



## Properties of the third order trinomial functional differential equations

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**Abstract.** The purpose of the paper is to study asymptotic properties of the third-order delay differential equation

$$\left( r_2(t) \left( r_1(t) (y'(t))^\gamma \right)' \right)' + p(t) (y'(t))^\gamma + q(t) f(y(\tau(t))) = 0. \quad (E)$$

Employing comparison principles with a suitable first order delay differential equation we shall establish criteria for all nonoscillatory solutions of (E) to converge to zero, while oscillation of a couple of first order delay differential equations yields oscillation of (E). An example is provided to illustrate the main results.

**Keywords:** third-order, functional, trinomial differential equations, transformation, oscillation, canonical form.

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### 1 Introduction

In this paper, we are dealing with the oscillation and asymptotic behavior of solutions of the third-order nonlinear delay differential equation

$$\left( r_2(t) \left( r_1(t) (y'(t))^\gamma \right)' \right)' + p(t) (y'(t))^\gamma + q(t) f(y(\tau(t))) = 0, \quad (E)$$


where  $r_2, r_1, p, q \in C(I, \mathbb{R})$ ,  $I = [t_0, \infty) \subset \mathbb{R}$ ,  $t_0 \geq 0$ ,  $f \in C(-\infty, \infty)$ . Throughout the paper, we will assume that the following conditions are fulfilled:

(H<sub>1</sub>)  $r_1(t)$ ,  $r_2(t)$ ,  $q(t)$  are positive functions,  $p(t)$  is nonnegative,

(H<sub>2</sub>)  $\tau(t) \in C^1(I, \mathbb{R})$  satisfies  $0 < \tau(t) \leq t$ ,  $\tau'(t) > 0$  and  $\lim_{t \rightarrow \infty} \tau(t) = \infty$ ,

(H<sub>3</sub>)  $\gamma$  is the quotient of two positive odd integers,

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(H<sub>4</sub>)  $xf(x) > 0$ ,  $f'(x) \geq 0$  for  $x \neq 0$ ,  $-f(-xy) \geq f(xy) \geq f(x)f(y)$  for  $xy > 0$ ,

(H<sub>5</sub>)  $R_2(t) = \int_{t_0}^t r_2^{-1}(s) ds \rightarrow \infty$  as  $t \rightarrow \infty$ .

By a solution of (E), we mean a function  $y(t)$  such that  $r_2(t) (r_1(t) (y'(t))^\gamma) \in C^1[T_y, \infty)$  for a certain  $T_y \geq t_0$  and  $y(t)$  satisfies (E) on the half-line  $[T_y, \infty)$ . Our attention is restricted to only such extendable solutions  $y(t)$  of (E) which satisfy  $\sup\{|y(t)| : t \geq T\} > 0$  for all  $T \geq T_y$ . Further, we make a standing hypothesis that (E) possesses such a solution. As customary, a solution  $y(t)$  of (E) is said to be *oscillatory* if it has arbitrarily large zeros on  $[T_y, \infty)$  and otherwise it is called to be *nonoscillatory*. Equation (E) itself is called *oscillatory* if all of its solutions are oscillatory.

As is well known, differential equations of third order have long been considered as valuable tools in the modeling of many phenomena in different areas of applied mathematics and physics. Indeed, it is worthwhile to mention their use in the study of entry-flow phenomenon [11], the propagation of electrical pulses in the nerve of a squid approximated by the famous Nagumo's equation [16], the feedback nuclear reactor problem [23] and so on.

Hence, a great deal of work has been done in recent decades and the investigation of oscillatory and asymptotic properties for these equations has taken the shape of a well-developed theory turned mainly toward functional differential equations. In fact, the development of oscillation theory for the third order differential equations began in 1961 with the appearance of the work of Hanan [10] and Lazer [15]. Since then, many authors contributed to the subject studying different classes of equations and applying various techniques, see, for instance, [1–23]. A systematic survey of the most significant efforts in this theory can be found in the excellent monographs of Swanson [21], Greguš [9] and the very recent-one of Padhi and Pati [19].

In fact, determination of trinomial delay differential equations of third order often depends on the close related second order differential equation. The case when this associated equation is oscillatory was object of research in [7]. Taking under the assumption the nonoscillation of the corresponding auxiliary equation, special cases of (E) has been considered in many papers.

The partial case of (E), namely

$$y'''(t) + p(t)y'(t) + q(t)y(\tau(t)) = 0$$

has been studied e.g., by present authors [6], Parhi and Padhi [17, 18].

Series of articles [3–5] deal with the case of (E) when  $\gamma = 1$ , i.e.,

$$\left( r_2(t) (r_1(t)y'(t)) \right)' + p(t)y'(t) + q(t)f(y(\tau(t))) = 0. \quad (E')$$

By means of a generalized Riccati transformation and integral averaging technique, authors have established some sufficient conditions which ensure that any solution of (E') oscillates or converges to zero. Further oscillation criteria have been obtained by establishing a useful comparison principle with either first or second order delay differential inequality, given in [1]. Another approach of investigation (E'), which depends on the sign of a particular functional, was proposed in [8] as a generalization of known results for ordinary case [15].

In spite of a substantial number of existing papers on asymptotic behavior of solutions of third order trinomial equation (E'), many interesting questions regarding oscillatory properties remain without answers. More exactly, existing literature does not provide any criteria which directly ensure oscillation of (E') as well as condition  $\int_{t_0}^{\infty} r_1^{-1}(s) ds = \infty$  has been always assumed to hold.

In view of the above motivation, our purpose in this paper is to extend the technique presented in [6] to cover also more general differential equation (E). We stress that our criteria does not require any condition on the function  $r_1(t)$ .

As convenient, all functional inequalities considered in this paper are assumed to hold eventually, that is, they are satisfied for all sufficiently large  $t$ .

We say that (E) has the *property (P)* if all of its nonoscillatory solutions  $y(t)$  satisfy the condition

$$y(t)y'(t) < 0. \quad (1.1)$$

As will be shown, the properties of (E) are closely connected with the positive solutions of the auxiliary second-order differential equation

$$(r_2(t)v'(t))' + \frac{p(t)}{r_1(t)}v(t) = 0, \quad (V)$$

as the following theorem says.

**Theorem 1.1.** *Let (V) possess a positive solution  $v(t)$ . Then the operator*

$$\mathcal{L}y = \left( r_2(t) \left( r_1(t) (y'(t))^\gamma \right)' \right)' + p(t) (y'(t))^\gamma$$

can be represented as

$$\mathcal{L}y \equiv \frac{1}{v(t)} \left( r_2(t)v^2(t) \left( \frac{r_1(t)}{v(t)} (y'(t))^\gamma \right)' \right)'. \quad (1.2)$$

*Proof.* It is straightforward to see that

$$\begin{aligned} \mathcal{L}y &\equiv \frac{1}{v(t)} \left( r_2(t)v^2(t) \left( \frac{r_1(t)}{v(t)} (y'(t))^\gamma \right)' \right)' \\ &= \frac{1}{v(t)} \left( r_2(t) \left( r_1(t) (y'(t))^\gamma \right)' v(t) - r_1(t)r_2(t) (y'(t))^\gamma v'(t) \right)' \\ &= \left( r_2(t) \left( r_1(t) (y'(t))^\gamma \right)' \right)' - \frac{r_1(t)(r_2(t)v'(t))'}{v(t)} (y'(t))^\gamma \\ &= \left( r_2(t) \left( r_1(t) (y'(t))^\gamma \right)' \right)' + p(t) (y'(t))^\gamma. \end{aligned}$$

□

**Corollary 1.2.** *If  $v(t)$  is a positive solution of (V), then (E) can be written as the binomial equation*

$$\left( r_2(t)v^2(t) \left( \frac{r_1(t)}{v(t)} (y'(t))^\gamma \right)' \right)' + q(t)v(t)f(y(\tau(t))) = 0. \quad (E_c)$$

It is convenient if the Eq. (E<sub>c</sub>) is in the canonical form, i.e.

$$\int_t^\infty \frac{ds}{r_2(s)v^2(s)} = \infty \quad (1.3)$$

and

$$\int_t^\infty \left( \frac{v(s)}{r_1(s)} \right)^{1/\gamma} ds = \infty, \quad (1.4)$$

because such equations (as will be shown later) have simpler structure of possible nonoscillatory solution.

In what follows, we first investigate the properties of the positive solutions of (V) and then, instead of studying properties of the trinomial equation (E), we will study the behavior of its pertaining binomial representation (E<sub>c</sub>).

The following result is a consequence of Sturm's comparison theorem and guarantees the existence of a nonoscillatory solution of (V).

**Lemma 1.3.** *Assume that*

$$R_2^2(t) \frac{r_2(t)}{r_1(t)} p(t) \leq \frac{1}{4}, \quad \text{for } t \geq t_0. \quad (1.5)$$

*Then (V) possesses a positive solution.*

To be sure that (V) possesses a positive solution, in what follows, we will assume that (1.5) holds.

For our next purposes, the following lemma will be useful.

**Lemma 1.4.** *Assume that (1.5) is fulfilled, then (V) always possesses a nonoscillatory solution satisfying (1.3).*

*Proof.* If  $v_1(t)$  is a positive solution of (V), such that

$$\int_t^\infty \frac{ds}{r_2(s)v_1^2(s)} < \infty,$$

then another linearly independent solution of (V) is given by

$$v_2(t) = v_1(t) \int_t^\infty \frac{ds}{r_2(s)v_1^2(s)}. \quad (1.6)$$

Really, taking (1.6) into account, it is easy to see that

$$\begin{aligned} (r_2(t)v_2'(t))' &= (r_2(t)v_1'(t))' \int_t^\infty \frac{ds}{r_2(s)v_1^2(s)} = -\frac{p(t)}{r_1(t)}v_1(t) \int_t^\infty \frac{ds}{r_2(s)v_1^2(s)} \\ &= -\frac{p(t)}{r_1(t)}v_2(t). \end{aligned}$$

Moreover,  $v_2(t)$  meets (1.3) by now. To see that, denote  $\mathcal{U}(t) = \int_t^\infty \frac{ds}{r_2(s)v_1^2(s)}$ , then  $\lim_{t \rightarrow \infty} \mathcal{U}(t) = 0$  and

$$\begin{aligned} \int_{t_0}^\infty \frac{1}{r_2(t)v_2^2(t)} dt &= \int_{t_0}^\infty \frac{\mathcal{U}^{-2}(t)}{r_2(t)v_1^2(t)} dt = -\int_{t_0}^\infty \frac{\mathcal{U}'(t)}{\mathcal{U}^2(t)} dt \\ &= \lim_{t \rightarrow \infty} \left( \frac{1}{\mathcal{U}(t)} - \frac{1}{\mathcal{U}(t_0)} \right) = \infty. \end{aligned}$$

□

Bringing together all the previous results, it is reasonable to conclude the following.

**Lemma 1.5.** *Let (1.5) hold. Then the trinomial equation (E) can be written in its binomial form (E<sub>c</sub>). Moreover, if (1.4) is satisfied, then (E<sub>c</sub>) is in canonical form.*

From now on, we are prepared to study the properties of (E) with the help of its equivalent representation (E<sub>c</sub>). In view of familiar Kiguradze's lemma [12], we have the following structure of nonoscillatory solutions of (E).

**Lemma 1.6.** *Let (1.5) hold and assume that  $v(t)$  is such positive solution of (V) that satisfies (1.3). If (1.4) is satisfied, then every positive solution of (E<sub>c</sub>) is either of degree 0, that is*

$$\begin{aligned} \frac{r_1(t)}{v(t)} (y'(t))^\gamma < 0, \quad r_2(t)v^2(t) \left( \frac{r_1(t)}{v(t)} (y'(t))^\gamma \right)' > 0, \\ \left( r_2(t)v^2(t) \left( \frac{r_1(t)}{v(t)} (y'(t))^\gamma \right)' \right)' < 0, \end{aligned} \quad (1.7)$$

or of degree 2, that is,

$$\begin{aligned} \frac{r_1(t)}{v(t)} (y'(t))^\gamma > 0, \quad r_2(t)v^2(t) \left( \frac{r_1(t)}{v(t)} (y'(t))^\gamma \right)' > 0, \\ \left( r_2(t)v^2(t) \left( \frac{r_1(t)}{v(t)} (y'(t))^\gamma \right)' \right)' < 0. \end{aligned} \quad (1.8)$$

In the case when (1.4) fails, there may exist one extra class, that is

$$\begin{aligned} \frac{r_1(t)}{v(t)} (y'(t))^\gamma < 0, \quad r_2(t)v^2(t) \left( \frac{r_1(t)}{v(t)} (y'(t))^\gamma \right)' < 0, \\ \left( r_2(t)v^2(t) \left( \frac{r_1(t)}{v(t)} (y'(t))^\gamma \right)' \right)' < 0. \end{aligned} \quad (1.9)$$

If we denote the classes of positive solutions of (E<sub>c</sub>) satisfying (1.7), (1.8) and (1.9) by  $\mathcal{N}_0$ ,  $\mathcal{N}_2$  and  $\mathcal{N}_*$ , respectively, Then the set  $\mathcal{N}$  of all positive solutions of (E<sub>c</sub>) (as well as (E)) has the following decomposition

$$\mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_2$$

provided that both (1.3) and (1.4) hold and

$$\mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_2 \cup \mathcal{N}_*$$

if (1.4) fails.

## 2 Canonical form

Since (E<sub>c</sub>) is in a canonical form, the set of all positive solutions of (E<sub>c</sub>) is given by

$$\mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_2.$$

Now we are prepared to provide criteria for property (P) of (E) and later also for oscillation of (E).

Let us denote

$$Q(t) = q(t)v(t)f \left( \int_{t_1}^{\tau(t)} \left( \frac{v(s)}{r_1(s)} \int_{t_1}^s \frac{1}{r_2(u)v^2(u)} du \right)^{1/\gamma} ds \right).$$

**Theorem 2.1.** *Let (1.5) hold and assume that  $v(t)$  is such positive solution of (V) that (1.3) and (1.4) are satisfied. If the first order nonlinear differential equation*

$$z'(t) + Q(t)f \left( z^{1/\gamma}(\tau(t)) \right) = 0 \quad (E_P)$$

is oscillatory, then (E) has property (P).

*Proof.* Assume that (E) has an eventually positive solution  $y(t)$ . Then  $y(t)$  is also solution of  $(E_c)$ . It follows from Lemma 1.6 that  $y(t)$  is either of degree 2 or degree 0. If  $y(t) \in \mathcal{N}_2$ , then by making use of the fact that

$$z(t) = r_2(t)v^2(t) \left( \frac{r_1(t)}{v(t)} (y'(t))^\gamma \right)' > 0$$

is decreasing, we have

$$\begin{aligned} \frac{r_1(t)}{v(t)} (y'(t))^\gamma &\geq \int_{t_1}^t \frac{1}{r_2(u)v^2(u)} \left( r_2(u)v^2(u) \left( \frac{r_1(u)}{v(u)} (y'(u))^\gamma \right)' \right) du \\ &\geq z(t) \int_{t_1}^t \frac{1}{r_2(u)v^2(u)} du. \end{aligned}$$

Integrating from  $t_1$  to  $t$ , we are led to

$$\begin{aligned} y(t) &\geq \int_{t_1}^t \left( \frac{v(s)}{r_1(s)} z(s) \int_{t_1}^s \frac{1}{r_2(u)v^2(u)} du \right)^{1/\gamma} ds \\ &\geq z^{1/\gamma}(t) \int_{t_1}^t \left( \frac{v(s)}{r_1(s)} \int_{t_1}^s \frac{1}{r_2(u)v^2(u)} du \right)^{1/\gamma} ds. \end{aligned}$$

Hence,

$$y(\tau(t)) \geq z^{1/\gamma}(\tau(t)) \int_{t_1}^{\tau(t)} \left( \frac{v(s)}{r_1(s)} \int_{t_1}^s \frac{1}{r_2(u)v^2(u)} du \right)^{1/\gamma} ds.$$

Combining the last inequality together with  $(E_c)$ , we obtain

$$\begin{aligned} -z'(t) &\geq q(t)v(t)f \left( z^{1/\gamma}(\tau(t)) \int_{t_1}^{\tau(t)} \left( \frac{v(s)}{r_1(s)} \int_{t_1}^s \frac{1}{r_2(u)v^2(u)} du \right)^{1/\gamma} ds \right) \\ &\geq q(t)v(t)f \left( \int_{t_1}^{\tau(t)} \left( \frac{v(s)}{r_1(s)} \int_{t_1}^s \frac{1}{r_2(u)v^2(u)} du \right)^{1/\gamma} ds \right) f \left( z^{1/\gamma}(\tau(t)) \right). \end{aligned}$$

Therefore, it is clear that  $z(t)$  is a positive solution of differential inequality

$$z'(t) + Q(t)f\left(z^{1/\gamma}(\tau(t))\right) \leq 0.$$

On the other hand, in view of Theorem 1 of Philos [20], the corresponding differential equation  $(E_p)$  also has a positive solution. This is a contradiction and we conclude that  $y(t)$  is of degree 0 and the first two inequalities of (1.7) implies property (P) of equation (E).  $\square$

Employing criteria for oscillation of  $(E_p)$  we immediately get criteria for property (P) of (E).

**Corollary 2.2.** *Let (1.5) hold and assume that  $v(t)$  is such positive solution of (V) that (1.3) and (1.4) are satisfied. Let  $f(u) = u^\gamma$ . Assume that*

$$\liminf_{t \rightarrow \infty} \int_{\tau(t)}^t q(u)v(u) \left( \int_{t_1}^{\tau(u)} \left( \frac{v(s)}{r_1(s)} \int_{t_1}^s \frac{1}{r_2(x)v^2(x)} dx \right)^{1/\gamma} ds \right)^\gamma du > \frac{1}{e}, \quad (C_1)$$

then (E) has the property (P).

**Corollary 2.3.** *Let (1.5) hold and assume that  $v(t)$  is such positive solution of (V) that (1.3) and (1.4) are satisfied. Let  $\gamma > 1$ ,  $f(u) = u$ . If*

$$\int_{t_0}^{\infty} q(t)v(t) \int_{t_1}^{\tau(t)} \left( \frac{v(s)}{r_1(s)} \int_{t_1}^s \frac{1}{r_2(u)v^2(u)} du \right)^{1/\gamma} ds dt = \infty, \quad (2.1)$$

then (E) has property (P).

**Corollary 2.4.** *Let (1.5) hold and assume that  $v(t)$  is such positive solution of (V) that (1.3) and (1.4) are satisfied. Suppose  $\gamma \in (0, 1)$ ,  $\theta \in (0, 1) > 0$ ,  $\tau(t) = \theta t$ ,  $f(u) \equiv u$ . If there exists*

$$\lambda > \ln(\gamma) / \ln(\theta),$$

such that

$$\liminf_{t \rightarrow \infty} \left( \left( q(t)v(t) \int_{t_1}^{\tau(t)} \left( \frac{v(s)}{r_1(s)} \int_{t_1}^s \frac{1}{r_2(u)v^2(u)} du \right)^{1/\gamma} ds \right) \exp(-t^\lambda) \right) > 0$$

holds, then (E) has property (P).

**Corollary 2.5.** *Let (1.5) hold and assume that  $v(t)$  is such positive solution of (V) that (1.3) and (1.4) are satisfied. Suppose  $\gamma \in (0, 1)$ ,  $\theta \in (0, 1) > 0$ ,  $\tau(t) = t^\theta$ ,  $f(u) \equiv u$ .*

*If there exists*

$$\lambda > \ln(\gamma) / \ln(\theta),$$

such that

$$\liminf_{t \rightarrow \infty} \left( \left( q(t)v(t) \int_{t_1}^{\tau(t)} \left( \frac{v(s)}{r_1(s)} \int_{t_1}^s \frac{1}{r_2(u)v^2(u)} du \right)^{1/\gamma} ds \right) \exp(-\ln^\lambda(t)) \right) > 0$$

holds, then (E) has property (P).

The sufficient conditions for oscillation of  $(E_p)$  in previous corollaries are recalled from [14], [13] and [22], respectively.

Now, we enhance our results to ensure stronger asymptotic behavior of the nonoscillatory solutions of  $(E)$ . We impose an additional condition on the coefficients of  $(E)$  to guarantee that every solution of  $(E)$  either oscillates or tends to zero as  $t \rightarrow \infty$ .

**Lemma 2.6.** *Assume that equation  $(E)$  possesses property  $(P)$ . If*

$$\int_{t_0}^{\infty} \left( \frac{v(u)}{r_1(u)} \int_u^{\infty} \frac{1}{r_2(s)v^2(s)} \int_s^{\infty} v(x)q(x) dx ds \right)^{1/\gamma} du = \infty, \quad (2.2)$$

then every nonoscillatory solution of  $(E)$  tends to zero as  $t \rightarrow \infty$ .

*Proof.* Let  $y(t)$  be an eventually positive solution of  $(E)$ . Recall  $(E)$  possesses property  $(P)$ , iff  $y(t)y'(t) < 0$ . It is clear that there exists a  $\lim_{t \rightarrow \infty} y(t) = \ell \geq 0$ . Assume for contradiction  $\ell > 0$ . On the other hand,  $y(t)$  is also a solution of  $(E_c)$  of degree 0. Using  $(H_4)$  in  $(E_c)$ , we have

$$- \left( r_2(t)v^2(t) \left( \frac{r_1(t)}{v(t)} (y'(t))^\gamma \right)' \right)' = v(t)q(t)f(y(\tau(t))) \geq f(\ell)v(t)q(t).$$

Then, integration of the previous inequality from  $t$  to  $\infty$  leads to

$$r_2(t)v^2(t) \left( \frac{r_1(t)}{v(t)} (y'(t))^\gamma \right)' \geq f(\ell) \int_t^{\infty} v(x)q(x) dx.$$

Integrating the last inequality from  $t$  to  $\infty$ , we conclude

$$-\frac{r_1(t)}{v(t)} (y'(t))^\gamma \geq f(\ell) \int_t^{\infty} \frac{1}{r_2(s)v^2(s)} \int_s^{\infty} v(x)q(x) dx ds.$$

Integrating once more the last inequality from  $t$  to  $\infty$ , we obtain

$$y(t) \leq y(t_1) - f^{1/\gamma}(\ell) \int_{t_1}^t \left( \frac{v(u)}{r_1(u)} \int_u^{\infty} \frac{1}{r_2(s)v^2(s)} \int_s^{\infty} v(x)q(x) dx ds \right)^{1/\gamma} du.$$

Letting  $t \rightarrow \infty$  and using (2.2), it is easy to see that  $\lim_{t \rightarrow \infty} y(t) = -\infty$ , which contradicts the fact that  $y(t)$  is a positive solution of  $(E_c)$ . Therefore, we deduce that  $\ell = 0$ . The proof is complete.  $\square$

Requiring oscillation of another suitable first order differential equation, we can obtain even oscillation of  $(E)$ .

**Theorem 2.7.** *Let (1.5) hold and assume that  $v(t)$  is such positive solution of  $(V)$  that (1.3) and (1.4) are satisfied. Suppose that there exists a function  $\xi(t) \in C^1([t_0, \infty))$  such that*

$$\xi'(t) \geq 0, \quad \xi(t) > t, \quad \eta(t) = \tau(\xi(\xi(t))) < t. \quad (2.3)$$

If both the first-order delay differential equations  $(E_p)$  and

$$z'(t) + \left( \frac{v(t)}{r_1(t)} \int_t^{\xi(t)} \frac{1}{r_2(s)v^2(s)} \int_s^{\xi(s)} v(x)q(x) dx ds \right)^{1/\gamma} f^{1/\gamma}(z[\eta(t)]) = 0 \quad (E_0)$$

are oscillatory, then  $(E)$  is oscillatory.



*Proof.* Let  $y(t)$  be an eventually positive solution of (E). It follows from Lemma 1.6 that either  $y(t) \in \mathcal{N}_0$  or  $y(t) \in \mathcal{N}_2$ . In view of the proof of Theorem 2.1, it is known that oscillation of  $(E_p)$  eliminates all solutions of degree 2. Therefore,  $y(t)$  is of degree 0.

An integration of  $(E_c)$  from  $t$  to  $\xi(t)$  yields

$$\begin{aligned} r_2(t)v^2(t) \left( \frac{r_1(t)}{v(t)} (y'(t))^\gamma \right)' &\geq \int_t^{\xi(t)} v(x)q(x)f(y[\tau(x)]) dx \\ &> f(y[\tau(\xi(t))]) \int_t^{\xi(t)} v(x)q(x) dx. \end{aligned}$$

Then

$$\left( \frac{r_1(t)}{v(t)} (y'(t))^\gamma \right)' \geq \frac{f(y[\tau(\xi(t))])}{r_2(t)v^2(t)} \int_t^{\xi(t)} v(x)q(x) dx.$$

Integrating the above inequality from  $t$  to  $\xi(t)$  once more, we have

$$\begin{aligned} -\frac{r_1(t)}{v(t)} (y'(t))^\gamma &\geq \int_t^{\xi(t)} \frac{f(y[\tau(\xi(s))])}{r_2(s)v^2(s)} \int_s^{\xi(s)} v(x)q(x) dx ds \\ &\geq f(y[\eta(t)]) \int_t^{\xi(t)} \frac{1}{r_2(s)v^2(s)} \int_s^{\xi(s)} v(x)q(x) dx ds. \end{aligned}$$

Finally, integration from  $t$  to  $\infty$  leads us

$$y(t) \geq \int_t^\infty \left( \frac{v(u)}{r_1(u)} f(y[\eta(u)]) \int_u^{\xi(u)} \frac{1}{r_2(s)v^2(s)} \int_s^{\xi(s)} v(x)q(x) dx ds \right)^{1/\gamma} du.$$

Let us denote the right-hand side of the above inequality by  $z(t)$ . Then  $y(t) \geq z(t) > 0$  and it is easy to verify that

$$\begin{aligned} 0 &= z'(t) + \left( \frac{v(t)}{r_1(t)} \int_t^{\xi(t)} \frac{1}{r_2(s)v^2(s)} \int_s^{\xi(s)} v(x)q(x) dx ds \right)^{1/\gamma} f^{1/\gamma}(y[\eta(t)]) \\ &\geq z'(t) + \left( \frac{v(t)}{r_1(t)} \int_t^{\xi(t)} \frac{1}{r_2(s)v^2(s)} \int_s^{\xi(s)} v(x)q(x) dx ds \right)^{1/\gamma} f^{1/\gamma}(z[\eta(t)]). \end{aligned}$$

Consequently, Theorem 1 of Philos [20] implies that the corresponding differential equation  $(E_0)$  has also a positive solution  $z(t)$ , which contradicts our assumption. We conclude that also  $\mathcal{N}_0 = \emptyset$  and thus, (E) is oscillatory. The proof is complete.  $\square$

**Corollary 2.8.** *Let (1.5) hold and assume that  $v(t)$  is such positive solution of (V) that (1.3) and (1.4) are satisfied. Let  $f(u) = u^\gamma$ . Suppose that there exists a function  $\xi(t) \in C^1([t_0, \infty))$  such that (2.3) holds. If, moreover,  $(C_1)$  is satisfied and*

$$\liminf_{t \rightarrow \infty} \int_{\eta(t)}^t \left( \frac{v(u)}{r_1(u)} \int_u^{\xi(u)} \frac{1}{r_2(s)v^2(s)} \int_s^{\xi(s)} v(x)q(x) dx ds \right)^{1/\gamma} du > \frac{1}{e} \quad (C_2)$$

holds, then (E) is oscillatory.

*Proof.* Conditions (C<sub>1</sub>) and (C<sub>2</sub>) implies that (E<sub>p</sub>) and (E<sub>0</sub>) are oscillatory. The assertion immediately follows from Theorem 2.7.  $\square$

## 2.1 Example

We support the criteria obtained by the following illustrative example.

**Example 2.9.** We consider the differential equation

$$\left(t^{1/4} (y'(t))^{1/3}\right)'' + \frac{3}{16t^{7/4}} (y'(t))^{1/3} + \frac{a}{t^{25/12}} y^{1/3}(\lambda t) = 0, \quad \lambda \in (0, 1). \quad (E_x)$$

Now, (V) takes the form

$$y''(t) + \frac{3}{16t^2} y(t) = 0$$

with couple of positive solutions  $v_1(t) = t^{1/4}$  and  $v_2(t) = t^{3/4}$ . For our considerations, we take  $v(t) = t^{1/4}$ , since it obeys both conditions (1.3) and (1.4). Some computations shows that (C<sub>1</sub>) reduces to

$$a \sqrt[3]{\frac{16}{5}} \lambda^{5/6} \ln\left(\frac{1}{\lambda}\right) > \frac{1}{e} \quad (2.4)$$

and by Corollary 2.2 this condition guarantees that (E<sub>x</sub>) has property (P). What is more, it is easy to verify that (2.2) holds true, which by Lemma 2.6 ensures, that every nonoscillatory solution of (E<sub>x</sub>) tends to zero as  $t \rightarrow \infty$ . For  $a = \frac{7}{9} \sqrt[3]{\lambda}$  one such solution is  $y(t) = 1/t$ .

On the other hand, we set  $\zeta(t) = \alpha t$ , where  $\alpha = (1 + \sqrt{\lambda})/(2\lambda)$ . Then condition (C<sub>2</sub>) transforms to

$$\frac{18a}{5} (1 - \alpha^{-5/6})(1 - \alpha^{-1/3}) \ln\left(\frac{1}{\lambda \alpha^2}\right) > \frac{1}{e}, \quad (2.5)$$

which by Corollary 2.8 guarantees oscillation of (E<sub>x</sub>).

For clearness for  $\lambda = 0.4$  condition  $a > 0.5847$  guarantees that every nonoscillatory solution of (E<sub>x</sub>) tends to zero as  $t \rightarrow \infty$ , for  $a = \sqrt[3]{1.08}$  one such solution is  $y(t) = 1/t^2$ , while the condition  $a > 16.1197$  guarantees oscillation of (E<sub>x</sub>).

## 3 Noncanonical form

Now, we consider (E<sub>c</sub>) in its noncanonical form, thus the set of all positive solutions of (E<sub>c</sub>) is given by

$$\mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_2 \cup \mathcal{N}_*.$$

To obtain oscillation of (E), we need to empty enymore the extra class  $\mathcal{N}_*$ . Let us denote

$$P(t) = \int_t^\infty \left(\frac{v(s)}{r_1(s)}\right)^{1/\gamma} ds.$$

**Theorem 3.1.** Let (1.5) hold and assume that  $v(t)$  is such positive solution of (V) that (1.3) is satisfied. Suppose that there exists a function  $\xi(t) \in C^1([t_0, \infty))$  such that (2.3) hold. If both the first-order delay differential equations (E<sub>p</sub>) and (E<sub>0</sub>) are oscillatory and

$$\int_{t_1}^\infty \left[ \frac{v(x)}{r_1(x)} \int_{t_1}^x \frac{1}{r_2(u)v^2(u)} \int_{t_1}^u q(s)v(s)f(P(\tau(s))) ds du \right]^{1/\gamma} dx = \infty \quad (3.1)$$

holds, then equation (E) is oscillatory.

*Proof.* To ensure oscillation of (E), assume for the sake of contradiction that  $y(t)$  is a positive solution of (E). Then  $y(t)$  is also solution of (E<sub>c</sub>). Using result of Theorem 2.7, oscillation of (E<sub>p</sub>) and (E<sub>0</sub>) guaranties that classes  $\mathcal{N}_0$  and  $\mathcal{N}_2$  are empty. So assume that  $y(t) \in \mathcal{N}_*$ . Therefore,  $y(t)$  is decreasing and integration from  $t$  to  $\infty$  yields

$$y(t) \geq - \int_t^\infty \left( \frac{v(s)}{r_1(s)} \right)^{1/\gamma} \left( \frac{r_1(s)}{v(s)} (y'(s))^\gamma \right)^{1/\gamma} ds \geq - \left( \frac{r_1(t)}{v(t)} (y'(t))^\gamma \right)^{1/\gamma} P(t).$$

Since  $-\left(\frac{r_1(t)}{v(t)} (y'(t))^\gamma\right)^{1/\gamma}$  is positive and increasing, there exists  $L > 0$  such that

$$-\left(\frac{r_1(t)}{v(t)} (y'(t))^\gamma\right)^{1/\gamma} > L.$$

Consequently

$$y(t) \geq LP(t).$$

Setting the above inequality into (E<sub>c</sub>), we have

$$\left( r_2(t)v^2(t) \left( \frac{r_1(t)}{v(t)} (y'(t))^\gamma \right)' \right)' + q(t)v(t)f(L)f(P(\tau(t))) \leq 0.$$

Integrating from  $t_1$  to  $t$ , we obtain

$$r_2(t)v^2(t) \left( \frac{r_1(t)}{v(t)} (y'(t))^\gamma \right)' + f(L) \int_{t_1}^t q(s)v(s)f(P(\tau(s))) ds \leq 0.$$

Repeating integration from  $t_1$  to  $t$ , we get

$$\frac{r_1(t)}{v(t)} (y'(t))^\gamma + f(L) \int_{t_1}^t \frac{1}{r_2(u)v^2(u)} \int_{t_1}^u q(s)v(s)f(P(\tau(s))) ds du \leq 0$$

or

$$y'(t) + f^{1/\gamma}(L) \left[ \frac{v(t)}{r_1(t)} \int_{t_1}^t \frac{1}{r_2(u)v^2(u)} \int_{t_1}^u q(s)v(s)f(P(\tau(s))) ds du \right]^{1/\gamma} \leq 0.$$

Finally, integrating once more,

$$y(t_1) \geq f^{1/\gamma}(L) \int_{t_1}^t \left[ \frac{v(x)}{r_1(x)} \int_{t_1}^x \frac{1}{r_2(u)v^2(u)} \int_{t_1}^u q(s)v(s)f(P(\tau(s))) ds du \right]^{1/\gamma} dx,$$

which contradicts with our assumption. The proof is complete.  $\square$

## 4 Summary

In this paper, we have extended the technique presented in [6] to cover a more general differential equation (E). Easily verifiable criteria are established to complement other known results for the case  $\gamma = 1$ . We point out that our main theorems do not require any restricted conditions to coefficient  $r_1(t)$  and can ensure oscillation of *all* solutions of (E).

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