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ON A NON-LOCAL BOUNDARY VALUE PROBLEM FOR LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. We establish new efficient conditions for the unique solvability of a non-local boundary value problem for first-order linear functional differential equations. Differential equations with argument deviations are also considered in which case further results are obtained. The results obtained reduce to those well-known for the ordinary differential equations.

1. Introduction

On the interval $[a, b]$, we consider the problem on the existence and uniqueness of a solution to the equation

$$u'(t) = \ell(u)(t) + q(t) \tag{1.1}$$

satisfying the non-local boundary condition

$$h(u) = c, \tag{1.2}$$

where $\ell\colon C([a, b]; \mathbb{R}) \to L([a, b]; \mathbb{R})$ and $h\colon C([a, b]; \mathbb{R}) \to \mathbb{R}$ are linear bounded operators, $q \in L([a, b]; \mathbb{R})$, and $c \in \mathbb{R}$. By a solution to the problem (1.1), (1.2) we understand an absolutely continuous function $u\colon [a, b] \to \mathbb{R}$ satisfying the equation (1.1) almost everywhere on the interval $[a, b]$ and verifying also the boundary condition (1.2).

The question on the solvability of various types of boundary value problems for functional differential equations and their systems is a classical topic in the theory of differential equations (see, e.g., [1, 3–5, 7–9, 11–14] and references therein). Many particular cases of the boundary condition (1.2) are studied in detail (namely, periodic, anti-periodic and multi-point conditions), but only a few efficient conditions is known in the case, where a general non-local boundary condition is considered. In the present paper, new efficient conditions are found sufficient for the unique solvability of the problem (1.1), (1.2). It is clear that the ordinary differential equation

$$u' = p(t)u + q(t), \tag{1.3}$$

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where $p, q \in L([a,b];\mathbb{R})$, is a particular case of the equation (1.1) and that the problem (1.3), (1.2) is uniquely solvable if and only if the condition

$$h\left(e^{\int_a^{\cdot} p(s)\mathrm{d}s}\right) \neq 0 \tag{1.4}$$

is satisfied. Below, we establish new solvability conditions for the problem (1.1), (1.2) in terms of norms of the operators appearing in (1.1) and (1.2) (see Theorems 2.1–2.4). Moreover, we apply these results to the differential equation with an argument deviation

$$u'(t) = p(t)u(\tau(t)) + q(t) \tag{1.5}$$

in which $p, q \in L([a,b];\mathbb{R})$ and $\tau\colon [a,b] \to [a,b]$ is a measurable function (see Theorems 2.5 and 2.6), and we show that the assumptions of the statements obtained reduce to the condition (1.4) in the case, where the equation (1.5) is the ordinary one (see Remark 2.6). All the main results are formulated in Section 2, their proofs are given in Section 3.

The following notation is used throughout the paper:

(1) $\mathbb{R}$ is the set of all real numbers, $\mathbb{R}_+ = [0, +\infty[$.
(2) $C([a,b];\mathbb{R})$ is the Banach space of continuous functions $u\colon [a,b] \to \mathbb{R}$ endowed with the norm $\|u\|_C = \max\{|u(t)| : t \in [a,b]\}$.
(3) $L([a,b];\mathbb{R})$ is the Banach space of Lebesgue integrable functions $p\colon [a,b] \to \mathbb{R}$ endowed with the norm $\|p\|_L = \int_a^b |p(s)|ds$.
(4) P_{ab} is set of linear operators $\ell\colon C([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$ mapping the set $C([a,b];\mathbb{R}_+)$ into the set $L([a,b];\mathbb{R}_+)$.
(5) PF_{ab} is the set of linear functionals $h : C([a,b];\mathbb{R}) \to \mathbb{R}$ mapping the set $C([a,b];\mathbb{R}_+)$ into the set $\mathbb{R}_+$.

2. Main Results

In theorems stated below, we assume that the operator ℓ admits the representation $\ell = \ell_0 - \ell_1$ with $\ell_0, \ell_1 \in P_{ab}$. This is equivalent to the fact that ℓ is not only bounded, but it is strongly bounded (see, e.g., [6, Ch.VII, §1.2]), i.e., that there exists a function $\eta \in L([a,b];\mathbb{R}_+)$ such that the condition

$$|\ell(v)(t)| \leq \eta(t)\|v\|_C \quad \text{for a. e. } t \in [a,b] \text{ and all } v \in C([a,b];\mathbb{R}).$$

is satisfied.

We first consider the case, where the boundary condition (1.2) is understood as a non-local perturbation of a two-point condition of an anti-periodic type. More precisely, we consider the boundary condition

$$u(a) + \lambda u(b) = h_0(u) - h_1(u) + c, \tag{2.1}$$

where $\lambda \geq 0$, $h_0, h_1 \in PF_{ab}$, and $c \in \mathbb{R}$. We should mention that there is no loss of generality in assuming this, because an arbitrary functional h can be represented in the form

$$h(v) \stackrel{\text{def}}{=} v(a) + \lambda v(b) - h_0(v) + h_1(v) \quad \text{for } v \in C([a,b];\mathbb{R}).$$

Note also that we have studied the problem (1.1), (2.1) with $\lambda < 0$ in the paper [10].

Theorem 2.1. *Let $h_0(1) < 1 + \lambda + h_1(1)$ and $\ell = \ell_0 - \ell_1$, where $\ell_0, \ell_1 \in P_{ab}$. Let, moreover,*

$$\lambda\big(\lambda - h_0(1)\big) \le \big(1 + h_1(1)\big)^2 \tag{2.2}$$

and either the conditions

$$\|\ell_0\| < 1 - h_0(1) - \big(\lambda + h_1(1)\big)^2, \tag{2.3}$$

$$\|\ell_1\| < 1 - \lambda - h_1(1) + 2\sqrt{1 - h_0(1) - \|\ell_0\|}\,, \tag{2.4}$$

be satisfied, or the conditions

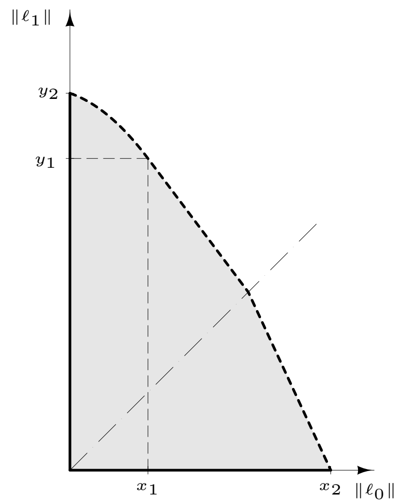
$$\|\ell_0\| \ge 1 - h_0(1) - \big(\lambda + h_1(1)\big)^2, \tag{2.5}$$

$$\|\ell_0\| + \big(\lambda + h_1(1)\big)\|\ell_1\| < 1 + \lambda - h_0(1) + h_1(1), \tag{2.6}$$

$$\big(1 + h_1(1)\big)\|\ell_0\| + \lambda\|\ell_1\| < 1 + \lambda - h_0(1) + h_1(1) \tag{2.7}$$

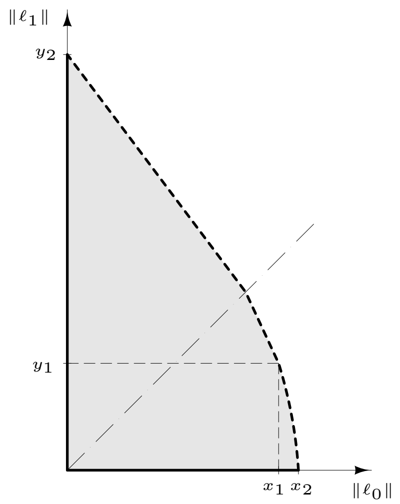
hold. Then the problem (1.1), (2.1) has a unique solution.

Remark 2.1. Geometrical meaning of the assumptions of Theorem 2.1 is illustrated on Fig. 2.1.



$$x_1 = 1 - h_0(1) - (\lambda + h_1(1))^2$$
$$x_2 = \frac{1 + \lambda - h_0(1) + h_1(1)}{1 + \lambda + h_1(1)}$$
$$y_1 = 1 + \lambda + h_1(1)$$
$$y_2 = 1 - \lambda - h_1(1) + 2\sqrt{1 - h_0(1)}$$

Figure 2.1



$$x_1 = 1 + \frac{1 + h_1(1)}{\lambda}$$
$$x_2 = 1 - \frac{1 + h_1(1)}{\lambda} + 2\sqrt{1 - \frac{1}{\lambda}\, h_0(1)}$$
$$y_1 = 1 - \frac{1}{\lambda}\, h_0(1) - \frac{(1 + h_1(1)r}{\lambda^2}$$
$$y_2 = \frac{1 + \lambda - h_0(1) + h_1(1)}{\lambda + h_1(1)}$$

Figure 2.2

Remark 2.2. Let $\ell = \ell_0 - \ell_1$ with $\ell_0, \ell_1 \in P_{ab}$. Define the operator $\varphi\colon C([a,b];\mathbb{R}) \to C([a,b];\mathbb{R})$ by setting

$$\varphi(w)(t) \stackrel{\text{def}}{=} w(a+b-t) \quad \text{for } t \in [a,b],\ w \in C([a,b];\mathbb{R}).$$

For $i = 0, 1$, we put

$$\tilde{\ell}_i(w)(t) \stackrel{\text{def}}{=} \ell_i(\varphi(w))(a+b-t) \quad \text{for a. e. } t \in [a,b] \text{ and all } w \in C([a,b];\mathbb{R})$$

and

$$\tilde{q}(t) \stackrel{\text{def}}{=} -q(a+b-t) \quad \text{for a. e. } t \in [a,b],$$

$$\tilde{h}(w) \stackrel{\text{def}}{=} h(\varphi(w)) \quad \text{for } w \in C([a,b];\mathbb{R}).$$

It is clear that if u is a solution to the problem (1.1), (1.2) then the function $v \stackrel{\text{def}}{=} \varphi(u)$ is a solution to the problem

$$v'(t) = \tilde{\ell}_1(v)(t) - \tilde{\ell}_0(v)(t) + \tilde{q}(t), \qquad \tilde{h}(v) = c, \tag{2.8}$$

and vice versa, if v is a solution to the problem (2.8) then the function $u \stackrel{\text{def}}{=} \varphi(v)$ is a solution to the problem (1.1), (1.2).

Using the transformation described in the previous remark, we can immediately derive from Theorem 2.1 the following statement.

Theorem 2.2. *Let $\lambda > 0$, $h_0(1) < 1+\lambda+h_1(1)$, and $\ell = \ell_0 - \ell_1$, where $\ell_0, \ell_1 \in P_{ab}$. Let, moreover,*

$$1 - h_0(1) \le \big(\lambda + h_1(1)\big)^2 \tag{2.9}$$

and either the conditions

$$\|\ell_1\| < 1 - \frac{1}{\lambda}\, h_0(1) - \frac{\big(1+h_1(1)\big)^2}{\lambda^2}\,, \tag{2.10}$$

$$\|\ell_0\| < 1 - \frac{1}{\lambda}\big(1 + h_1(1)\big) + 2\sqrt{1 - \frac{1}{\lambda}\, h_0(1) - \|\ell_1\|} \tag{2.11}$$

be satisfied, or

$$\|\ell_1\| \ge 1 - \frac{1}{\lambda}\, h_0(1) - \frac{\big(1+h_1(1)\big)^2}{\lambda^2} \tag{2.12}$$

and the conditions (2.6) *and* (2.7) *hold. Then the problem* (1.1), (2.1) *has a unique solution.*

Remark 2.3. Geometrical meaning of the assumptions of Theorem 2.2 is illustrated on Fig. 2.2.

Remark 2.4. It is easy to verify that, for any $\lambda \ge 0$ and $h_0, h_1 \in PF_{ab}$, at least one of the conditions (2.2) and (2.9) is fulfilled and thus Theorems 2.1 and 2.2 cover all cases.

Theorems 2.1 and 2.2 yield

Corollary 2.1. *Let $\lambda > 0$, $h_0(1) < 1+\lambda+h_1(1)$ and $\ell = \ell_0 - \ell_1$, where $\ell_0, \ell_1 \in P_{ab}$. If, moreover, the conditions* (2.2), (2.6), (2.7), *and* (2.9) *are fulfilled, then the problem* (1.1), (2.1) *has a unique solution.*

In the case, where $\lambda = 0$ in (2.1), we consider the problem

$$u'(t) = \ell(u)(t) + q(t), \qquad u(a) = h_0(u) - h_1(u) + c \tag{2.13}$$

and from Theorem 2.1 we get

Corollary 2.2. *Let $h_0(1) < 1 + h_1(1)$ and $\ell = \ell_0 - \ell_1$, where $\ell_0, \ell_1 \in P_{ab}$. Let, moreover, either the conditions*

$$\|\ell_0\| < 1 - h_0(1) - h_1(1)^2, \tag{2.14}$$

$$\|\ell_1\| < 1 - h_1(1) + 2\sqrt{1 - h_0(1) - \|\ell_0\|}\,, \tag{2.15}$$

be satisfied, or the conditions

$$1 - h_0(1) - h_1(1)^2 \le \|\ell_0\| < 1 - \frac{h_0(1)}{1 + h_1(1)}\,, \tag{2.16}$$

$$\|\ell_0\| + h_1(1)\|\ell_1\| < 1 - h_0(1) + h_1(1) \tag{2.17}$$

hold. Then the problem (2.13) *has a unique solution.*

Now we give two statements dealing with the unique solvability of the problem (1.1), (1.2). We assume in Theorems 2.3 and 2.4 that $h = h^+ - h^-$ with $h^+, h^- \in PF_{ab}$. There is no loss of generality in assuming this, because every linear bounded functional $h\colon C([a, b]) \to \mathbb{R}$ can be expressed in such a form.

Theorem 2.3. *Let $h(1) > 0$, $h = h^+ - h^-$ with $h^+, h^- \in PF_{ab}$, and $\ell = \ell_0 - \ell_1$, where $\ell_0, \ell_1 \in P_{ab}$. Let, moreover, the conditions*

$$\|\ell_0\| + h^+(1)\|\ell_1\| < h(1)$$

and

$$h^+(1)\|\ell_0\| + \|\ell_1\| < h(1)$$

be fulfilled. Then the problem (1.1), (1.2) *has a unique solution.*

Theorem 2.4. *Let $h(1) < 0$, $h = h^+ - h^-$ with $h^+, h^- \in PF_{ab}$, and $\ell = \ell_0 - \ell_1$, where $\ell_0, \ell_1 \in P_{ab}$. Let, moreover, the conditions*

$$\|\ell_0\| + h^-(1)\|\ell_1\| < |h(1)|$$

and

$$h^-(1)\|\ell_0\| + \|\ell_1\| < |h(1)|$$

be fulfilled. Then the problem (1.1), (1.2) *has a unique solution.*

Remark 2.5. Geometrical meaning of the assumptions of Theorems 2.3 and 2.4 is illustrated, respectively, on Fig. 2.3 and Fig. 2.4.

It is clear that, from Theorems 2.1–2.4, we can immediately obtain conditions guaranteeing the unique solvability of the problem (1.5), (1.2), whenever we replace the terms $\|\ell_0\|$ and $\|\ell_1\|$ appearing therein, respectively, by the terms $\int_a^b [p(s)]_+ \mathrm{d}s$ and $\int_a^b [p(s)]_- \mathrm{d}s$. In what follows, we establish two theorems, which can be also derived from Theorems 2.3 and 2.4, and which require that the deviation $\tau(t) - t$ is "small" enough. In order to simplify formulation of statements, we put

$$p_0(t) = \sigma(t)[p(t)]_+ \int_t^{\tau(t)} [p(s)]_+ e^{\int_t^{\tau(s)} p(\xi)\mathrm{d}\xi} \mathrm{d}s +$$

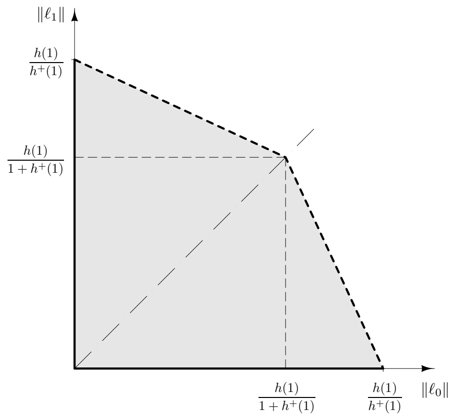


FIGURE 2.3

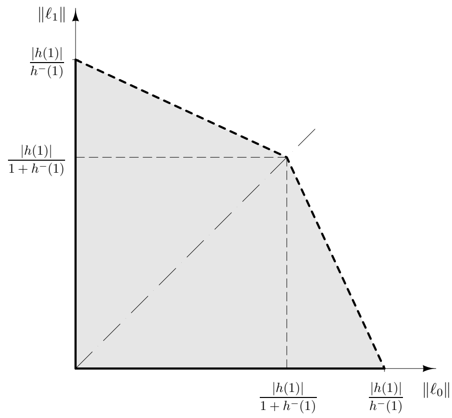


FIGURE 2.4

$$
\begin{aligned}
&+\sigma(t)[p(t)]_-\int_t^{\tau(t)}[p(s)]_-e^{\int_t^{\tau(s)}p(\xi)\mathrm{d}\xi}\mathrm{d}s+\\
&+\big(1-\sigma(t)\big)[p(t)]_+\int_{\tau(t)}^{t}[p(s)]_-e^{\int_t^{\tau(s)}p(\xi)\mathrm{d}\xi}\mathrm{d}s+\\
&+\big(1-\sigma(t)\big)[p(t)]_-\int_{\tau(t)}^{t}[p(s)]_+e^{\int_t^{\tau(s)}p(\xi)\mathrm{d}\xi}\mathrm{d}s \quad \text{for a. e. } t\in[a,b] \qquad (2.18)
\end{aligned}
$$

and

$$
\begin{aligned}
p_1(t)=\sigma(t)[p(t)]_+&\int_t^{\tau(t)}[p(s)]_-e^{\int_t^{\tau(s)}p(\xi)\mathrm{d}\xi}\mathrm{d}s+\\
&+\sigma(t)[p(t)]_-\int_t^{\tau(t)}[p(s)]_+e^{\int_t^{\tau(s)}p(\xi)\mathrm{d}\xi}\mathrm{d}s+\\
&+\big(1-\sigma(t)\big)[p(t)]_+\int_{\tau(t)}^{t}[p(s)]_+e^{\int_t^{\tau(s)}p(\xi)\mathrm{d}\xi}\mathrm{d}s+\\
&+\big(1-\sigma(t)\big)[p(t)]_-\int_{\tau(t)}^{t}[p(s)]_-e^{\int_t^{\tau(s)}p(\xi)\mathrm{d}\xi}\mathrm{d}s \quad \text{for a. e. } t\in[a,b], \qquad (2.19)
\end{aligned}
$$

where

$$\sigma(t)=\frac{1}{2}\big(1+\operatorname{sgn}(\tau(t)-t)\big) \quad \text{for a. e. } t\in[a,b].$$

Moreover, having $h^+, h^- \in PF_{ab}$, we denote

$$\mu_0=h^+\left(e^{\int_a^{\cdot}p(s)\mathrm{d}s}\right) \qquad \text{and} \qquad \mu_1=h^-\left(e^{\int_a^{\cdot}p(s)\mathrm{d}s}\right). \qquad (2.20)$$

Theorem 2.5. *Let $h=h^+-h^-$ with $h^+, h^- \in PF_{ab}$. Let, moreover, $\mu_0>\mu_1$ and the conditions*

$$\int_a^b p_0(s)\mathrm{d}s+\mu_0\int_a^b p_1(s)\mathrm{d}s<\mu_0-\mu_1$$

and

$$\mu_0 \int_a^b p_0(s)\mathrm{d}s + \int_a^b p_1(s)\mathrm{d}s < \mu_0 - \mu_1$$

be fulfilled, where the functions p_0, p_1 and the numbers μ_0, μ_1 are defined, respectively, by the relations (2.18), (2.19) *and* (2.20). *Then the problem* (1.5), (1.2) *has a unique solution.*

Theorem 2.6. *Let $h = h^+ - h^-$ with $h^+, h^- \in PF_{ab}$. Let, moreover, $\mu_0 < \mu_1$ and the conditions*

$$\int_a^b p_0(s)\mathrm{d}s + \mu_1 \int_a^b p_1(s)\mathrm{d}s < \mu_1 - \mu_0$$

and

$$\mu_1 \int_a^b p_0(s)\mathrm{d}s + \int_a^b p_1(s)\mathrm{d}s < \mu_1 - \mu_0$$

be fulfilled, where the functions p_0, p_1 and the numbers μ_0, μ_1 are defined, respectively, by the relations (2.18), (2.19) *and* (2.20). *Then the problem* (1.5), (1.2) *has a unique solution.*

Remark 2.6. Theorems 2.5 and 2.6 yield, in particular, that the problem (1.3), (1.2) is uniquely solvable if $\mu_0 \neq \mu_1$, i. e., if the condition (1.4) holds. However, it is well-known that, in the framework of the ordinary differential equations, the condition (1.4) is not only sufficient, but also necessary for the unique solvability of the problem (1.3), (1.2).

3. Proofs

It is well-known that the linear problem has the Fredholm property, i. e., the following assertion holds (see, e. g., [2,4]; in the case, where the operator ℓ is strongly bounded, see also [1, 14]).

Lemma 3.1. *The problem* (1.1), (1.2) *has a unique solution for an arbitrary $q \in L([a, b]; \mathbb{R})$ and every $c \in \mathbb{R}$ if and only if the corresponding homogeneous problem*

$$u'(t) = \ell(u)(t), \qquad h(u) = 0 \tag{3.1}$$

has only the trivial solution.

Proof of Theorem 2.1. According to Lemma 3.1, to prove the theorem it is sufficient to show that the homogeneous problem

$$u'(t) = \ell_0(u)(t) - \ell_1(u)(t), \tag{3.2}$$

$$u(a) + \lambda u(b) = h_0(u) - h_1(u) \tag{3.3}$$

has only the trivial solution. Assume that, on the contrary, u is a nontrivial solution to the problem (3.2), (3.3).

First suppose that u changes its sign. Put

$$M = \max\{u(t) : t \in [a, b]\}, \qquad m = -\min\{u(t) : t \in [a, b]\}, \tag{3.4}$$

and choose $t_M, t_m \in [a, b]$ such that

$$u(t_M) = M, \qquad u(t_m) = -m. \tag{3.5}$$

It is clear that

$$M > 0, \qquad m > 0. \tag{3.6}$$

We can assume without loss of generality that $t_M < t_m$. The integration of the equality (3.2) from t_M to t_m, from a to t_M, and from t_m to b, in view of (3.4), (3.5), and the assumption $\ell_0, \ell_1 \in P_{ab}$, yields

$$M + m = \int_{t_M}^{t_m} \ell_1(u)(s)\,\mathrm{d}s - \int_{t_M}^{t_m} \ell_0(u)(s)\,\mathrm{d}s \leq MB_1 + mA_1, \tag{3.7}$$

$$\begin{aligned} M - u(a) + u(b) + m = \int_a^{t_M} \ell_0(u)(s)\,\mathrm{d}s - \int_a^{t_M} \ell_1(u)(s)\,\mathrm{d}s + \\ + \int_{t_m}^b \ell_0(u)(s)\,\mathrm{d}s - \int_{t_m}^b \ell_1(u)(s)\,\mathrm{d}s \leq MA_2 + mB_2, \end{aligned} \tag{3.8}$$

where

$$A_1 = \int_{t_M}^{t_m} \ell_0(1)(s)\,\mathrm{d}s, \qquad A_2 = \int_a^{t_M} \ell_0(1)(s)\,\mathrm{d}s + \int_{t_m}^b \ell_0(1)(s)\,\mathrm{d}s,$$

$$B_1 = \int_{t_M}^{t_m} \ell_1(1)(s)\,\mathrm{d}s, \qquad B_2 = \int_a^{t_M} \ell_1(1)(s)\,\mathrm{d}s + \int_{t_m}^b \ell_1(1)(s)\,\mathrm{d}s.$$

On the other hand, from the boundary condition (3.3), in view of the relations (3.5), (3.6) and the assumption $h_0, h_1 \in PF_{ab}$, we get

$$u(a) - u(b) = -(1+\lambda)u(b) + h_0(u) - h_1(u) \leq (1+\lambda)m + Mh_0(1) + mh_1(1)$$

and

$$\begin{aligned} u(a) - u(b) = \Big(1 + \frac{1}{\lambda}\Big)u(a) - \frac{1}{\lambda}h_0(u) + \frac{1}{\lambda}h_1(u) \leq \\ \leq \Big(1 + \frac{1}{\lambda}\Big)M + m\frac{1}{\lambda}\,h_0(1) + M\frac{1}{\lambda}\,h_1(1). \end{aligned}$$

Hence, it follows from the relation (3.8) that

$$M - \lambda m \leq MA_2 + mB_2 + Mh_0(1) + mh_1(1) \tag{3.9}$$

and

$$m - \frac{1}{\lambda}\,M \leq MA_2 + mB_2 + m\frac{1}{\lambda}\,h_0(1) + M\frac{1}{\lambda}\,h_1(1). \tag{3.10}$$

We first assume that $\|\ell_0\| \geq 1$. Then the conditions (2.6) and (2.7) are supposed to be satisfied. It is clear that the inequality (2.7) implies $\lambda > 0$ and $\|\ell_1\| < 1 - \frac{1}{\lambda}h_0(1)$ and thus

$$B_1 < 1, \qquad B_2 < 1 - \frac{1}{\lambda}\,h_0(1).$$

Using these inequalities and the relations (3.6), from (3.7) and (3.10) we obtain

$$0 < M(1 - B_1) \leq m(A_1 - 1),$$

$$0 < m\Big(1 - \frac{1}{\lambda}\,h_0(1) - B_2\Big) \leq M\Big(A_2 + \frac{1}{\lambda}\big(1 + h_1(1)\big)\Big),$$

which yields that

$$(1 - B_1)\Big(1 - \frac{1}{\lambda}\,h_0(1) - B_2\Big) \leq (A_1 - 1)\Big(A_2 + \frac{1}{\lambda}\big(1 + h_1(1)\big)\Big). \tag{3.11}$$

Obviously,

$$(1-B_1)\Big(1-\frac{1}{\lambda}h_0(1)-B_2\Big) \geq 1-\frac{1}{\lambda}h_0(1)-\|\ell_1\|. \tag{3.12}$$

On the other hand, by virtue of (2.2), it follows from the inequality (2.7) that

$$\|\ell_0\| < 1+\frac{\lambda-h_0(1)}{1+h_1(1)} \leq 1+\frac{1}{\lambda}\big(1+h_1(1)\big),$$

and thus we obtain

$$\begin{aligned}(A_1-1)\Big(A_2+\frac{1}{\lambda}\big(1+h_1(1)\big)\Big) &\leq (\|\ell_0\|-1)A_2+(A_1-1)\frac{1}{\lambda}\big(1+h_1(1)\big) \leq \\ &\leq \frac{1}{\lambda}\big(1+h_1(1)\big)(A_1+A_2-1) \leq \frac{1}{\lambda}\big(1+h_1(1)\big)(\|\ell_0\|-1).\end{aligned} \tag{3.13}$$

Now, from (3.11), (3.12), and (3.13) we get

$$1+\lambda-h_0(1)+h_1(1) \leq \big(1+h_1(1)\big)\|\ell_0\|+\lambda\|\ell_1\|.$$

which contradicts the inequality (2.7).

Now assume that $\|\ell_0\|<1$. Then, in view of the relations (3.6), the inequalities (3.7) and (3.9) yield

$$0 < m\big(1-A_1\big) \leq M\big(B_1-1\big),$$
$$M\big(1-h_0(1)-A_2\big) \leq m\big(B_2+\lambda+h_1(1)\big)$$

and thus we get $\|\ell_1\| \geq B_1 > 1$ and

$$\big(1-A_1\big)\big(1-h_0(1)-A_2\big) \leq \big(B_1-1\big)\big(B_2+\lambda+h_1(1)\big). \tag{3.14}$$

Obviously,

$$\big(1-A_1\big)\big(1-h_0(1)-A_2\big) \geq 1-h_0(1)-\|\ell_0\|. \tag{3.15}$$

If $\|\ell_0\| \geq 1-h_0(1)-\big(\lambda+h_1(1)\big)^2$ then the conditions (2.6) and (2.7) are supposed to be satisfied. Therefore, we obtain from the inequality (2.6) that $\|\ell_1\| \leq 1+\lambda+h_1(1)$ and thus it is easy to verify that

$$\begin{aligned}\big(B_1-1\big)\big(B_2+\lambda+h_1(1)\big) &\leq (\|\ell_1\|-1)B_2+(B_1-1)(\lambda+h_1(1)) \leq \\ &\leq \big(\lambda+h_1(1)\big)\big(B_1+B_2-1\big) \leq \big(\lambda+h_1(1)\big)\big(\|\ell_1\|-1\big).\end{aligned} \tag{3.16}$$

Now, it follows from (3.14), (3.15), and(3.16) that

$$1+\lambda-h_0(1)+h_1(1) \leq \|\ell_0\|+\big(\lambda+h_1(1)\big)\|\ell_1\|,$$

which contradicts the inequality (2.6).

If $\|\ell_0\| < 1-h_0(1)-\big(\lambda+h_1(1)\big)^2$ then, taking the above-mentioned condition $\|\ell_1\|>1$ and the obvious inequality

$$\big(B_1-1\big)\big(B_2+\lambda+h_1(1)\big) \leq \frac{1}{4}\big(\|\ell_1\|-1+\lambda+h_1(1)\big)^2$$

into account, from the relations (3.14) and (3.15) we get

$$1-\lambda-h_1(1)+2\sqrt{1-h_0(1)-\|\ell_0\|} \leq \|\ell_1\|,$$

which contradicts the inequality (2.4).

Now suppose that u does not change its sign. Then, without loss of generality, we can assume that

$$u(t) \geq 0 \quad \text{for } t \in [a, b]. \tag{3.17}$$

Put

$$M_0 = \max\{u(t) : t \in [a, b]\}, \qquad m_0 = \min\{u(t) : t \in [a, b]\}, \tag{3.18}$$

and choose $t_{M_0}, t_{m_0} \in [a, b]$ such that

$$u(t_{M_0}) = M_0, \qquad u(t_{m_0}) = m_0. \tag{3.19}$$

It is clear that

$$M_0 > 0, \qquad m_0 \geq 0, \tag{3.20}$$

and either

$$t_{M_0} \geq t_{m_0}, \tag{3.21}$$

or

$$t_{M_0} < t_{m_0}. \tag{3.22}$$

Notice that if the assumptions of the theorem are fulfilled, then both inequalities

$$A + \big(\lambda + h_1(1)\big)B < 1 + \lambda - h_0(1) + h_1(1) \tag{3.23}$$

and

$$\big(1 + h_1(1)\big)A + \lambda B < 1 + \lambda - h_0(1) + h_1(1) \tag{3.24}$$

hold, where $A = \|\ell_0\|$ and $B = \|\ell_1\|$.

The integration of the equality (3.2) from a to t_{M_0} and from t_{M_0} to b, in view of the relations (3.17), (3.18), and (3.19) and the assumption $\ell_0, \ell_1 \in P_{ab}$, yields

$$M_0 - u(a) = \int_a^{t_{M_0}} \ell_0(u)(s)\,\mathrm{d}s - \int_a^{t_{M_0}} \ell_1(u)(s)\,\mathrm{d}s \leq M_0 A$$

and

$$M_0 - u(b) = \int_{t_{M_0}}^b \ell_1(u)(s)\,\mathrm{d}s - \int_{t_{M_0}}^b \ell_0(u)(s)\,\mathrm{d}s \leq M_0 B.$$

The last two inequalities yield

$$M_0(1 + \lambda) - u(a) - \lambda u(b) \leq M_0(A + \lambda B)$$

and thus, using (3.3), (3.18), and the assumption $h_0, h_1 \in PF_{ab}$, we get

$$m_0 h_1(1) \leq M_0\big(A + \lambda B + h_0(1) - 1 - \lambda\big). \tag{3.25}$$

First suppose that (3.21) holds. The integration of the equality (3.2) from t_{m_0} to t_{M_0}, in view of (3.17), (3.18), and (3.19) and the assumption $\ell_0, \ell_1 \in P_{ab}$, results in

$$M_0 - m_0 = \int_{t_{m_0}}^{t_{M_0}} \ell_0(u)(s)\,\mathrm{d}s - \int_{t_{m_0}}^{t_{M_0}} \ell_1(u)(s)\,\mathrm{d}s \leq M_0 A,$$

i. e.,

$$M_0\big(1 - A\big) \leq m_0.$$

From this inequality and (3.25) we obtain

$$\big(1 + h_1(1)\big)A + \lambda B \geq 1 + \lambda - h_0(1) + h_1(1),$$

which contradicts the inequality (3.24).

Now assume that (3.22) holds. The integration of the equality (3.2) from t_{M_0} to t_{m_0}, in view of (3.17), (3.18), and (3.19) and the assumption $\ell_0, \ell_1 \in P_{ab}$, yields

$$M_0 - m_0 = \int_{t_{M_0}}^{t_{m_0}} \ell_1(u)(s)\,\mathrm{d}s - \int_{t_{M_0}}^{t_{m_0}} \ell_0(u)(s)\,\mathrm{d}s \le M_0 B,$$

i. e.,

$$M_0(1-B) \le m_0.$$

The last inequality, together with (3.25), results in

$$A + \big(\lambda + h_1(1)\big)B \ge 1 + \lambda - h_0(1) + h_1(1),$$

which contradicts the inequality (3.23).

The contradictions obtained prove that the homogeneous problem (3.2), (3.3) has only the trivial solution. □

Proof of Theorem 2.2. The assertion of the theorem can be derived from Theorem 2.1 using the transformation described in Remark 2.2. □

Proof of Corollary 2.1. The validity of the corollary follows immediately from Theorems 2.1 and 2.2. □

Proof of Corollary 2.2. It is clear that the assumptions of Theorem 2.1 with $\lambda = 0$ are satisfied. □

Proof of Theorem 2.3. Let the functionals h_0 and h_1 be defined by the formulae

$$h_0(v) \overset{\text{def}}{=} v(a) + h^-(v), \quad h_1(v) = h^+(v) \quad \text{for } v \in C([a,b];\mathbb{R}).$$

By virtue of Corollary 2.2, the problem (1.1), (1.2) is uniquely solvable under the assumptions

$$\|\ell_0\| < 1 - \frac{1 + h^-(1)}{1 + h^+(1)}, \qquad \|\ell_0\| + h^+(1)\|\ell_1\| < h^+(1) - h^-(1).$$

Moreover, using the transformation described in Remark 2.2, it is not difficult to verify that the problem (1.1), (1.2) is uniquely solvable also under the assumptions

$$\|\ell_1\| < 1 - \frac{1 + h^-(1)}{1 + h^+(1)}, \qquad \|\ell_1\| + h^+(1)\|\ell_0\| < h^+(1) - h^-(1).$$

Combining these two cases we obtain the required assertion. □

Proof of Theorem 2.4. The validity of the theorem follows from Theorem 2.3 and fact that the problem

$$u'(t) = \ell(u)(t) + q(t), \qquad h(u) = c$$

has a unique solution for every $q \in L([a,b];\mathbb{R})$ and $c \in \mathbb{R}$ if and only if the problem

$$v'(t) = \ell(v)(t) + q(t), \qquad -h(v) = c$$

has a unique solution for every $q \in L([a,b];\mathbb{R})$ and $c \in \mathbb{R}$. □

Proof of Theorem 2.5. According to Lemma 3.1, to prove the theorem it is sufficient to show that the homogeneous problem

$$u'(t) = p(t)u(\tau(t)), \qquad h(u) = 0 \tag{3.26}$$

has only the trivial solution.

Let u be an arbitrary solution to the problem (3.26). Then it is easy to verify by direct calculation that the function

$$v(t) = u(t)e^{-\int_a^t p(s)ds} \quad \text{for } t \in [a, b]$$

is a solution to the problem

$$v'(t) = \ell(v)(t), \qquad \tilde{h}(v) = 0, \tag{3.27}$$

where the operators ℓ and $\tilde{h}$ are defined by the relations

$$\ell(w)(t) \stackrel{\text{def}}{=} p(t) \int_t^{\tau(t)} p(s)e^{\int_t^{\tau(s)} p(\xi)\mathrm{d}\xi} w(\tau(s))\mathrm{d}s$$
$$\text{for a. e. } t \in [a, b] \text{ and all } w \in C([a, b]; \mathbb{R})$$

and

$$\tilde{h}(w) \stackrel{\text{def}}{=} h\left(w(\cdot)e^{\int_a^{\cdot} p(s)\mathrm{d}s}\right) \quad \text{for } w \in C([a, b]; \mathbb{R}).$$

The operator ℓ can be expressed in the form $\ell = \ell_0 - \ell_1$, where $\ell_0, \ell_1 \in P_{ab}$ are such that $\ell_0(1) \equiv p_0$ and $\ell_1(1) \equiv p_1$ and, moreover, the functional $\tilde{h}$ admits the representation $\tilde{h} = \tilde{h}^+ - \tilde{h}^-$ in which $\tilde{h}^+, \tilde{h}^- \in PF_{ab}$ are such that $\tilde{h}^+(1) = \mu_0$ and $\tilde{h}^-(1) = \mu_1$.

Consequently, by virtue of Theorem 2.3, the problem (3.27) has only the trivial solution and thus $u \equiv 0$. This means that the problem (3.26) has only the trivial solution. □

Proof of Theorem 2.6. The proof is analogous to those of Theorem 2.5, only Theorem 2.4 must be used instead of Theorem 2.3. □

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