(t), t\in[0,1],
$$

where $\varphi(t)$ is infinitely differentiable function on $R$, defined by

$$
\varphi(t)=A\begin{cases} e^{\frac{kt^2}{t^2-t_0^2}}, & t<t_0,\\ \\ 0, & t\ge t_0\end{cases}
$$

here, $k$ is a constant satisfies

$$
0<k\le\frac{(t_0^3-tt_0^2)^2+t^2t_0^2(t_0^2-t^2)+2tt_0^2(t_0^3-t^3)}{2t_0^6}, t\in[0,t_0),
$$

and $A$ is a positive constant satisfies

$$
\frac{2A\varepsilon t_0^2k}{(\varepsilon^2-t_0^2)^2}e^{\frac{k\varepsilon^2}{\varepsilon^2-t_0^2}}\ge -f^{'}(\varepsilon),\quad \varepsilon \text{ is the positive constant obtained in (2.1).}
$$

By calculation(applying the Law of L'Hospital), we easily obtain that

$$\varphi(t_0) = 0, \varphi^{'}(t_0) = 0, \varphi^{''}(t_0) = 0$$

and that,

$$\begin{aligned}
\varphi^{''}(t) &= -A(e^{\frac{kt^2}{t^2-t_0^2}} \frac{2tt_0^2 k}{(t^2-t_0^2)^2})' \\
&= -2Ake^{\frac{kt^2}{t^2-t_0^2}} \frac{t_0^6 + 2t^2t_0^4 - 3t^4t_0^2 - 2t^2t_0^4k}{(t-t_0)^4} \\
&= -2Ake^{\frac{kt^2}{t^2-t_0^2}} \frac{(t_0^3 - tt_0^2)^2 + t^2t_0^2(t_0^2-t^2) + 2tt_0^2(t_0^3-t^3) - 2t^2t_0^4k}{(t-t_0)^4},
\end{aligned}$$

for $t \in [0, t_0)$. Since $0 < k \leq \frac{(t_0^3-tt_0^2)^2+t^2t_0^2(t_0^2-t^2)+2tt_0^2(t_0^3-t^3)}{2t_0^6}$, so, for $t \in [0, t_0)$, there is

$$\begin{aligned}
&(t_0^3 - tt_0^2)^2 + t^2t_0^2(t_0^2-t^2) + 2tt_0^2(t_0^3-t^3) - 2t^2t_0^4k \\
&\geq (t_0^3 - tt_0^2)^2 + t^2t_0^2(t_0^2-t^2) + 2tt_0^2(t_0^3-t^3) - 2t_0^6k \geq 0.
\end{aligned}$$

Hence, the Riemann-Liouville fractional integral $I^{2-\alpha}\varphi^{''}(t_0) \leq 0$. And that, it follows from $\frac{2A\varepsilon t_0^2 k}{(\varepsilon^2-t_0^2)^2} e^{\frac{k\varepsilon^2}{\varepsilon^2-t_0^2}} \geq -f^{'}(\varepsilon)$ that $h^{'}(\varepsilon) \leq 0$.

Since $h \in C^2(0,1)$, $h(t_0) = h^{'}(t_0) = 0$, there are

$$|h(t)| = |h(t) - h(t_0)| \leq |h^{'}(\overline{t})|(t_0 - t) = |h^{'}(\overline{t}) - h^{'}(t_0)|(t_0 - t) \leq c_1(t_0 - t)^2,$$

$$|h^{'}(t)| = |h^{'}(t) - h^{'}(t_0)| \leq c_2(t_0 - t), t \in [\varepsilon, t_0],$$

where $c_1 > 0, c_2 > 0$ are positive constants, and $\overline{t} \in (t, t_0)$. By the same arguments as case $(i)$, we can obtain that

$$\begin{aligned}
D^\alpha h(t_0) = &\frac{1}{\Gamma(2-\alpha)} \int_0^\varepsilon (t_0 - s)^{1-\alpha} h^{''}(s) ds + \frac{1}{\Gamma(2-\alpha)} \int_\varepsilon^{t_0} (t_0 - s)^{1-\alpha} h^{''}(s) ds \\
= &I_1 + I_2,
\end{aligned}$$

$$I_2 = -\frac{(1-\alpha)(t_0-\varepsilon)^{-\alpha}h(\varepsilon)}{\Gamma(2-\alpha)} - \frac{\alpha(1-\alpha)}{\Gamma(2-\alpha)} \int_\varepsilon^{t_0} (t_0 - s)^{-\alpha-1} h(s) ds - \frac{(t_0-\varepsilon)^{1-\alpha}h^{'}(\varepsilon)}{\Gamma(2-\alpha)},$$

which leads to the relation

$$I_2 \geq 0$$

that together with (2.1) produce $D^\alpha h(t_0) \geq 0$. Hence, we have

$$-D^\alpha f(t_0) + I^{2-\alpha}\varphi^{''}(t_0) \geq 0,$$

which implies that

$$D^{\alpha}f(t_0) \le I^{2-\alpha}\varphi^{''}(t_0) \le 0.$$

Thus, we complete this proof.

**Theorem 2.2** Let a function $f \in C^2(0,1) \cap C[0,1]$, attain its minimum over the interval $[0,1]$ at the point $t_0, t_0 \in (0,1]$. Then the Caputo derivative of the function $f$ is nonnegative at the point $t_0$ for any $\alpha$, $D^{\alpha}f(t_0) \ge 0, 1 < \alpha \le 2$.

**Proof** For given function $f \in C^2(0,1) \cap C[0,1]$, Since $f^{''} \in L_1(0,t_0)$, hence,

$$\forall \delta > 0, \exists \quad 0 < \varepsilon < t_0 \quad \text{such that} \quad |\frac{1}{\Gamma(2-\alpha)}\int_a^{\varepsilon}(t_0-s)^{1-\alpha}f^{''}(s)ds| \le \delta. \tag{2.3}$$

For $\varepsilon >$ obtained in (2.3), let us consider the following two cases: Case $(i)$: $f^{'}(\varepsilon) \le 0$; Case $(ii)$: $f^{'}(\varepsilon) > 0$.

For case $(i)$, we consider an auxiliary function

$$h(t) = f(t) - f(t_0), t \in [0,1].$$

Because the function $f$ attains its minimum over the interval $[0,1]$ at the point $t_0, t_0 \in (0,1]$, the Caputo derivative is a linear operator and $D^{\alpha}c \equiv 0$($c$ being a constant), hence, function $h$ possesses the following properties:

$$\begin{cases} h(t) \ge 0, t \in [0,1]; h(t_0) = 0; h^{'}(t_0) = f^{'}(t_0) = 0; \\ \\ D^{\alpha}h(t) = D^{\alpha}f(t), t \in (0,1]. \end{cases} \tag{2.4}$$

Obviously, $h^{'}(\varepsilon) = f^{'}(\varepsilon) \le 0$. Since $h \in C^2(0,1)$, $h(t_0) = h^{'}(t_0) = 0$, there are

$$|h(t)| = |h(t) - h(t_0)| \le |h^{'}(\bar{t})|(t_0 - t) = |h^{'}(\bar{t}) - h^{'}(t_0)|(t_0 - t) \le c_1(t_0 - t)^2,$$

$$|h^{'}(t)| = |h^{'}(t) - h^{'}(t_0)| \le c_2(t_0 - t), t \in [\varepsilon, t_0],$$

where $c_1 > 0, c_2 > 0$ are positive constants, and $\bar{t} \in (t, t_0)$. And that

$$\begin{aligned} D^{\alpha}h(t_0) &= \frac{1}{\Gamma(2-\alpha)}\int_0^{t_0}(t-s)^{1-\alpha}h^{''}(s)ds \\ &= \frac{1}{\Gamma(2-\alpha)}\int_0^{\varepsilon}(t_0-s)^{1-\alpha}h^{''}(s)ds + \frac{1}{\Gamma(2-\alpha)}\int_{\varepsilon}^{t_0}(t_0-s)^{1-\alpha}h^{''}(s)ds \\ &= I_1 + I_2 \end{aligned}$$

is valid for $\varepsilon$ in (2.3); On the other hand, we have

$$\begin{aligned}
I_2 = & \quad \frac{1}{\Gamma(2-\alpha)} \int_{\varepsilon}^{t_0} (t_0 - s)^{1-\alpha} h^{''}(s) ds \\
= & \quad \frac{1-\alpha}{\Gamma(2-\alpha)} \int_{\varepsilon}^{t_0} (t_0 - s)^{-\alpha} h^{'}(s) ds - \frac{(t_0-\varepsilon)^{1-\alpha} h^{'}(\varepsilon)}{\Gamma(2-\alpha)} \\
= & \quad -\frac{(1-\alpha)(t_0-\varepsilon)^{-\alpha} h(\varepsilon)}{\Gamma(2-\alpha)} - \frac{\alpha(1-\alpha)}{\Gamma(2-\alpha)} \int_{\varepsilon}^{t_0} (t_0 - s)^{-\alpha-1} h(s) ds \\
& \quad -\frac{(t_0-\varepsilon)^{1-\alpha} h^{'}(\varepsilon)}{\Gamma(2-\alpha)},
\end{aligned}$$

which leads to the relation

$$I_2 \geq 0$$

that together with (2.3) complete the proof of the theorem.

We consider case $(ii)$ in the remaining part of the proof. Here, we consider the following auxiliary function

$$h(t) = f(t) - f(t_0) + \varphi(t), t \in [0,1],$$

where $\varphi(t)$ is infinitely differentiable function on $R$, defined by

$$\varphi(t) = A \begin{cases} e^{\frac{kt^2}{t^2-t_0^2}}, & t < t_0, \\ 0, & t \geq t_0 \end{cases}$$

here, $k$ is a constant satisfies

$$0 < k \leq \frac{(t_0^3 - tt_0^2)^2 + t^2 t_0^2 (t_0^2 - t^2) + 2tt_0^2 (t_0^3 - t^3)}{2t_0^6}, t \in [0, t_0),$$

and $A$ is a positive constant satisfies

$$\frac{2A\varepsilon t_0^2 k}{(\varepsilon^2 - t_0^2)^2} e^{\frac{k\varepsilon^2}{\varepsilon^2 - t_0^2}} \geq f^{'}(\varepsilon), \quad \varepsilon \text{ is the positive constant obtained in (2.3).}$$

By calculation(applying the Law of L'Hospital), we easily obtain that

$$\varphi(t_0) = 0, \varphi^{'}(t_0) = 0, \varphi^{''}(t_0) = 0,$$

and that,

$$\begin{aligned}
& \varphi^{''}(t) = -A\left(e^{\frac{kt^2}{t^2-t_0^2}} \frac{2tt_0^2 k}{(t^2-t_0^2)^2}\right)^{'} \\
= & \quad -2Ake^{\frac{kt^2}{t^2-t_0^2}} \frac{t_0^6 + 2t^2 t_0^4 - 3t^4 t_0^2 - 2t^2 t_0^4 h}{(t-t_0)^4} \\
= & \quad -2Ake^{\frac{kt^2}{t^2-t_0^2}} \frac{(t_0^3 - tt_0^2)^2 + t^2 t_0^2 (t_0^2 - t^2) + 2tt_0^2 (t_0^3 - t^3) - 2t^2 t_0^4 k}{(t-t_0)^4},
\end{aligned}$$

for $t \in [0, t_0)$. Since $0 < k \leq \frac{(t_0^3 - tt_0^2)^2 + t^2 t_0^2 (t_0^2 - t^2) + 2tt_0^2 (t_0^3 - t^3)}{2t_0^6}$, so, for $t \in [0, t_0)$, there is

$$(t_0^3 - tt_0^2)^2 + t^2 t_0^2 (t_0^2 - t^2) + 2tt_0^2 (t_0^3 - t^3) - 2t^2 t_0^4 k$$

$$\geq (t_0^3 - tt_0^2)^2 + t^2 t_0^2 (t_0^2 - t^2) + 2tt_0^2 (t_0^3 - t^3) - 2t_0^6 k \geq 0.$$

Hence the Riemann-Liouville fractional integral $I^{2-\alpha} \varphi''(t_0) \leq 0$. Since $h \in C^2(0,1)$, $h(t_0) = h'(t_0) = 0$, there are

$$|h(t)| = |h(t) - h(t_0)| \leq |h'(\bar{t})|(t_0 - t) = |h'(\bar{t}) - h'(t_0)|(t_0 - t) \leq c_1 (t_0 - t)^2,$$

$$|h'(t)| = |h'(t) - h'(t_0)| \leq c_2 (t_0 - t), t \in [\varepsilon, t_0],$$

where $c_1 > 0, c_2 > 0$ are positive constants, and $\bar{t} \in (t, t_0)$. Thus, by the same arguments as case $(i)$, we can obtain that

$$\begin{aligned} D^\alpha g(t_0) = & \quad \frac{1}{\Gamma(2-\alpha)} \int_0^\varepsilon (t_0 - s)^{1-\alpha} h''(s) ds + \frac{1}{\Gamma(2-\alpha)} \int_\varepsilon^{t_0} (t_0 - s)^{1-\alpha} h''(s) ds \\ = & \quad I_1 + I_2, \end{aligned}$$

$$\begin{aligned} I_2 = & \quad -\frac{(1-\alpha)(t_0 - \varepsilon)^{-\alpha} h(\varepsilon)}{\Gamma(2-\alpha)} - \frac{\alpha(1-\alpha)}{\Gamma(2-\alpha)} \int_\varepsilon^{t_0} (t_0 - s)^{-\alpha-1} h(s) ds \\ & \quad -\frac{(t_0 - \varepsilon)^{1-\alpha} h'(\varepsilon)}{\Gamma(2-\alpha)}, \end{aligned}$$

which leads to the relation

$$I_2 \geq 0$$

that together with (2.3) produce $D^\alpha h(t_0) \geq 0$. Hence, we have

$$D^\alpha f(t_0) + I^{2-\alpha} \varphi''(t_0) \geq 0,$$

which implies that

$$D^\alpha f(t_0) \geq -I^{2-\alpha} \varphi''(t_0) \geq 0.$$

Thus, we complete this proof.

## 3 Existence result

In this section, we shall apply the lower and upper solutions method to consider the existence of solution to problem (1.1).

**Definition 3.1** we call a function $\alpha(t)$ a lower solution for problem (1.1), if $\alpha \in C^2([0,1], R)$ and

$$\begin{cases} D^\delta \alpha(t) + g(t, \alpha) \geq 0, t \in (0,1), 1 < \delta \leq 2, \\ \alpha(0) \leq a, \alpha(1) \leq b. \end{cases} \tag{3.1}$$

Similarly, we call a function $\beta(t)$ an upper solution for problem (1.1), if $\beta \in C^2([0,1], R)$ and

$$\begin{cases} D^\delta \beta(t) + g(t,\beta) \le 0, t \in (0,1), 1 < \delta \le 2, \\ \beta(0) \ge a, \beta(1) \ge b. \end{cases} \tag{3.2}$$

The following theorem is our main result.

**Theorem 3.1** Assume that $g : [0,1] \times R \to R$ is a continuous differential function respect to all variables, and that $g_u^{'}(t,u)$ is continuous in $t$ for all $u \in R$. Moreover, assume that $\alpha(t), \beta(t)$ are lower solution and upper solution of problem (1.1), such that $\alpha(t) \le \beta(t), t \in [0,1]$ and $g_u^{'}(t,\alpha) \le 0, g_u^{'}(t,\beta) \le 0$ for all $t \in [0,1]$. Then problem (1.1) has at least one solution $u(t) \in C[0,1]$ such that $\alpha(t) \le u(t) \le \beta(t), t \in [0,1]$.

**Proof** First of all, let us consider the the following modified boundary value problem

$$\begin{cases} D^\delta u(t) + g^*(t,u(t)) = 0, t \in (0,1), 1 < \delta \le 2, \\ u(0) = a, u(1) = b, \end{cases} \tag{3.3}$$

where

$$g^*(t,u) = \begin{cases} g(t,\alpha(t)) + e^{M_1 sin\frac{u-\alpha}{M_1} g_u^{'}(t,\alpha)} - \frac{\alpha - u}{N_1(1+u^2)} + \frac{e^{sin(cos(\frac{\alpha-u}{N_1}+\frac{3\pi}{2}))}}{1+\alpha^2} - \frac{2+\alpha^2}{1+\alpha^2}, & \text{if} \quad u < \alpha(t), \\ g(t,u(t)), \quad \text{if} \quad \alpha(t) \le u \le \beta(t), \\ g(t,\beta(t)) - e^{M_2 sin\frac{\beta-u}{M_2} g_u^{'}(t,\beta)} - \frac{\beta - u}{N_2(1+u^2)} - \frac{e^{sin(cos(\frac{u-\beta}{N_2}+\frac{3\pi}{2}))}}{1+\beta^2} + \frac{2+\beta^2}{1+\beta^2}, & \text{if} \quad u > \beta(t). \end{cases} \tag{3.4}$$

where $M_1, M_2 > 0$, such that $-\pi < \frac{u-\alpha}{M_1} < -\frac{3\pi}{2}$ and $-\pi < \frac{\beta-u}{M_2} < -\frac{3\pi}{2}$ for all $u-\alpha < 0, \beta - u < 0$; $N_1, N_2 < 0$, such that $-2\pi < \frac{\alpha-u}{N_1} < -\frac{3\pi}{2}$ and $-2\pi < \frac{u-\beta}{N_2} < -\frac{3\pi}{2}$ for $\alpha - u > 0$, $u - \beta > 0$. Obviously, from the continuity assumption to $g$, function $g^*$ is a continuous differential function with respect to all variables on $(t,x) \in [0,1] \times R$. In fact, we can obtain that

$$g_t^{*'}(t,u) = \begin{cases} g_t^{'}(t,\alpha(t)) + M_1 e^{M_1 sin\frac{u-\alpha}{M_1} g_u^{'}(t,\alpha)} sin\frac{u-\alpha}{M_1} g_{ut}^{''}(t,\alpha), & \text{if} \quad u < \alpha(t), \\ g_t^{'}(t,u(t)), & \text{if} \quad \alpha(t) \le u \le \beta(t), \\ g_t^{'}(t,\beta(t)) - M_2 e^{M_2 sin\frac{\beta-u}{M_2} g_u^{'}(t,\beta)} sin\frac{\beta-u}{M_2} g_{ut}^{''}(t,\beta), & \text{if} \quad u > \beta(t). \end{cases}$$

and

$$g_u^{*'}(t,u)=\begin{cases} e^{M_1 sin\frac{u-\alpha}{M_1}g_u'(t,\alpha)}cos\frac{u-\alpha}{M_1}g_u'(t,\alpha)-\frac{u^2-2\alpha u-1}{N_1(1+u^2)^2}+ \\ \frac{e^{sin(cos(\frac{\alpha-u}{N_1}+\frac{3\pi}{2}))}}{N_1(1+\alpha^2)}cos(cos(\frac{\alpha-u}{N_1}+\frac{3\pi}{2}))sin(\frac{\alpha-u}{N_1}+\frac{3\pi}{2})), & \text{if} \quad u<\alpha(t), \\ g_u'(t,u(t)), & \text{if} \quad \alpha(t)\le u\le\beta(t), \\ e^{M_2 sin\frac{\beta-u}{M_2}g_u'(t,\beta)}cos\frac{\beta-u}{M_2}g_u'(t,\beta)-\frac{u^2-2\beta u-1}{N_2(1+u^2)^2}+ \\ \frac{e^{sin(cos(\frac{u-\beta}{N_2}+\frac{3\pi}{2}))}}{N_2(1+\beta^2)}cos(cos(\frac{u-\beta}{N_2}+\frac{3\pi}{2}))sin(\frac{u-\beta}{N_2}+\frac{3\pi}{2})), & \text{if} \quad u>\beta(t). \end{cases}$$

We claim that if $u(t)\in C[0,1]$ is any solution of (3.3), then $\alpha(t)\le u(t)\le\beta(t)$ for all $t\in[0,1]$ and hence $u$ is a solution of (1.1) which satisfies $\alpha(t)\le u(t)\le\beta(t), t\in[0,1]$.

In fact, from the assumptions of theorem, $u''(t)=D^{2-\delta}g^*(t,u(t))=I^{\delta-1}(g_t^{*'}(t,u)+g_u^{*'}(t,u)u'(t))\in C[0,1]$(because $u'(t)=c_2+I^{\delta-1}g^*(t,u)\in C[0,1], c_2\in R$), that is, $u''\in C^2[0,1]$. Now, by $\alpha(0)\le u(0),\alpha(1)\le u(1)$, we suppose by contradiction that there is $t_0\in(0,1)$ such that

$$w(t_0)=\alpha(t_0)-u(t_0)=\max_{t\in[0,1]}(\alpha(t)-u(t))=\max_{t\in[0,1]}w(t)>0.$$

Then, by Theorem 2.1, there is

$$D^\delta w(t_0)\le 0. \tag{3.5}$$

Moreover, by the previous assumptions, we know that

$$-\pi<\frac{u(t_0)-\alpha(t_0)}{M_1}<-\frac{3\pi}{2};\quad -\frac{\pi}{2}<\frac{\alpha(t_0)-u(t_0)}{N_1}+\frac{3\pi}{2}<0,$$

hence,

$$M_1 sin\frac{u(t_0)-\alpha(t_0)}{M_1}g_u'(t_0,\alpha)\ge 0;\quad sin(cos(\frac{\alpha(t_0)-u(t_0)}{N_1}+\frac{3\pi}{2}))\ge 0.$$

Thus, we have

$$\begin{aligned} D^\delta w(t_0)= &\quad D^\delta\alpha(t_0)-D^\delta u(t_0) \\ \ge &\quad -g(t_0,\alpha(t_0))+g(t_0,\alpha(t_0))+e^{M_1 sin\frac{u(t_0)-\alpha(t_0)}{M_1}g_u'(t_0,\alpha)}-\frac{\alpha(t_0)-u(t_0)}{N_1(1+u^2)}+ \\ &\quad \frac{e^{sin(cos(\frac{\alpha(t_0)-u(t_0)}{N_1}+\frac{3\pi}{2}))}}{1+\alpha^2}-\frac{2+\alpha^2}{1+\alpha^2} \\ \ge &\quad 1-\frac{\alpha(t_0)-u(t_0)}{N_1(1+u^2)}+\frac{1}{1+\alpha^2}-\frac{2+\alpha^2}{1+\alpha^2}>0, \end{aligned}$$

which is a contradiction with (3.5). The same argument, with obvious changes works in the proof of $\alpha(t) \le u(t)$ in $[0,1]$, we can obtain that $u(t) \le \beta(t)$ in $[0,1]$. Indeed, by $\beta(0) \ge u(0), \beta(1) \ge u(1)$, we suppose by contradiction that there is $t_0 \in (0,1)$ such that

$$w(t_0) = \beta(t_0) - u(t_0) = \min_{t\in[0,1]} (\beta(t) - u(t)) = \min_{t\in[0,1]} w(t) < 0.$$

Then, by Theorem 2.2, there is

$$D^\delta w(t_0) \ge 0.$$

Moreover, by the previous assumptions, we know that

$$-\pi < \frac{\beta(t_0) - u(t_0)}{M_2} < -\frac{3\pi}{2}; \quad -\frac{\pi}{2} < \frac{u(t_0) - \beta(t_0)}{N_2} + \frac{3\pi}{2} < 0,$$

hence,

$$M_2 sin\frac{\beta(t_0) - u(t_0)}{M_2} g_u^{'}(t_0, \beta) \ge 0; \quad sin(cos(\frac{u(t_0) - \beta(t_0)}{N_2} + \frac{3\pi}{2})) \ge 0.$$

Thus, we have

$$\begin{aligned}
D^\delta w(t_0) = \quad & D^\delta \beta(t_0) - D^\delta u(t_0) \\
\le \quad & -g(t_0, \beta(t_0)) + g(t_0, \beta(t_0)) - e^{M_2 sin\frac{\beta(t_0)-u(t_0)}{M_2} g_u^{'}(t_0,\beta)} - \frac{\beta(t_0) - u(t_0)}{N_2(1+u^2)} - \\
& \frac{e^{sin(cos(\frac{u(t_0)-\beta(t_0)}{N_2} + \frac{3\pi}{2}))}}{1+\beta^2} + \frac{2+\beta^2}{1+\beta^2} \\
\le \quad & -1 - \frac{\beta(t_0) - u(t_0)}{N_2(1+u^2)} - \frac{1}{1+\beta^2} + \frac{2+\beta^2}{1+\beta^2} < 0,
\end{aligned}$$

which produces a contradiction.

Then, the claim is proved and now it is sufficient to prove that problem (3.3) has at least one solution.

From the standard argument, we can know that the solution of (3.3) has the form

$$u(t) = \int_0^1 G(t,s) g^*(s, u(s)) ds + a + (b-a)t. \tag{3.6}$$

where

$$G(t,s) = \begin{cases} \frac{t(1-s)^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, 0 \le s \le t \le 1, \\ \\ \frac{t(1-s)^{\alpha-1}}{\Gamma(\alpha)}, 0 \le t \le s \le 1. \end{cases}$$

In fact, we may consider the solution of the linear problem of (3.3)

$$\begin{cases} D^\delta u(t) + \rho(t) = 0, t \in (0,1), \\ \\ u(0) = a, u(1) = b, \end{cases} \tag{3.7}$$

where $\rho(t) \in C[0,1]$. Applying the fractional integral $I^\delta$ on both sides of equation in (3.7) and Using the following relationship (Lemma 2.22[1]): $I^\delta D^\delta f(t) = f(t) - c_1 - c_2 t, c_i \in R, i = 1,2$ for $f(t) \in AC([0,T]$ or $f(t) \in C([0,T])$ and $1 < \delta \leq 2$, we obtain

$$u(t) = c_1 + c_2 t - I^\delta \rho(t),$$

for some constants $c_i, i = 1,2$. By boundary value conditions of problem (3.7), we can calculate out that $c_1 = a$, $c_2 = b - a + I^\delta \rho(1)$, Consequently, the solution of problem (3.7) is

$$\begin{aligned} u(t) = \quad & a + (b-a)t + tI^\delta \rho(1) - I^\delta \rho(t) \\ = \quad & a + (b-a)t + \frac{1}{\Gamma(\delta)} \int_0^1 t(1-s)^{\delta-1} \rho(s) ds - \frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} \rho(s) ds \\ = \quad & a + (b-a)t + \int_0^t \frac{t(1-s)^{\delta-1} - (t-s)^{\delta-1}}{\Gamma(\delta)} \rho(s) ds + \frac{1}{\Gamma(\delta)} \int_t^1 t(1-s)^{\delta-1} \rho(s) ds \\ = \quad & \int_0^1 G(t,s) \rho(s) ds + a + (b-a)t. \end{aligned}$$

Hence, the solution $u$ of problem (3.7) is $u(t) = \int_0^1 G(t,s)\rho(s)ds + a + (b-a)t$, $t \in [0,1]$, which means that the solution of (3.3) has the form of (3.6).

Now, consider the operator $T : C[0,1] \to C[0,1]$ by

$$Tu(t) = \int_0^1 G(t,s) g^*(s, u(s)) ds + a + (b-a)t. \tag{3.8}$$

From the definition of function $g^*(t,u)$ that $T : C[0,1] \to C[0,1]$ is well defined, continuous, and $T(\Omega)$ is a bounded (here, $\Omega$ is a bounded subset of $C[0,1]$). It is well know that $u \in C[0,1]$ is a solution to (3.3) if and only if $u$ is a fixed point of operator $T$. Moreover, we can prove that $T(\Omega)$ is relatively compact. Indeed, we can obtain that

$$\begin{aligned} & |\frac{d}{dt} Tu(t)| \\ = \quad & |b - a + \frac{1}{\Gamma(\delta)} \int_0^1 (1-s)^{\delta-1} g^*(s,u(s)) ds - \frac{1}{\Gamma(\delta-1)} \int_0^t (t-s)^{\delta-2} g^*(s,u(s)) ds| \\ \leq \quad & |b-a| + \frac{M}{\Gamma(1+\delta)} + \frac{M}{\Gamma(\delta)}, \end{aligned}$$

where $M = \max_{t\in[0,1],u\in\Omega} |g^*(t,u)| + 1$. Hence, this is sufficient to ensure the the relatively compact of $T(\Omega)$ via the Ascoli-Arzela theorem.

We let

$$\Omega = \{u \in C[0,1]; \|u\| < R\},$$

where

$$R > \{3|a|, 3|b-a|, \frac{3L}{\Gamma(\delta)} \int_0^1 (s+1)(1-s)^{\delta-1} ds\},$$

here,

$$L = \max_{t\in[0,1],\alpha(t)\le u(t)\le\beta(t)} |g(t,u(t))| + e^{L_1} + \max\{\frac{\|\alpha\|+1}{|N_1|}, \frac{\|\beta\|+1}{|N_2|}\} + e + 2.$$

$$L_1 = \max\{\max_{t\in[0,1]} M_1|g_u^{'}(t,\alpha)|, \max_{t\in[0,1]} M_2|g_u^{'}(t,\beta)|\}.$$

Then, for $u \in \Omega$, we have

$$\begin{aligned} |Tu(t)| \le & \quad |a| + |b-a| + \frac{L}{\Gamma(\delta)} \int_0^1 (s+1)(1-s)^{\delta-1} ds \\ \le & \quad \frac{R}{3} + \frac{R}{3} + \frac{R}{3} = R, \end{aligned}$$

which implies that $T(\Omega) \subseteq \Omega$. Therefore, we see that, the existence of a fixed point for the operator $T$ follows from the Schauder fixed theorem.

Finally, we give an example.

**Example.** We consider the following boundary value problem

$$\begin{cases} D^{\frac{3}{2}}u - u^3 + 1 = 0, \quad t \in (0,1), \\ \\ u(0) = u(1) = 0. \end{cases} \tag{3.9}$$

Let $g(t,u) = 1 - u^3$. Obviously, we can check that $\alpha(t) \equiv 0$ is a lower solution for problem (3.9), and $\beta(t) = 3$ is an upper solution for problem (3.9). And that, $g_t^{'}(t,u) = g_{ut}^{''}(t,u) = 0$, $g_u^{'}(t,u) = -3u^2$, $g_u^{'}(t,\alpha) = 0, g_u^{'}(t,\beta) = -27$, hence, function $g$ satisfies the assumption condition of theorem 3.1. Then, the theorem 3.1 assures that problem (3.9) has at least one solution $u^* \in C[0,1]$ with $0 \le u^*(t) \le 3, t \in [0,1]$.

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