Electronic Journal of Qualitative Theory of Differential Equations
2015, No. **46**, 1–24; doi: 10.14232/ejqtde.2015.1.46 http://www.math.u-szeged.hu/ejqtde/

Nonlinear nonhomogeneous Neumann eigenvalue problems

Pasquale Candito[1], **Roberto Livrea**[✉1] and **Nikolaos S. Papageorgiou**[2]

[1]Department DICEAM, University of Reggio Calabria, 89100 - Reggio Calabria, Italy
[2]National Technical University, Department of Mathematics, Zografou Campus, Athens 15780, Greece

Received 8 April 2015, appeared 28 July 2015
Communicated by Gabriele Bonanno

Abstract. We consider a nonlinear parametric Neumann problem driven by a nonhomogeneous differential operator with a reaction which is $(p-1)$-superlinear near $\pm\infty$ and exhibits concave terms near zero. We show that for all small values of the parameter, the problem has at least five solutions, four of constant sign and the fifth nodal. We also show the existence of extremal constant sign solutions.

Keywords: superlinear reaction, concave terms, maximum principle, extremal constant sign solutions, nodal solution, critical groups.

2010 Mathematics Subject Classification: 35J20, 35J60, 35J92, 58E05.

1 Introduction

Let $\Omega \in \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. The aim of this work is to study the existence and multiplicity of solutions with a precise sign information, for the following nonlinear nonhomogeneous parametric (eigenvalue) Neumann problem:

$$-\operatorname{div} a(Du(z)) = f(z,u(z),\lambda) \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega\,. \tag{P_λ}$$

Here $n(\cdot)$ stands for the outward unit normal on $\partial\Omega$. Also, $a\colon \mathbb{R}^N \to \mathbb{R}^N$ is a continuous and strictly monotone map which satisfies certain other regularity conditions listed in hypotheses $H(a)$ below. These hypotheses are general enough to incorporate as a special case several differential operators of interest, such as the p-Laplacian ($1 < p < \infty$), the (p,q)-Laplacian (that is, the sum of a p-Laplacian and a q-Laplacian with $1 < q < p < \infty$) and the generalized p-mean curvature differential operator. The variable $\lambda > 0$ is a parameter (eigenvalue) which in general enters in the equation in a nonlinear fashion. The nonlinearity of the right-hand side (the reaction of the problem) $f(z,x,\lambda)$ is a Carathéodory function in $(z,x) \in \Omega \times \mathbb{R}$ (that is for all $x \in \mathbb{R}$, $\lambda > 0$, $z \mapsto f(z,x,\lambda)$ is measurable and for a.a. $z \in \Omega$, all $\lambda > 0$, $x \mapsto f(z,x,\lambda)$ is continuous). We assume that $x \mapsto f(z,x,\lambda)$ exhibits $(p-1)$-superlinear growth near $\pm\infty$, while near zero we assume the presence of a concave

✉Corresponding author. Email: roberto.livrea@unirc.it

term (that is, of a $(p-1)$-sublinear term). So, in the reaction $f(z,x,\lambda)$, we can have the competing effects of two different kinds of nonlinearities ("concave–convex" nonlinearities). Such problems were first investigated by Ambrosetti–Brezis–Cerami [2] who deal with semilinear (that is, $p=2$) equations. Their work was extended to equations driven by the Dirichlet p-Laplacian, by García Azorero–Manfredi–Peral Alonso [9] and by Guo–Zhang [13]. In all three works the reaction has the special form

$$f(z,x,\lambda)=\lambda|x|^{q-2}x+|x|^{p-2}x \quad \text{with } 1<q<p<p^*=\begin{cases}\frac{Np}{N-p} & \text{if } p<N,\\ +\infty & \text{if } p\geq N.\end{cases}$$

More general reactions were considered by Hu–Papageorgiou [14] and by Marano–Papageorgiou [18]. Both papers deal with Dirichlet problems driven by the p-Laplacian. For the Neumann problem, we mention the work of Papageorgiou–Smyrlis [24], where the differential operator is

$$u\mapsto -\Delta_p u+\beta(z)u \quad \text{for all } u\in W^{1,p}(\Omega)\ (1<p<\infty),$$

with Δ_p being the p-Laplace differential operator defined by

$$\Delta_p u=\operatorname{div}(|\nabla u|^{p-2}\nabla u) \quad \text{for all } u\in W^{1,p}(\Omega)$$

and $\beta\in L^\infty(\Omega)$, $\beta\geq 0$, $\beta\neq 0$. So, in this case the differential operator is coercive. This is not the case in problem (P_λ). Moreover, the reaction in [24] has the form

$$f(z,x,\lambda)=\lambda|x|^{q-2}x+g(z,x)$$

with $1<q<p$ and $g(z,x)$ is a Carathéodory function which is $(p-1)$-superlinear in $x\in\mathbb{R}$. Papageorgiou–Smyrlis [24] look for positive solutions and they prove a bifurcation-type theorem describing the set of positive solutions as the parameter $\lambda>0$ varies.

Our approach is variational based on the critical point theory. We also use suitable truncation and perturbation techniques and Morse theory (critical groups).

2 Mathematical background – hypotheses

In this section, we present the main mathematical tools which we will use in the sequel and state the hypotheses on the data of problem (P_λ). We also present some straightforward but useful consequences of the hypotheses.

Let X be a Banach space and X^* its topological dual. By $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair (X^*,X). Let $\varphi\in C^1(X)$. We say that φ satisfies the Cerami condition (the C-condition for short), if the following is true:

Every sequence $\{x_n\}_{n\geq 1}\subseteq X$ *s.t.* $\{\varphi(x_n)\}_{n\geq 1}\subseteq\mathbb{R}$ *is bounded and*

$$(1+\|x_n\|)\varphi'(x_n)\to 0 \quad \text{in } X^* \text{ as } n\to\infty,$$

admits a strongly convergent subsequence.

This is a compactness type condition on the functional φ, more general than the Palais–Smale condition. It compensates for the fact that the ambient space X need not be locally compact (being in general infinite dimensional). The C-condition suffices to prove a deformation theorem and then from it derive the minimax theory for the critical values of φ. Prominent in that theory is the so-called "mountain pass theorem" (see [3]).

Theorem 2.1. *If $\varphi \in C^1(X)$ satisfies the C-condition, $x_0, x_1 \in X$ with $\|x_1 - x_0\| > \rho > 0$*

$$\max\{\varphi(x_0), \varphi(x_1)\} < \inf\{\varphi(x) : \|x - x_0\| = \rho\} = m_\rho$$

and $c = \inf_{\gamma\in\Gamma} \max_{0\le t\le 1} \varphi(\gamma(t))$, where $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = x_0,\ \gamma(1) = x_1\}$, then $c \ge m_\rho$ and c is a critical value of φ.

The analysis of problem (P_λ), in addition to the Sobolev space $W^{1,p}(\Omega)$, will also involve the Banach space $C^1(\bar{\Omega})$. This is an ordered Banach space with positive cone

$$C_+ = \{u \in C^1(\bar{\Omega}) : u(z) \ge 0 \text{ for all } z \in \bar{\Omega}\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \{u \in C^1(\bar{\Omega}) : u(z) > 0 \text{ for all } z \in \bar{\Omega}\}.$$

Now, let us introduce the hypotheses on the map $a(\cdot)$.

Let $\xi \in C^1(0,\infty)$ with $\xi(t) > 0$ for all $t > 0$ and assume that

$$0 < \hat{c} \le \frac{t\xi'(t)}{\xi(t)} \le c_0 \quad \text{and} \quad c_1 t^{p-1} \le \xi(t) \le c_2(1 + t^{p-1}), \tag{2.1}$$

for all $t > 0$, with $c_1 > 0$.

The hypotheses on the map $a(\cdot)$ are the following:

$H(a)$: $a(y) = a_0(|y|)y$ *for all* $y \in \mathbb{R}^N$ *with* $a_0(t) > 0$ *for all* $t > 0$ *and*

(i) $a_0 \in C^1(0,\infty)$, $t \mapsto ta_0(t)$ *is strictly increasing on* $(0,\infty)$, $ta_0(t) \to 0^+$ *as* $t \to 0^+$ *and*

$$\lim_{t\to 0^+} \frac{ta_0'(t)}{a_0(t)} = c > -1;$$

(ii) $\|\nabla a(y)\| \le c_3 \frac{\xi(|y|)}{|y|}$ *for all* $y \in \mathbb{R}^N \setminus \{0\}$ *and some* $c_3 > 0$;

(iii) $(\nabla a(y)h, h)_{\mathbb{R}^N} \ge \frac{\xi(|y|)}{|y|}|h|^2$ *for all* $y \in \mathbb{R}^N \setminus \{0\}$, *all* $h \in \mathbb{R}^N$;

(iv) *if* $G_0(t) = \int_0^t sa_0(s)ds$ *for all* $t \ge 0$, *then* $pG_0(t) - t^2a_0(t) \ge -\gamma$ *for all* $t \ge 0$ *and some* $\gamma > 0$;

(v) *there exists* $\tau \in (1,p)$ *such that* $t \mapsto G_0(t^{1/\tau})$ *is convex,* $\lim_{t\to 0^+} \frac{G_0(t)}{t^\tau} = 0$ *and*

$$t^2a_0(t) - \tau G_0(t) \ge \tilde{c}t^p$$

for all $t > 0$ *an some* $\tilde{c} > 0$.

Remark 2.2. Evidently G_0 is strictly convex and strictly increasing. We set $G(y) = G_0(|y|)$ for all $y \in \mathbb{R}^N$. Then G is convex and it is differentiable at every $y \in \mathbb{R}^N \setminus \{0\}$. Also

$$\nabla G(y) = G_0'(|y|)\frac{y}{|y|} = a_0(|y|)y = a(y) \ \text{ for all } y \in \mathbb{R}^N \setminus \{0\}, \ \ 0 \in \partial G(0),$$

implies that G is the primitive of the map a.

The convexity of G and the fact that $G(0) = 0$, imply

$$G(y) \le (a(y), y)_{\mathbb{R}^N} = a_0(|y|)|y|^2 \tag{2.2}$$

for all $y \in \mathbb{R}^N$.

The next lemma is a straightforward consequence of the above hypotheses and summarizes the main properties of the map a, which we will use in the sequel.

Lemma 2.3. *If hypotheses $H(a)$ hold, then*

(a) $y \mapsto a(y)$ is continuous and strictly monotone, hence maximal monotone too;

(b) $|a(y)| \leq c_4(1+|y|^{p-1})$ for all $y \in \mathbb{R}^N$ and some $c_4 > 0$;

(c) $(a(y),y)_{\mathbb{R}^N} \geq \frac{c_1}{p-1}|y|^p$ for all $y \in \mathbb{R}^N$.

Lemma 2.3 and (2.1), (2.2), lead to the following growth estimates for the primitive G.

Corollary 2.4. *If hypotheses $H(a)$ hold, then*

$$\frac{c_1}{p(p-1)}|y|^p \leq G(y) \leq c_5(1+|y|^p)$$

for all $y \in \mathbb{R}^N$ with $c_5 > 0$.

Example 2.5. The following maps satisfy hypotheses $H(a)$:

(a) $a(y) = |y|^{p-2}y$ with $1 < p < \infty$. This map corresponds to the p-Laplacian

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u) \quad \text{for all } u \in W^{1,p}(\Omega).$$

(b) $a(y) = |y|^{p-2}y + \mu|y|^{q-2}y$ with $1 < q < p$ and $\mu > 0$. This map corresponds to a sum of a p-Laplacian and a q-Laplacian, that is:

$$\Delta_p u + \mu\Delta_q u \quad \text{for all } u \in W^{1,p}(\Omega).$$

Such differential operators arise in many physical applications (see [23] and the references therein).

(c) $a(y) = (1+|y|^2)^{\frac{p-2}{2}}y$ with $1 < p < \infty$. This map corresponds to the generalized p-mean curvature differential operator

$$\operatorname{div}\left[(1+|\nabla u|^2)^{\frac{p-2}{2}}\nabla u\right] \quad \text{for all } u \in W^{1,p}(\Omega).$$

(d) $a(y) = |y|^{p-2}y + \frac{|y|^{p-2}y}{1+|y|^p}$ with $1 < p < \infty$.

We introduce the following nonlinear map $A\colon W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ defined by

$$\langle A(u), y\rangle = \int_\Omega (a(\nabla u), \nabla y)_{\mathbb{R}^N}\, dz \tag{2.3}$$

for all $u, y \in W^{1,p}(\Omega)$.

The next result is a particular case of a more general result proved by Gasinski–Papageorgiou [11].

Proposition 2.6. *If hypotheses $H(a)$ hold, then the map $A\colon W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ defined by (2.3) is bounded (that is, it maps bounded sets to bounded sets), demicontinuous, strictly monotone (hence maximal monotone too) and of type $(S)_+$, that is*

$$u_n \rightharpoonup u \quad \text{in } W^{1,p}(\Omega) \quad \text{and} \quad \limsup_{n\to\infty}\langle A(u_n), u_n - u\rangle \leq 0 \Rightarrow u_n \to u \quad \text{in } W^{1,p}(\Omega).$$

Consider a Carathéodory function $f_0\colon \Omega \times \mathbb{R} \to \mathbb{R}$ satisfying

$$|f_0(z,x)| \leq a(z)(1+|x|^{r-1}) \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R},$$

with $a \in L^\infty(\Omega)_+$, $1 < r < p^*$. We set $F_0(z,x) = \int_0^x f_0(z,s)\,ds$ and consider the C^1-functional $\varphi_0\colon W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_0(u) = \int_\Omega G(\nabla u(z))\,dz - \int_\Omega F_0(z,u(z))\,dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

In the sequel by $\|\cdot\|_{1,p}$ we denote the norm of the Sobolev space $W^{1,p}(\Omega)$, that is

$$\|u\|_{1,p} = \left[\|u\|_p^p + \|\nabla u\|_p^p\right]^{1/p}.$$

The following result is due to Motreanu–Papageorgiou [21].

Proposition 2.7. *If $u_0 \in W^{1,p}(\Omega)$ is a local $C^1(\bar{\Omega})$-minimizer of φ_0, that is there exists $\rho_0 > 0$ such that*

$$\varphi_0(u_0) \leq \varphi_0(u_0+h) \quad \text{for all } h \in C^1(\bar{\Omega}) \text{ with } \|h\|_{C^1(\bar{\Omega})} \leq \rho_0,$$

then $u_0 \in C^{1,\beta}(\bar{\Omega})$ for some $\beta \in (0,1)$ and it is also a local $W^{1,p}(\Omega)$-minimizer of φ_0, that is there exists $\rho_1 > 0$ such that

$$\varphi_0(u_0) \leq \varphi_0(u_0+h) \quad \text{for all } h \in W^{1,p}(\bar{\Omega}) \text{ with } \|h\|_{1,p} \leq \rho_1.$$

Remark 2.8. The first such result relating local minimizers, was proved by Brezis–Nirenberg [4], for the spaces $C_0^1(\bar{\Omega}) = \{u \in C^1(\bar{\Omega}) : u_{|_{\partial\Omega}} = 0\}$ and $H_0^1(\Omega)$ and with $G(y) = \frac{1}{2}|y|^2$ for all $y \in \mathbb{R}^N$ (it corresponds to the Dirichlet Laplacian).

Now let X be a Banach space and $Y_2 \subseteq Y_1 \subseteq X$. For every integer $k \geq 0$ by $H_k(Y_1,Y_2)$ we denote the kth-singular homology group with integer coefficients for the pair (Y_1,Y_2). Recall that $H_k(Y_1,Y_2) = 0$ for all $k < 0$.

Given $\varphi \in C^1(X)$ and $c \in \mathbb{R}$, we introduce the following sets:

$$\varphi^c = \{x \in X : \varphi(x) \leq c\}, \qquad K_\varphi = \{x \in X : \varphi'(x) = 0\}, \qquad K_\varphi^c = \{x \in K_\varphi : \varphi(x) = c\}.$$

The critical groups of φ at an isolated critical point $x_0 \in X$ with $\varphi(x_0) = c$ (that is, $x_0 \in K_\varphi^c$) are defined by

$$C_k(\varphi,x_0) = H_k(\varphi^c \cap \mathcal{U}, \varphi^c \cap \mathcal{U} \setminus \{x_0\}) \quad \text{for all } k \geq 0,$$

where $\mathcal{U}$ is a neighborhood of $x_0 \in X$ such that $K_\varphi \cap \varphi^c \cap \mathcal{U} = \{x_0\}$. The excision property of singular homology implies that the above definition of critical groups is independent of the choice of the neighborhood $\mathcal{U}$.

Next we introduce the hypotheses on the reaction $f(z,x,\lambda)$.

$H(f)$: $f\colon \Omega \times \mathbb{R} \times (0,+\infty) \to \mathbb{R}$ *is a function such that for every $\lambda > 0$, $(z,x) \mapsto f(z,x,\lambda)$ is Carathéodory, $f(z,0,\lambda) = 0$ for a.a. $z \in \Omega$ and*

(i) *for every $\rho > 0$ and $\lambda > 0$, there exists $a_\rho(\lambda) \in L^\infty(\Omega)_+$ such that $\lambda \mapsto \|a_\rho(\lambda)\|_\infty$ is bounded on bounded sets and*

$$|f(z,x,\lambda)| \leq a_\rho(\lambda)(z) \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \leq \rho;$$

(ii) *if* $F(z,x,\lambda)=\int_0^x f(z,s,\lambda)\,ds$, *then for all* $\lambda>0$

$$\lim_{x\to\pm\infty}\frac{F(z,x,\lambda)}{|x|^p}=+\infty \quad \text{uniformly for a.a. } z\in\Omega$$

and there exist $r(\lambda)\in(p,p^*)$ *with* $r(\lambda)\to r_0\in(p,p^*)$ *as* $\lambda\to 0^+$ *and* $\hat\eta_\infty(\lambda),\eta_\infty(\lambda)\in L^\infty(\Omega)$ *such that*

$$\hat\eta_\infty(\lambda)(z)\le\liminf_{x\to\pm\infty}\frac{f(z,x,\lambda)}{|x|^{r(\lambda)-2}x}\le\limsup_{x\to\pm\infty}\frac{f(z,x,\lambda)}{|x|^{r(\lambda)-2}x}\le\eta_\infty(\lambda)(z)$$

uniformly for a.a. $z\in\Omega$ *and* $\lambda\mapsto\|\hat\eta_\infty(\lambda)\|_\infty,\|\eta_\infty(\lambda)\|_\infty$ *are bounded on bounded sets in* $(0,+\infty)$;

(iii) *for every* $\lambda>0$, *there exists* $\theta(\lambda)\in\big(\max\{(r(\lambda)-p)\frac{N}{p},1\},p^*\big)$, *and* $\beta_0(\lambda)$ *such that*

$$0<\beta_0(\lambda)\le\liminf_{x\to\pm\infty}\frac{f(z,x,\lambda)x-pF(z,x,\lambda)}{|x|^{\theta(\lambda)}} \quad \text{uniformly for a.a. } z\in\Omega;$$

(iv) *for every* $\lambda>0$, *there exist* $q(\lambda),\mu(\lambda)\in(1,\tau)$ *(see hypotheses* $H(a)$ *(v)) with* $q(\lambda)\le\mu(\lambda)$ *and* $\delta_0(\lambda)\in(0,1)$, $\hat c_0(\lambda)>0$ *such that* $q(\lambda)\to q_0\in(1,p)$ *as* $\lambda\to 0^+$ *and*

$$\hat c_0(\lambda)|x|^{\mu(\lambda)}\le f(z,x,\lambda)x\le q(\lambda)F(z,x,\lambda) \quad \text{for a.a. } z\in\Omega,\ \text{all } |x|\le\delta_0(\lambda),$$

there exist $\beta(\lambda)$, $\beta_1,\beta_2>0$, *with* $\beta(\lambda)\to 0^+$ *as* $\lambda\to 0^+$ *such that*

$$f(z,x,\lambda)x\le\beta(\lambda)|x|^{q(\lambda)}+\beta_1|x|^{r^*}-\beta_2|x|^p \quad \text{for a.a. } z\in\Omega,\ \text{all } x\in\mathbb{R},$$

with $r(\lambda)\le r^*<p^*$ *(see (ii)) and there exits a function* $\eta_0(\cdot,\lambda)\in L^\infty(\Omega)_+$ *such that*

$$\|\eta_0(\cdot,\lambda)\|_\infty\to 0 \quad \text{as } \lambda\to 0^+,$$

$$\limsup_{x\to 0}\frac{F(z,x,\lambda)}{|x|^{q(\lambda)}}\le\eta_0(z,\lambda) \quad \text{uniformly for a.a. } z\in\Omega.$$

Remark 2.9. Hypotheses $H(f)$ (ii), (iii) imply that for a.a. $z\in\Omega$ and all $\lambda>0$, the reaction $f(z,\cdot,\lambda)$ is $(p-1)$-superlinear near $\pm\infty$. Usually such problems are studied using the Ambrosetti–Rabinowitz condition (see [3]). Our hypothesis here is more general and incorporates in our framework superlinear functions with "slower" growth near $\pm\infty$, which fail to satisfy the Ambrosetti–Rabinowitz condition (see the examples below). On this issue, see also [16], [19] and the references therein.

Example 2.10. The following functions satisfy hypotheses $H(f)$. For the sake of simplicity we drop the z-dependence:

$$f_1(x,\lambda)=\lambda|x|^{q-2}x+|x|^{r-2}x-|x|^{p-2}x \quad \text{with } 1<q<\tau<p<r<p^*,$$

$$f_2(x,\lambda)=\begin{cases}|x|^{r-2}x-|x|^{p-2}x-\sigma(\lambda) & \text{if } x<-\rho(\lambda)\\ \lambda|x|^{q-2}x-|x|^{p-2}x & \text{if } -\rho(\lambda)\le x\le\rho(\lambda)\\ |x|^{r-2}x-|x|^{p-2}x+\sigma(\lambda) & \text{if } \rho(\lambda)<x\end{cases}$$

with $1<q<\tau<p<r<p^*$, $\rho(\lambda)\in(0,1]$, $\rho(\lambda)\to 0^+$ as $\lambda\to 0^+$, $\sigma(\lambda)=(\lambda-\rho(\lambda)^{r-q})\rho(\lambda)^{q-1}$;

$$f_3(x,\lambda)=\lambda|x|^{q-2}x+|x|^{p-2}x\left(\ln|x|-\frac{p-1}{p}\right)$$

with $1<q<\tau<p$.

Note that $f_4(\cdot,\lambda)$ does not satisfy the Ambrosetti–Rabinowitz condition.

We introduce the following truncations-perturbations of the reaction $f(z,x,\lambda)$:

$$\hat{f}_+(z,x,\lambda) = \begin{cases} 0 & \text{if } x \leq 0 \\ f(z,x,\lambda) + \beta_2 x^{p-1} & \text{if } 0 < x \end{cases} \tag{2.4}$$

and

$$\hat{f}_-(z,x,\lambda) = \begin{cases} f(z,x,\lambda) + \beta_2 |x|^{p-2}x & \text{if } x < 0 \\ 0 & \text{if } 0 \leq x. \end{cases} \tag{2.5}$$

Both are Carathéodory functions. We set

$$\hat{F}_\pm(z,x,\lambda) = \int_0^x \hat{f}_\pm(z,s,\lambda)\,ds$$

and introduce the C^1-functionals $\hat{\varphi}^\lambda_\pm \colon W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\varphi}^\lambda_\pm(u) = \int_\Omega G(\nabla u(z))\,dz + \frac{\beta_2}{p}\|u\|_p^p - \int_\Omega \hat{F}_\pm(z,u(z),\lambda)\,dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

Also, by $\varphi_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ we denote the energy functional for problem (P_λ) defined by

$$\varphi_\lambda(u) = \int_\Omega G(\nabla u(z))\,dz - \int_\Omega F(z,u(z),\lambda)\,dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

Clearly $\varphi_\lambda \in C^1(W^{1,p}(\Omega))$.

We conclude this section by fixing our notation. For $x \in \mathbb{R}$ let $x^\pm = \max\{\pm x, 0\}$. Then, given $u \in W^{1,p}(\Omega)$, we set $u^\pm(\cdot) = u(\cdot)^\pm$. We have

$$u = u^+ - u^-, \qquad |u| = u^+ + u^- \quad \text{and} \quad u^+, u^-, |u| \in W^{1,p}(\Omega).$$

Give $h(z,x)$ a jointly measurable function (for example, a Carathéodory function), we define

$$N_h(u)(\cdot) = h(\cdot, u(\cdot)) \quad \text{for all } u \in W^{1,p}(\Omega).$$

Finally by $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$.

3 Solutions of constant sign

In this section we show that for $\lambda > 0$ small, problem (P_λ) has at least four nontrivial solutions of constat sign (two positive and two negative).

First we establish the compactness properties of the functionals $\hat{\varphi}^\lambda_\pm$ and φ_λ.

Proposition 3.1. *If hypotheses $H(a)$ and $H(f)$ hold and $\lambda > 0$, then the functionals $\hat{\varphi}^\lambda_\pm$ satisfy the C-condition.*

Proof. We do the proof for the functional $\hat{\varphi}^\lambda_+$, the proof for $\hat{\varphi}^\lambda_-$ being similar. So, we consider a sequence $\{u_n\}$ in $W^{1,p}(\Omega)$ such that

$$|\hat{\varphi}^\lambda_+(u_n)| \leq M_1 \tag{3.1}$$

for some $M_1 > 0$, all $n \geq 1$,

$$(1 + \|u_n\|_{1,p})(\hat{\varphi}^\lambda_+)'(u_n) \to 0 \quad \text{in } W^{1,p}(\Omega)^* \quad \text{as } n \to \infty. \tag{3.2}$$

From (3.2) we have

$$|\langle(\hat{\varphi}_+^\lambda)'(u_n),h\rangle| \leq \frac{\varepsilon_n\|h\|_{1,p}}{1+\|u_n\|_{1,p}}$$

for all $h \in W^{1,p}(\Omega)$ with $\varepsilon_n \to 0^+$. Hence

$$\left|\langle A(u_n),h\rangle + \beta_2\int_\Omega |u_n|^{p-2}u_n h\,dz - \int_\Omega \hat{f}_+(z,u_n,\lambda)h\,dz\right| \leq \frac{\varepsilon_n\|h\|_{1,p}}{1+\|u_n\|_{1,p}} \tag{3.3}$$

for all $n \geq 1$. In (3.3) we choose $h = -u_n^- \in W^{1,p}(\Omega)$. Then, in view of (2.4),

$$\int_\Omega \left(a(-\nabla u_n^-),-\nabla u_n^-)\right)_{\mathbb{R}^N} dz + \beta_2\|u_n^-\|_p^p \leq \varepsilon_n \quad \text{for all } n \geq 1,$$

so that, because of Lemma 2.3,

$$\frac{c_1}{p-1}\|\nabla u_n^-\|_p^p + \beta_2\|u_n^-\|_p^p \leq \varepsilon_n,$$

that is

$$u_n^- \to 0 \quad \text{in } W^{1,p}(\Omega). \tag{3.4}$$

Next, in (3.3) we choose $h = u_n^+ \in W^{1,p}(\Omega)$. Then

$$-\int_\Omega \left(a(\nabla u_n^+),\nabla u_n^+)\right)_{\mathbb{R}^N} dz + \int_\Omega f(z,u_n^+,\lambda)u_n^+\,dz \leq \varepsilon_n \quad \text{for all } n \geq 1. \tag{3.5}$$

Also, from (3.1), (3.4) and Corollary 2.4, we have

$$\int_\Omega pG(\nabla u_n^+)\,dz - \int_\Omega pF(z,u_n^+,\lambda)\,dz \leq M_2 \quad \text{for some } M_2 > 0, \text{ all } n \geq 1. \tag{3.6}$$

We add (3.5) and (3.6) and obtain

$$\int_\Omega \left[pG(\nabla u_n^+) - \left(a(\nabla u_n^+),\nabla u_n^+)\right)_{\mathbb{R}^N}\right] dz + \int_\Omega \left[f(z,u_n^+,\lambda) - pF(z,u_n,\lambda)\right] dz \leq M_3, \tag{3.7}$$

for some $M_3 > 0$, all $n \geq 1$. Hence, $H(a)$ (iv) assures that, for all $n \geq 1$

$$\int_\Omega \left[f(z,u_n^+,\lambda) - pF(z,u_n^+,\lambda)\right] dz \leq \hat{M}_3. \tag{3.8}$$

Hypotheses $H(f)$ (i), (iii) imply that we can find $b_1(\lambda) \in (0,\beta_0(\lambda))$ and $c_6(\lambda) > 0$ such that

$$b_1(\lambda)|x|^{\theta(\lambda)} - c_6(\lambda) \leq f(z,x,\lambda)x - pF(z,x,\lambda) \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}. \tag{3.9}$$

Using (3.9) in (3.8), we obtain that

$$\{u_n^+\} \subseteq L^{\theta(\lambda)}(\Omega) \text{ is bounded.} \tag{3.10}$$

Note that in hypothesis $H(f)$ (iii) without any loss of generality, we may assume that $1 \leq \theta(\lambda) < r(\lambda)$. First suppose that $N \neq p$ and let $t \in (0,1)$ be such that

$$\frac{1}{r(\lambda)} = \frac{1-t}{\theta(\lambda)} + \frac{t}{p^*}. \tag{3.11}$$

The interpolation inequality (see, for example, Gasinski–Papageorgiou [10, p. 905]) implies

$$\|u_n^+\|_{r(\lambda)} \leq \|u_n^+\|_{\theta(\lambda)}^{1-t}\|u_n^+\|_{p^*}^t.$$

Thus, from (3.10) and the Sobolev embedding theorem

$$\|u_n^+\|_{r(\lambda)}^{r(\lambda)} \leq M_4 \|u_n^+\|_{1,p}^{tr(\lambda)} \quad \text{for some } M_4 > 0, \text{ all } n \geq 1. \tag{3.12}$$

Hypotheses $H(f)$ (i), (ii) imply that we can find $c_7(\lambda) > 0$ such that

$$f(z,x,\lambda) \leq c_7(\lambda)(1 + |x|^{r(\lambda)-1}) \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}. \tag{3.13}$$

In (3.3) we choose $h = u_n^+ \in W^{1,p}(\Omega)$. Then

$$\int_\Omega \left(a(\nabla u_n^+), \nabla u_n^+\right)_{\mathbb{R}^N} dz - \int_\Omega f(z, u_n^+, \lambda) u_n^+ \, dz \leq \varepsilon_n \quad \text{for all } n \geq 1.$$

Hence, from Lemma 2.3, (3.13) and (3.12), there exist $c_8(\lambda), c_9(\lambda) > 0$ such that for all $n \geq 1$

$$\begin{aligned} \frac{c_1}{p-1}\|\nabla u_n^+\|_p^p &\leq c_8(\lambda)(1 + \|u_n^+\|_{r(\lambda)}^{r(\lambda)}) \\ &\leq c_9(\lambda)(1 + \|u_n^+\|_{1,p}^{tr(\lambda)}). \end{aligned} \tag{3.14}$$

Recall that $u \mapsto \|u\|_{\theta(\lambda)} + \|\nabla u\|_p$ is an equivalent norm on the space $W^{1,p}(\Omega)$ (see, for example, [10, p. 227]). Then, (3.10) and (3.14) imply

$$\|u_n^+\|_{1,p}^p \leq c_{10}(\lambda)\left(1 + \|u_n^+\|_{1,p}^{tr(\lambda)}\right) \quad \text{for some } c_{10}(\lambda) > 0, \text{ all } n \geq 1. \tag{3.15}$$

From hypothesis $H(f)$ (iii) and after a simple calculation involving (3.11), we show that $\mathrm{tr}(\lambda) < p$. So, from (3.15) we infer that

$$\{u_n^+\} \subseteq W^{1,p}(\Omega) \text{ is bounded.} \tag{3.16}$$

If $N = p$, then by the Sobolev embedding theorem we know that $W^{1,p}(\Omega) \hookrightarrow L^s(\Omega)$ continuously (in fact compactly) for all $s \in [1, \infty)$. Then, for $s > r(\lambda) \geq \theta(\lambda) \geq 1$ sufficiently large, reasoning as in (3.11) and recalling hypothesis $H(f)$ (iii), one has that

$$tr(\lambda) = \frac{(r(\lambda) - \theta(\lambda))s}{s - \theta(\lambda)} < p.$$

Therefore, the previous argument remains valid and so we reach again (3.16).

From (3.4) and (3.16) it follows that

$$\{u_n\} \subseteq W^{1,p}(\Omega) \text{ is bounded.}$$

At this point, we may assume that there exists $u \in W^{1,p}(\Omega)$ such that

$$u_n \rightharpoonup u \quad \text{in } W^{1,p}(\Omega) \quad \text{and} \quad u_n \to u \quad \text{in } L^{r(\lambda)}(\Omega). \tag{3.17}$$

We return to (3.3), choose $h = u_n - u$ and pass to the limit as $n \to \infty$ and use (3.17). Then

$$\lim_{n\to\infty} \langle A(u_n), u_n - u \rangle = 0,$$

and Proposition 2.6 implies that $u_n \to u$ in $W^{1,p}(\Omega)$. This proves that the functional $\hat{\varphi}_+^\lambda$ satisfies the C-condition. □

With minor changes in the proof, we can also have the following result.

Proposition 3.2. *If hypotheses $H(a)$ and $H(f)$ hold and $\lambda > 0$, then the functional φ_λ satisfies the C-condition.*

Next we show that for all small values of the parameter $\lambda > 0$, the functionals $\hat{\varphi}^\lambda_\pm$ satisfy the mountain pass geometry (see Theorem 2.1).

Proposition 3.3. *If hypotheses $H(a)$ and $H(f)$ hold, then there exist $\lambda^*_\pm > 0$ such that for every $\lambda \in (0, \lambda^*_\pm)$, we can find $\rho^\lambda_\pm > 0$ for which we have*

$$\inf\left[\hat{\varphi}^\lambda_\pm(u) : \|u\|_{1,p} = \rho^\lambda_\pm\right] = \hat{m}^\lambda_\pm > 0.$$

Proof. By virtue of hypothesis $H(f)$ (iv) we see that given any $\lambda > 0$

$$F(z,x,\lambda) \leq \frac{\beta(\lambda)}{q(\lambda)}|x|^{q(\lambda)} + \frac{\beta_1}{r^*}|x|^{r^*} - \frac{\beta_2}{p}|x|^p \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}. \tag{3.18}$$

For all $u \in W^{1,p}(\Omega)$, because of Corollary 2.4 and (2.4) we have

$$\begin{aligned}\hat{\varphi}^\lambda_+(u) &= \int_\Omega G(\nabla u)\,dz + \frac{\beta_2}{p}\|u\|^p_p - \int_\Omega \hat{F}_+(z,u,\lambda)\,dz \\ &\geq \frac{c_1}{p(p-1)}\|\nabla u\|^p_p + \frac{\beta_2}{p}\|u\|^p_p - \frac{\beta_2}{p}\|u^+\|^p_p - \int_\Omega F(z,u^+,\lambda)\,dz. \end{aligned} \tag{3.19}$$

If in (3.19) we use (3.18), we obtain

$$\begin{aligned}\hat{\varphi}^\lambda_+(u) &\geq \frac{c_1}{p(p-1)}\|\nabla u\|^p_p + \frac{\beta_2}{p}\|u\|^p_p - \frac{\beta(\lambda)}{q(\lambda)}\|u^+\|^{q(\lambda)}_{q(\lambda)} - \frac{\beta_1}{r^*}\|u^+\|^{r^*}_{r^*} \\ &\geq \left[c_{11} - (c_{12}(\lambda)\|u\|^{q(\lambda)-p}_{1,p} + c_{13}\|u\|^{r^*-p}_{1,p})\right]\|u\|^p_{1,p}, \end{aligned} \tag{3.20}$$

with $c_{11},\ c_{13} > 0$ independent of λ and $c_{12}(\lambda) \to 0$ as $\lambda \to 0^+$. We introduce the function $\gamma_\lambda\colon (0,\infty) \to (0,\infty)$ defined by

$$\gamma_\lambda(t) = c_{12}(\lambda)t^{q(\lambda)-p} + c_{13}t^{r^*-p} \quad \text{for all } t > 0.$$

Recall that $1 < q(\lambda) < p < r(\lambda) \leq r^* < p^*$. Hence

$$\gamma_\lambda(t) \to +\infty \quad \text{as } t \to 0^+ \text{ and as } t \to +\infty.$$

Therefore we can find $t_0 = t_0(\lambda) \in (0,\infty)$ such that

$$\gamma_\lambda(t_0) = \min\left[\gamma_\lambda(t) : t > 0\right].$$

In particular,

$$\gamma'_\lambda(t_0) = (q(\lambda)-p)c_{12}(\lambda)t_0^{q(\lambda)-p-1} + (r^*-p)c_{13}t_0^{r^*-p-1} = 0,$$

hence

$$t_0 = t_0(\lambda) = \left[\frac{(p-q(\lambda))c_{12}(\lambda)}{(r^*-p)c_{13}}\right]^{\frac{1}{r^*-q(\lambda)}}$$

and a simple calculation leads to

$$\gamma_\lambda(t_0) = [c_{12}(\lambda)]^{\frac{q(\lambda)-p}{r^*-q(\lambda)}}c_{14}(\lambda),$$

with $\lambda \mapsto c_{14}(\lambda)$ bounded on bounded intervals. Note that using the hypotheses on $q(\cdot)$ and r^*, we have

$$\gamma_\lambda(t_0) \to 0^+ \quad \text{as } \lambda \to 0^+$$

So, choosing $\lambda_+^* > 0$ small, we have

$$\gamma_\lambda(t_0) < c_{11} \quad \text{for all } \lambda \in (0, \lambda_+^*).$$

Then, from (3.20) it follows that for all $\lambda \in (0, \lambda_+^*)$ we have

$$\hat{\varphi}_+^\lambda(u) \geq \hat{m}_+^\lambda > 0 \quad \text{for all } u \in W^{1,p}(\Omega) \quad \text{with } \|u\|_{1,p} = t_0(\lambda) = \rho_+^\lambda.$$

In a similar fashion we show the existence of $\lambda_-^* > 0$ such that

$$\hat{\varphi}_-^\lambda(u) \geq \hat{m}_-^\lambda > 0 \quad \text{for all } u \in W^{1,p}(\Omega) \quad \text{with } \|u\|_{1,p} = t_0(\lambda) = \rho_-^\lambda,$$

and the proof is complete. □

The next proposition completes the mountain pass geometry for the functionals $\hat{\varphi}_\pm^\lambda$. It is an immediate consequence of the p-superlinear hypothesis $H(f)$ (ii).

Proposition 3.4. *If hypotheses $H(a)$ and $H(f)$ hold, $\lambda > 0$ and $u \in \operatorname{int} C_+$, then $\hat{\varphi}_\pm^\lambda(tu) \to -\infty$ as $t \to \pm\infty$.*

Now we can use variational methods to produce constant sign solutions for problem (P_λ) when $\lambda > 0$ is small.

Proposition 3.5. *If hypotheses $H(a)$ and $H(f)$ hold, then*

(a) for every $\lambda \in (0, \lambda_+^*)$ problem (P_λ) has at least two positive solutions

$$u_0, \hat{u} \in \operatorname{int} C_+$$

with $\hat{u}$ being a local minimizer of φ_λ and $\varphi_\lambda(\hat{u}) < 0 < \varphi_\lambda(u_0)$;

(b) for every $\lambda \in (0, \lambda_-^*)$ problem (P_λ) has at least two negative solutions

$$v_0, \hat{v} \in -\operatorname{int} C_+$$

with $\hat{v}$ being a local minimizer of φ_λ and $\varphi_\lambda(\hat{v}) < 0 < \varphi_\lambda(v_0)$;

(c) if $\lambda^* = \min\{\lambda_+^*, \lambda_-^*\}$ and $\lambda \in (0, \lambda^*)$, then problem ($P_\lambda$) has at least four nontrivial solutions of constant sign

$$u_0, \hat{u} \in \operatorname{int} C_+, \quad v_0, \hat{v} \in -\operatorname{int} C_+$$

with $\hat{u}, \hat{v}$ local minimizers of φ_λ and $\varphi_\lambda(\hat{v}), \varphi_\lambda(\hat{u}) < 0 < \varphi_\lambda(u_0), \varphi_\lambda(v_0)$.

Proof. (a) For $\lambda \in (0, \lambda_+^*)$, let ρ_+^λ be as postulated by Proposition 3.3 and consider $\bar{B}_{\rho_+^\lambda} = \{u \in W^{1,p}(\Omega) : \|u\|_{1,p} \leq \rho_+^\lambda\}$, which clearly is weakly compact in $W^{1,p}(\Omega)$. Moreover, since $\hat{\varphi}_+^\lambda$ is sequentially weakly lower semicontinuous in $W^{1,p}(\Omega)$, one has that there exists $\hat{u} \in \bar{B}_{\rho_+^\lambda}$ such that

$$\hat{\varphi}_+^\lambda(\hat{u}) = \inf\left[\hat{\varphi}_+^\lambda(u) : \|u\|_{1,p} \leq \rho_+^\lambda\right] \leq \hat{m}_+^\lambda.$$

On the other hand, for $\delta_0 \in (0,1)$ as in hypothesis $H(f)$ (iv) and $\xi \in (0, \delta_0(\lambda))$ small (take $|\xi| < \rho_+^\lambda / |\Omega|_N^{1/p}$), we obtain

$$\hat{\varphi}_+^\lambda(\xi) = -\int_\Omega F(z,\xi,\lambda)\,dz < 0.$$

Therefore, because of Proposition 3.3, we can deduce that

$$\hat{u} \in B_{\rho_+^\lambda} = \{u \in W^{1,p}(\Omega) : \|u\|_{1,p} < \rho_+^\lambda\},$$

and

$$(\hat{\varphi}_+^\lambda)'(\hat{u}) = 0.$$

So, it follows that

$$A(\hat{u}) + \beta_2 |\hat{u}|^{p-2}\hat{u} = N_{\hat{f}_+^\lambda}(\hat{u}), \tag{3.21}$$

where $\hat{f}_+^\lambda(z,x) = \hat{f}_+(z,x,\lambda)$. On (3.21) we act with $-\hat{u}^- \in W^{1,p}(\Omega)$ and using (2.4) and Corollary 2.4, we obtain $\hat{u} \geq 0$, $\hat{u} \neq 0$. Then, again because of (2.4), (3.21) we have

$$-\operatorname{div} a(\nabla \hat{u}(z)) = f(z,\hat{u}(z),\lambda) \quad \text{a.e. in } \Omega, \qquad \frac{\partial \hat{u}}{\partial n} = 0 \quad \text{on } \partial\Omega,$$

(see [11]). From [26], we know that $\hat{u} \in L^\infty(\Omega)$. So, we can apply the regularity result of Lieberman [17] and infer that $\hat{u} \in C_+ \setminus \{0\}$. From hypotheses $H(f)$ (i), (iv), we see that for every $\lambda > 0$ and $\rho > 0$, we can find $\xi_\rho(\lambda) > 0$ such that

$$f(z,x,\lambda)x + \xi_\rho(\lambda)|x|^p \geq 0 \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \leq \rho.$$

Let $\rho = \|\hat{u}\|_\infty$ and let $\xi_\rho(\lambda) > 0$ as above. Then

$$-\operatorname{div} a(\nabla \hat{u}(z)) + \xi_\rho(\lambda)\hat{u}(z)^p = f(z,\hat{u}(z),\lambda) + \xi_\rho(\lambda)\hat{u}(z)^p \geq 0 \quad \text{a.e. in } \Omega,$$

that is

$$\operatorname{div} a(\nabla \hat{u}(z)) \leq \xi_\rho(\lambda)\hat{u}(z)^p \quad \text{a.e. in } \Omega. \tag{3.22}$$

Let $\chi(t) = t a_0(t)$ for all $t > 0$. Then, from $H(a)$ (iii)

$$t\chi'(t) = t^2 a_0'(t) + t a_0(t) \geq c_1 t^{p-1},$$

hence, by integration one has

$$\int_0^t s\chi'(s)\,ds \geq \tilde{c}t^p \quad \text{for all } t > 0. \tag{3.23}$$

From (3.22), (3.23) and the strong maximum principle of Pucci–Serrin [25, p. 111] we have

$$\hat{u}(z) > 0 \text{ for all } z \in \Omega.$$

So, we can apply the boundary point theorem of Pucci–Serrin [25, p. 120] and have

$$\hat{u} \in \operatorname{int} C_+. \tag{3.24}$$

From (2.4) it is clear that

$$\varphi_{\lambda_{|C_+}} = \hat{\varphi}^\lambda_{+_{|C_+}}.$$

From this equality and (3.24) it follows that $\hat{u}$ is a local $C^1(\bar{\Omega})$-minimizer of φ_λ. Invoking Proposition 2.7, we have that $\hat{u}$ is a local $W^{1,p}(\Omega)$-minimizer of φ_λ.

Now we look for the second positive solution. Propositions 3.1, 3.3 and 3.4 permit the use of Theorem 2.1 on the functional $\hat{\varphi}^\lambda_+$. So, we can find $u_0 \in W^{1,p}(\Omega)$ such that

$$\hat{\varphi}^\lambda_+(\hat{u}) < 0 = \hat{\varphi}^\lambda_+(0) < \hat{m}^\lambda_+ \leq \hat{\varphi}^\lambda_+(u_0) \quad \text{and} \quad (\hat{\varphi}^\lambda_+)'(u_0) = 0. \tag{3.25}$$

From (3.25) it follows that $u_0 \notin \{0, \hat{u}\}$, it solves problem (P_λ) and by the nonlinear regularity theory we have $u_0 \in C_+ \setminus \{0\}$ (see [17, 26]). In fact, as above, using the results of Pucci–Serrin [25, pp. 111, 120], we conclude that $u_0 \in \operatorname{int} C_+$.

(b) Working in a similar fashion, this time with the function $\hat{\varphi}^\lambda_-$, for $\lambda \in (0, \lambda^*_-)$ we produce two negative solutions for problem (P_λ)

$$v_0,\ \hat{v} \in -\operatorname{int} C_+.$$

Moreover, $\hat{v}$ is a local minimizer of φ_λ and $\varphi_\lambda(\hat{v}) < 0 < \varphi_\lambda(v_0)$.

(c) Follows from parts (a) and (b). □

4 Nodal solutions

In this section, we produce a fifth nontrivial solution of (P_λ), with $\lambda \in (0, \lambda^*)$, which is nodal (sign changing). The idea is first to generate the extremal nontrivial constant sign solutions, that is the smallest nontrivial positive solution u^*_λ and the biggest nontrivial negative solution v^*_λ of (P_λ). Then look for a nontrivial solution in the order interval $[v^*_\lambda, u^*_\lambda] = \{u \in W^{1,p}(\Omega) : v^*_\lambda(z) \leq u(z) \leq u^*_\lambda(z) \text{ a.e. in } \Omega\}$ distinct from v^*_λ and u^*_λ. Necessarily, this solution will be nodal.

Hypotheses $H(f)$ (i), (ii), (iv) imply that we can find $c_{15} > 0$ such that

$$f(z,x,\lambda)x \geq \hat{c}_0(\lambda)|x|^{\mu(\lambda)} - c_{15}|x|^{r(\lambda)} \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}, \text{ all } \lambda \in (0, \lambda^*). \tag{4.1}$$

This unilateral growth estimate on the reaction $f(z,\cdot,\lambda)$ leads to the following parametric auxiliary Neumann problem

$$\begin{cases} -\operatorname{div} a(\nabla u(z)) = \hat{c}_0(\lambda)|u(z)|^{\mu(\lambda)-2}u(z) - c_{15}|u(z)|^{r(\lambda)-2}u(z) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \\ 1 < \mu(\lambda) < p < r(\lambda) < p^*. \end{cases} \tag{S_λ}$$

For this auxiliary problem, we have the following existence and uniqueness result for nontrivial solutions of constant sign.

Proposition 4.1. *If hypotheses $H(a)$ hold and $\lambda > 0$, then problem (S_λ) has a unique positive solution $\bar{u}_\lambda \in \operatorname{int} C_+$ and since problem (S_λ) is odd $\bar{v}_\lambda = -\bar{u}_\lambda \in -\operatorname{int} C_+$ is the unique negative solution of (S_λ).*

Proof. First we establish the existence of a positive solution. To this end, let $\psi^+_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ be the C^1-functional defined by

$$\psi^+_\lambda(u) = \int_\Omega G(\nabla u(z))\,dz + \frac{1}{p}\|u\|^p - \frac{\hat{c}_0(\lambda)}{\mu(\lambda)}\|u^+\|^{\mu(\lambda)}_{\mu(\lambda)} + \frac{c_{15}}{r(\lambda)}\|u^+\|^{r(\lambda)}_{r(\lambda)} - \frac{1}{p}\|u^+\|^p_p.$$

Recall that $1 < \mu(\lambda) < p < r(\lambda)$ (see (S_λ)). So, using Corollary 2.4, we see that ψ_λ^+ is coercive. Also, using the Sobolev embedding theorem, we can check that ψ_λ^+ is sequentially weakly lower semicontinuous. Hence, by the Weierstrass theorem, we can find $\bar{u}_\lambda \in W^{1,p}(\Omega)$ such that

$$\psi_\lambda^+(\bar{u}_\lambda) = \inf\{\psi_\lambda^+(u) : u \in W^{1,p}(\Omega)\}. \tag{4.2}$$

Since $\mu(\lambda) < r(\lambda)$, for $\xi \in (0,1)$ small we have $\psi_\lambda^+(\xi) < 0$ and so, because of (4.2),

$$\psi_\lambda^+(\bar{u}_\lambda) < 0 = \psi_\lambda^+(0),$$

hence $\bar{u}_\lambda \neq 0$. From (4.2), we have that $\bar{u}_\lambda$ is a critical point of ψ_λ^+, namely

$$A(\bar{u}_\lambda) + |\bar{u}_\lambda|^{p-2}\bar{u}_\lambda = \hat{c}_0(\lambda)(\bar{u}_\lambda^+)^{\mu(\lambda)-1} - c_{15}(\bar{u}_\lambda^+)^{r(\lambda)-1} + (\bar{u}_\lambda^+)^{p-1}. \tag{4.3}$$

On (4.3) we act with $-\bar{u}_\lambda^- \in W^{1,p}(\Omega)$ and obtain $\bar{u}_\lambda \geq 0$, $\bar{u}_\lambda \neq 0$. Hence $\bar{u}_\lambda$ is a positive solution of (S_λ). Nonlinear regularity theory implies $\bar{u}_\lambda \in C_+ \setminus \{0\}$. We have

$$-\operatorname{div} a(\nabla \bar{u}_\lambda(z)) = \hat{c}_0(\lambda)\bar{u}_\lambda(z)^{\mu(\lambda)-1} - c_{15}\bar{u}_\lambda(z)^{r(\lambda)-1} \quad \text{a.e. in } \Omega,$$

thus

$$\operatorname{div} a(\nabla \bar{u}_\lambda(z)) \leq c_{15}\|\bar{u}_\lambda\|_\infty^{r(\lambda)-p}\bar{u}_\lambda(z)^{p-1} \quad \text{a.e. in } \Omega,$$

and from [25, p. 111, 120] we conclude that

$$\bar{u}_\lambda \in \operatorname{int} C_+.$$

So, we have established the existence of a positive solution $\bar{u}_\lambda \in \operatorname{int} C_+$ for problem (S_λ).

Next we show the uniqueness of this positive solution. To this end, consider the integral functional $\sigma_\lambda^+ : L^1(\Omega) \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\sigma_\lambda^+(u) = \begin{cases} \int_\Omega G(\nabla u^{1/\tau})\,dz & \text{if } u \geq 0,\ u^{1/\tau} \in W^{1,p}(\Omega) \\ +\infty & \text{otherwise.} \end{cases}$$

Let $u_1, u_2 \in \operatorname{dom}\sigma_\lambda^+ = \{u \in L^1(\Omega) : \sigma_\lambda^+(u) < +\infty\}$ (the effective domain of σ_λ^+) and let $t \in [0,1]$. We set

$$y = ((1-t)u_1 + tu_2)^{1/\tau}, \qquad v_1 = u_1^{1/\tau}, \qquad v_2 = u_2^{1/\tau}.$$

From [5, Lemma 1], we have

$$\|\nabla y(z)\| \leq [(1-t)\|\nabla v_1(z)\|^\tau + t\|\nabla v_2(z)\|^\tau]^{1/\tau} \quad \text{a.e. in } \Omega,$$

and exploiting the monotonicity of G_0 and hypothesis $H(a)$ (v)

$$\begin{aligned} G(\nabla y(z)) = G_0(\|\nabla y(z)\|) &\leq G_0\left(((1-t)\|\nabla v_1(z)\|^\tau + t\|\nabla v_2(z)\|^\tau)^{1/\tau}\right) \\ &\leq (1-t)G_0(\|\nabla v_1(z)\|) + tG_0(\|\nabla v_2(z)\|) \end{aligned}$$

for a.a. $z \in \Omega$, that is σ_λ^+ is convex.

Also, by Fatou's lemma σ_λ^+ is lower semicontinuous. Now, let $u \in W^{1,p}(\Omega)$ be a positive solution of problem (S_λ). From the first part of the proof, we have $u \in \operatorname{int} C_+$. So, if $h \in C^1(\bar{\Omega})$ and $t \in (-1,1)$ with $|t|$ small, we have

$$u^\tau + th \in \operatorname{int} C_+ \cap \operatorname{dom}\sigma_\lambda^+.$$

Therefore, the Gâteaux derivative of σ_λ^+ at u^τ in the direction h and can be computed using the chain rule

$$(\sigma_\lambda^+)'(u^\tau)(h) = \frac{1}{\tau}\int_\Omega \frac{-\operatorname{div} a(\nabla u)}{u^{\tau-1}} h\,dz$$

for all $h \in W^{1,p}(\Omega)$ (recall that $C^1(\overline{\Omega})$ is dense in $W^{1,p}(\Omega)$). Similarly, if $v \in W^{1,p}(\Omega)$ is another positive solution of (S_λ), then $v \in \operatorname{int} C_+$ and as above

$$(\sigma_\lambda^+)'(v^\tau)(h) = \frac{1}{\tau}\int_\Omega \frac{-\operatorname{div} a(\nabla v)}{v^{\tau-1}} h\,dz$$

for all $h \in W^{1,p}(\Omega)$. Since σ_λ^+ is convex, $(\sigma_\lambda^+)'(\cdot)$ is monotone. Therefore

$$\begin{aligned} 0 &\leq \int_\Omega \left[\frac{-\operatorname{div} a(\nabla u)}{u^{\tau-1}} + \frac{\operatorname{div} a(\nabla v)}{v^{\tau-1}}\right](u^\tau - v^\tau)\,dz \\ &= \int_\Omega \left[\hat{c}_0(\lambda)\left(\frac{1}{u^{\tau-\mu(\lambda)}} - \frac{1}{v^{\tau-\mu(\lambda)}}\right) + c_{15}\left(v^{r(\lambda)-\tau} - u^{r(\lambda)-\tau}\right)\right](u^\tau - v^\tau)\,dz. \end{aligned} \tag{4.4}$$

Since, $\mu(\lambda) < \tau < r(\lambda)$, from (4.4) it follows that

$$u = v,$$

hence $\hat{u}_\lambda \in \operatorname{int} C_+$ is the unique positive solution of problem (S_λ).

Equation (S_λ) is odd. Therefore $\hat{v}_\lambda = -\hat{u}_\lambda \in -\operatorname{int} C_+$ is the unique negative solution of (S_λ). □

For every $\lambda > 0$, let

$$\begin{aligned} S_+(\lambda) &= \{u : u \text{ is a positive solution of } (P_\lambda)\}, \\ S_-(\lambda) &= \{u : u \text{ is a negative solution of } (P_\lambda)\}. \end{aligned}$$

From Proposition 3.5, we know that

$$\begin{aligned} &\lambda \in (0, \lambda_+^*) \Rightarrow S_+(\lambda) \neq \emptyset,\ S_+(\lambda) \subseteq \operatorname{int} C_+, \\ &\lambda \in (0, \lambda_-^*) \Rightarrow S_-(\lambda) \neq \emptyset,\ S_-(\lambda) \subseteq -\operatorname{int} C_+. \end{aligned}$$

Moreover, as in [8] we have that

- $S_+(\lambda)$ is downward directed (that is, if $u_1, u_2 \in S_+(\lambda)$, then there exists $u \in S_+(\lambda)$ such that $u \leq u_1$, $u \leq u_2$).
- $S_-(\lambda)$ is upward directed (that is, if $v_1, v_2 \in S_-(\lambda)$, then there exists $v \in S_-(\lambda)$ such that $v_1 \leq v$, $v_2 \leq v$).

Proposition 4.2. *If hypotheses $H(a)$ and $H(f)$ hold, then*

(a) for all $\lambda \in (0, \lambda_+^*)$ and all $u \in S_+(\lambda)$, we have $\hat{u}_\lambda \leq u$;

(b) for all $\lambda \in (0, \lambda_-^*)$ and all $v \in S_-(\lambda)$, we have $v \leq \hat{v}_\lambda$.

Proof. (a) Let $u \in S_+(\lambda)$ ($\lambda \in (0, \lambda^*_+)$) and define

$$k^+_\lambda(z,x) = \begin{cases} 0 & \text{if } x < 0, \\ \hat{c}_0(\lambda)x^{\mu(\lambda)-1} - c_{15}x^{r(\lambda)-1} + x^{p-1} & \text{if } 0 \le x \le u(z), \\ \hat{c}_0(\lambda)u(z)^{\mu(\lambda)-1} - c_{15}u(z)^{r(\lambda)-1} + u(z)^{p-1} & \text{if } u(z) \le x. \end{cases} \tag{4.5}$$

This is a Carathéodory function. We set $K^+_\lambda(z,x) = \int_0^x k^+_\lambda(z,s)\,ds$ and consider the C^1-functional $\gamma^+_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\gamma^+_\lambda(u) = \int_\Omega G(\nabla(u(z)))\,dz + \frac{1}{p}\|u\|^p_p - \int_\Omega K^+_\lambda(z,u(z))\,dz$$

for all $u \in W^{1,p}(\Omega)$. From (4.5) it is clear that γ^+_λ coercive. Also, using the Sobolev embedding theorem, we see that γ^+_λ is sequentially weakly lower semicontinuous. So, by the Weierstrass theorem, we can find $\tilde{u}_\lambda \in W^{1,p}(\Omega)$ such that

$$\gamma^+_\lambda(\tilde{u}_\lambda) = \inf\{\gamma^+_\lambda(u) : u \in W^{1,p}(\Omega)\}. \tag{4.6}$$

Since $\mu(\lambda) < p < r(\lambda)$, for $\xi \in (0,1)$ small (namely $0 < \xi \le \min_{\bar{\Omega}} u$, recall $u \in \operatorname{int} C_+$) we have

$$\gamma^+_\lambda(\xi) < 0,$$

thus, from (4.6)

$$\gamma^+_\lambda(\tilde{u}_\lambda) < 0 = \gamma^+_\lambda(0)$$

and $\tilde{u}_\lambda \neq 0$. Again from (4.6) we have that $\tilde{u}_\lambda$ is a critical point of γ^+_λ, namely

$$A(\tilde{u}_\lambda) + |\tilde{u}_\lambda|^{p-2}\tilde{u}_\lambda = N_{k^+_\lambda}(\tilde{u}_\lambda.) \tag{4.7}$$

On (4.7) we act with $-\tilde{u}^-_\lambda \in W^{1,p}(\Omega)$. Using (4.5), we obtain that $\tilde{u}_\lambda \ge 0$, $\tilde{u}_\lambda \neq 0$. Also on (4.7) we act with $(\tilde{u}_\lambda - u)^+ \in W^{1,p}(\Omega)$. Then, making use of (4.5), (4.1) and recalling that $u \in S_+(\lambda)$

$$\begin{aligned} &\langle A(\tilde{u}_\lambda), (\tilde{u}_\lambda - u)^+\rangle + \int_\Omega \tilde{u}^{p-1}_\lambda(\tilde{u}_\lambda - u)^+\,dz \\ &\quad = \int_\Omega k^+_\lambda(z,\tilde{u}_\lambda)(\tilde{u}_\lambda - u)^+\,dz \\ &\quad = \int_\Omega \left[\hat{c}_0(\lambda)u^{\mu(\lambda)-1} - c_{15}u^{r(\lambda)-1} + u^{p-1}\right](\tilde{u}_\lambda - u)^+\,dz \\ &\quad \le \int_\Omega f(z,u,\lambda)(\tilde{u}_\lambda - u)^+\,dz + \int_\Omega u^{p-1}(\tilde{u}_\lambda - u)^+\,dz, \\ &\quad = \langle A(u), (\tilde{u}_\lambda - u)^+\rangle + \int_\Omega u^{p-1}(\tilde{u}_\lambda - u)^+\,dz, \end{aligned}$$

that implies

$$\langle A(\tilde{u}_\lambda) - A(u), (\tilde{u}_\lambda - u)^+\rangle + \int_\Omega (\tilde{u}^{p-1}_\lambda - u^{p-1})(\tilde{u}_\lambda - u)^+\,dz \le 0,$$

that is

$$|\{\tilde{u}_\lambda > u\}|_N = 0,$$

hence

$$\tilde{u}_\lambda \le u.$$

So, we have proved that

$$\tilde{u}_\lambda \in [0,u] = \{y \in W^{1,p}(\Omega) : 0 \leq y(z) \leq u(z) \text{ a.e. in } \Omega\}, \quad \tilde{u}_\lambda \neq 0.$$

Because of (4.5) and (4.7) one has that

$$A(\tilde{u}_\lambda) = c_0(\lambda)\tilde{u}_\lambda^{\mu(\lambda)-1} - c_{15}\tilde{u}_\lambda^{r(\lambda)-1},$$

and Proposition 4.1 assures that

$$\tilde{u}_\lambda = \hat{u}_\lambda \in \operatorname{int} C_+,$$

hence

$$\hat{u}_\lambda \leq u \quad \text{for all } u \in S_+(\lambda).$$

(b) In a similar fashion, we show that $v \leq \hat{v}_\lambda$ for all $v \in S_-(\lambda)$ $(\lambda \in (0, \lambda_-^*))$. □

Now we can generate the extremal nontrivial constant sign solutions for problem (P_λ) $(\lambda \in (0, \lambda^*))$.

Proposition 4.3. *If hypotheses $H(a)$ and $H(f)$ hold and $\lambda \in (0, \lambda^*)$, then problem (P_λ) has a smallest positive solution $u_\lambda^* \in \operatorname{int} C_+$ and a biggest negative solution $v_\lambda^* \in -\operatorname{int} C_+$.*

Proof. Since we are looking for the smallest positive solution and $S_+(\lambda)$ is downward directed, without any loss of generality, we may assume that

$$0 \leq u(z) \leq c_{16} \tag{4.8}$$

for some $c_{16} > 0$, all $z \in \bar{\Omega}$ and all $u \in S_+(\lambda)$.

From [7, p. 336], we know that we can find $\{u_n\} \subseteq S_+(\lambda)$ such that

$$\inf S_+(\lambda) = \inf_{n \geq 1} u_n.$$

For every $n \geq 1$ we have

$$A(u_n) = N_{f_\lambda}(u_n). \tag{4.9}$$

From (4.8), (4.9), Corollary 2.4 and hypothesis $H(f)$ (i), it follows that

$$\{u_n\} \subset W^{1,p}(\Omega) \text{ is bounded.}$$

So, we may assume that

$$u_n \rightharpoonup u_\lambda^* \quad \text{in } W^{1,p}(\Omega) \quad \text{and} \quad u_n \to u_\lambda^* \quad \text{in } L^{r(\lambda)}(\Omega). \tag{4.10}$$

On (4.9) we act with $u_n - u_\lambda^* \in W^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (4.10). Then

$$\lim_{n \to \infty} \langle A(u_n), u_n - u_\lambda^* \rangle = 0,$$

and Proposition 2.6 leads to

$$u_n \to u_\lambda^* \quad \text{in } W^{1,p}(\Omega). \tag{4.11}$$

Hence, if in (4.9) we pass to the limit as $n \to \infty$ and use (4.11), then

$$A(u_\lambda^*) = N_{f_\lambda}(u_\lambda^*). \tag{4.12}$$

Also, from Proposition 4.2, we have $\hat{u}_\lambda \leq u_n$ for all $n \geq 1$, hence $\hat{u}_\lambda \leq u_\lambda^*$ and so $u_\lambda^* \neq 0$. Therefore, in view of (4.12),

$$u_\lambda^* \in S_+(\lambda) \quad \text{and} \quad u_\lambda^* = \inf S_+(\lambda).$$

Similarly, we produce $v_\lambda^* \in -\operatorname{int} S_-(\lambda)$ the biggest negative solution of (P_λ). □

According to the plan outlined in the beginning of this section, now we look for a nontrivial solution of (P_λ) ($\lambda \in (0, \lambda^*)$) in the order interval $[v_\lambda^*, u_\lambda^*]$. Such a solution will be obtained using Theorem 2.1. To show that this solution is nontrivial, we will use critical groups. For this purpose we compute the critical groups of φ_λ at the origin. Such a computation was first done by Moroz [20] for Dirichlet problems with $a(y) = y$ for all $y \in \mathbb{R}^N$ (semilinear equations) and with a reaction satisfying stronger hypotheses. The result of Moroz was extended to problems with the p-Laplacian (that is $a(y) = |y|^{p-2}y$ for all $y \in \mathbb{R}^N$ with $1 < p < \infty$) by Jiu–Su [15]. Our result here extends both the aforementioned works. We point out that the Neumann case presents additional difficulties due to the failure of the Poincaré inequality.

Proposition 4.4. *If hypotheses $H(a)$ and $H(f)$ hold and $\lambda \in (0, \lambda^*)$, then $C_k(\varphi_\lambda, 0) = 0$ for all $k \geq 0$.*

Proof. From (4.1) it follows that

$$F(z,x,\lambda) \geq \frac{\hat{c}_0(\lambda)}{\mu(\lambda)}|x|^{\mu(\lambda)} - \frac{c_{15}}{r(\lambda)}|x|^{r(\lambda)} \tag{4.13}$$

for a.a $z \in \Omega$, all $x \in \mathbb{R}$. Also, hypothesis $H(a)$ (v) and Corollary 2.4, imply that

$$G(y) \leq c_{17}(|y|^\tau + |y|^p) \tag{4.14}$$

for some $c_{17} > 0$, all $y \in \mathbb{R}^N$.
Let $u \in W^{1,p}(\Omega)$ and $t \in (0,1)$. We have

$$\begin{aligned}\varphi_\lambda(tu) &= \int_\Omega G(t\nabla u)\,dz - \int_\Omega F(z,tu,\lambda)\,dz \\ &\leq c_{17}t^\tau\left(\|\nabla u\|_\tau^\tau + \|\nabla u\|_p^p\right) - \frac{\hat{c}_0(\lambda)}{\mu(\lambda)}t^{\mu(\lambda)}\|u\|_{\mu(\lambda)}^{\mu(\lambda)} + \frac{c_{15}}{r(\lambda)}t^{r(\lambda)}\|u\|_{r(\lambda)}^{r(\lambda)},\end{aligned} \tag{4.15}$$

where we used (4.13), (4.14) and the fact that $\tau < p$, $t \in (0,1)$.
Since $\mu(\lambda) < \tau < p < r(\lambda)$, from (4.15) we see that we can find $t^* = t^*(\lambda, u) \in (0,1)$ small such that

$$\varphi_\lambda(tu) < 0 \quad \text{for all } t \in (0, t^*). \tag{4.16}$$

Let $u \in W^{1,p}(\Omega)$, $0 < \|u\|_{1,p} < 1$ and $\varphi_\lambda(u) = 0$. Then, because $\varphi_\lambda(u) = 0$,

$$\begin{aligned}\frac{d}{dt}\varphi_\lambda(tu)\Big|_{t=1} &= \langle \varphi_\lambda'(u), u\rangle \\ &= \langle A(u), u\rangle - \int_\Omega f(z,u,\lambda)u\,dz \\ &= \int_\Omega [(a(\nabla u), \nabla u)_{\mathbb{R}^N} - \tau G(\nabla u)]\,dz + (\tau - q(\lambda))\int_\Omega F(z,u,\lambda)\,dz \\ &\quad + \int_\Omega [q(\lambda)F(z,u,\lambda) - f(z,u,\lambda)u]\,dz.\end{aligned} \tag{4.17}$$

Hypotheses $H(f)$ (ii), (iii) and (iv) imply that, for some $c_{18} = c_{18}(\lambda) > 0$, a.a. $z \in \Omega$ and all $x \in \mathbb{R}$ one has

$$q(\lambda)F(z,x,\lambda) - f(z,x,\lambda)x \geq -c_{18}|x|^{r(\lambda)}. \tag{4.18}$$

We return to (4.17) and use (4.13), (4.18) and hypothesis $H(a)$ (v). Then, recalling that $0 < \|u\|_{1,p} < 1$ and $\mu(\lambda) < p$

$$\begin{aligned}\frac{d}{dt}\varphi_\lambda(tu)\Big|_{t=1} &\geq \tilde{c}\|\nabla u\|_p^p + (\tau - q(\lambda))\frac{\hat{c}_0(\lambda)}{\mu(\lambda)}\|u\|_{\mu(\lambda)}^{\mu(\lambda)} - \hat{c}_{19}\|u\|_{r(\lambda)}^{r(\lambda)} \\ &\geq \tilde{c}\|\nabla u\|_p^p + (\tau - q(\lambda))\frac{\hat{c}_0(\lambda)}{\mu(\lambda)}\|u\|_{\mu(\lambda)}^{p} - c_{19}\|u\|_{1,p}^{r(\lambda)}, \end{aligned} \tag{4.19}$$

for some $c_{19} = c_{19}(\lambda) > 0$. We know that $u \mapsto \|u\|_{\mu(\lambda)} + \|\nabla u\|_p$ is an equivalent norm on $W^{1,p}(\Omega)$ (see for example [10, p. 227]). So, from (4.19) we have that, for some $c_{20} = c_{20}(\lambda) > 0$

$$\frac{d}{dt}\varphi_\lambda(tu)\Big|_{t=1} \geq c_{20}\|u\|_{1,p}^p - c_{19}\|u\|_{1,p}^{r(\lambda)}.$$

Since $p < r(\lambda)$, it follows that for $\rho \in (0,1)$ small we have

$$\frac{d}{dt}\varphi_\lambda(tu)\Big|_{t=1} > 0 \quad \text{for all } u \in W^{1,p}(\Omega) \quad \text{with } 0 < \|u\|_{1,p} < \rho,\ \varphi_\lambda(u) = 0. \tag{4.20}$$

Let $u \in W^{1,p}(\Omega)$ with $0 < \|u\|_{1,p} < \rho$, $\varphi_\lambda(u) = 0$. We show that

$$\varphi_\lambda(tu) \leq 0 \quad \text{for all } t \in [0,1]. \tag{4.21}$$

Arguing by contradiction, suppose that we can find $t_0 \in (0,1)$ such that $\varphi_\lambda(t_0 u) > 0$. Since φ_λ is continuous, we have

$$t_* = \min\{t : t_0 \leq t \leq 1,\ \varphi_\lambda(tu) = 0\} > t_0 > 0.$$

It follows that

$$\varphi_\lambda(tu) > 0 \quad \text{for all } t \in [t_0, t_*). \tag{4.22}$$

Let $y = t_* u$. Then $0 < \|y\|_{1,p} \leq \|u\|_{1,p} \leq \rho$ and $\varphi_\lambda(y) = 0$. So, from (4.20) we infer that

$$\frac{d}{dt}\varphi_\lambda(ty)\Big|_{t=1} > 0. \tag{4.23}$$

From (4.22) we have that for all $t \in [t_0, t_*)$

$$\varphi_\lambda(y) = \varphi_\lambda(t_* u) = 0 < \varphi_\lambda(tu)$$

hence

$$\frac{d}{dt}\varphi_\lambda(ty)\Big|_{t=1} = t_* \frac{d}{dt}\varphi_\lambda(tu)\Big|_{t=t_*} = t_* \lim_{t \to t_*^-} \frac{\varphi_\lambda(tu)}{t - t_*} \leq 0,$$

which is in contradiction with (4.23). This proves (4.21).

We can always choose $\rho \in (0,1)$ small such that $K_{\varphi_\lambda} \cap \bar{B}_\rho = \{0\}$. We consider the deformation $h\colon [0,1] \times (\varphi_\lambda^0 \cap \bar{B}_\rho) \to \varphi_\lambda^0 \cap \bar{B}_\rho$ defined by

$$h(t,u) = (1-t)u.$$

From (4.21) it is clear that this deformation is well-defined. So, $\varphi_\lambda^0 \cap \bar{B}_\rho$ is contractible in itself.

Let $u \in \bar{B}_\rho$ with $\varphi_\lambda(u) > 0$. We show that there exists unique $t(u) \in (0,1)$ such that

$$\varphi_\lambda(t(u)u) = 0.$$

From (4.15) and Bolzano's theorem, we see that such a $t(u) \in (0,1)$ exists. We need to show its uniqueness. Arguing by contradiction, suppose that we can find

$$0 < t_1 = t_1(u) < t_2 = t_2(u) < 1 \quad \text{such that} \quad \varphi_\lambda(t_1 u) = \varphi_\lambda(t_2 u) = 0.$$

From (4.21) we have

$$\theta(t) = \varphi_\lambda(t t_2 u) \leq 0 \quad \text{for all } t \in [0,1],$$

hence $\frac{t_1}{t_2} \in (0,1)$ is a maximizer of $\theta(\cdot)$ and

$$\frac{d}{dt}\theta(t)\Big|_{t=\frac{t_1}{t_2}} = 0.$$

Thus we derive that

$$\frac{t_1}{t_2}\,\frac{d}{dt}\varphi_\lambda(t t_2 u)\Big|_{t=\frac{t_1}{t_2}} = \frac{d}{dt}\varphi_\lambda(t t_1)\Big|_{t=1} = 0,$$

which contradicts (4.20). So we have proved the uniqueness of $t(u) \in (0,1)$.

By virtue of this uniqueness of $t(u) \in (0,1)$, we have

$$\varphi_\lambda(tu) < 0 \quad \text{for all } t \in (0, t(u)) \quad \text{and} \quad \varphi_\lambda(tu) > 0 \quad \text{for all } t \in (t(u), 1].$$

Now, let $\gamma_1 \colon \bar{B}_\rho \setminus \{0\} \to (0,1]$ be defined by

$$\gamma_1(u) = \begin{cases} 1 & \text{if } u \in \bar{B}_\rho \setminus \{0\},\ \varphi_\lambda(u) \leq 0 \\ t(u) & \text{if } u \in \bar{B}_\rho \setminus \{0\},\ \varphi_\lambda(u) > 0. \end{cases}$$

It is easy to see that γ_1 is continuous. Let $k \colon \bar{B}_\rho \setminus \{0\} \to (\varphi_\lambda^0 \cap \bar{B}_\rho) \setminus \{0\}$ be defined by

$$k(u) = \gamma_1(u) u$$

for all $x \in \bar{B}_\rho \setminus \{0\}$. Evidently k is continuous and

$$k\Big|_{(\varphi_\lambda^0 \cap \bar{B}_\rho)\setminus\{0\}} = \mathrm{id}\Big|_{(\varphi_\lambda^0 \cap \bar{B}_\rho)\setminus\{0\}}.$$

So, it follows that $(\varphi_\lambda^0 \cap \bar{B}_\rho) \setminus \{0\}$ is a retract of $\bar{B}_\rho \setminus \{0\}$ and the latter is contractible. Hence $(\varphi_\lambda^0 \cap \bar{B}_\rho) \setminus \{0\}$ is contractible in itself (see [6, p. 333]). So, recalling that $\varphi_\lambda^0 \cap \bar{B}_\rho$ is contractible in itself, from [12, p. 389], we have

$$H_k\left(\varphi_\lambda^0 \cap \bar{B}_\rho, (\varphi_\lambda^0 \cap \bar{B}_\rho) \setminus \{0\}\right) = 0 \quad \text{for all } k \geq 0$$

so that

$$C_k(\varphi_\lambda, 0) = 0 \quad \text{for all } k \geq 0.$$

□

Now, we are ready to produce a nodal solution for problem (P_λ), $\lambda \in (0, \lambda^*)$.

Proposition 4.5. *If hypotheses $H(a)$ and $H(f)$ hold and $\lambda \in (0, \lambda^*)$, then problem (P_λ) admits a nodal solution $y_\lambda \in [v_\lambda^*, u_\lambda^*] \cap C^1(\bar{\Omega})$.*

Proof. Let $u_\lambda^* \in \operatorname{int} C_+$ and $v_\lambda^* \in -\operatorname{int} C_+$ be the two extremal constant sign solutions produced in Proposition 4.3. We introduce the following truncation-perturbation of the reaction in problem (P_λ):

$$\gamma_\lambda(z,x) = \begin{cases} f(z, v_\lambda^*(z), \lambda) + |v_\lambda^*(z)|^{p-2} v_\lambda^*(z) & \text{if } x < v_\lambda^*(z) \\ f(z,x,\lambda) + |x|^{p-2}x & \text{if } v_\lambda^*(z) \le x \le u_\lambda^* \\ f(z, u_\lambda^*(z), \lambda) + u_\lambda^*(z)^{p-1} & \text{if } u_\lambda^*(z) < x. \end{cases} \tag{4.24}$$

This is a Carathéodory function. Let $\Gamma_\lambda(z,x) = \int_0^x \gamma_\lambda(z,s)\,ds$ and consider the C^1-functional $e_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$e_\lambda(u) = \int_\Omega G(\nabla u(z))\,dz + \frac{1}{p}\|u\|_p^p - \int_\Omega \Gamma_\lambda(z,u(z))\,dz$$

for all $u \in W^{1,p}(\Omega)$. In addition, we consider the positive and negative truncations of $\gamma_\lambda(z,\cdot)$, namely we introduce the Carathéodory functions

$$\gamma_\lambda^\pm(z,x) = \gamma_\lambda(z, \pm x^\pm).$$

We set $\Gamma_\lambda^\pm(z,x) = \int_0^x \gamma_\lambda^\pm(z,s)ds$ and consider the C^1-functionals $e_\lambda^\pm : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$e_\lambda^\pm(u) = \int_\Omega G(\nabla u(z))\,dz + \frac{1}{p}\|u\|_p^p - \int_\Omega \Gamma_\lambda^\pm(z,u(z))\,dz$$

for all $u \in W^{1,p}(\Omega)$. Reasoning as in the proof of Proposition 4.2, we can show that

$$K_{e_\lambda} \subseteq [v_\lambda^*, u_\lambda^*], \qquad K_{e_\lambda^+} \subseteq [0, u_\lambda^*], \qquad K_{e_\lambda^-} \subseteq [v_\lambda^*, 0].$$

The extremality of $u_\lambda^* \in \operatorname{int} C_+$ and $v_\lambda^* \in -\operatorname{int} C_+$, implies that

$$K_{e_\lambda} \subseteq [v_\lambda^*, u_\lambda^*], \qquad K_{e_\lambda^+} = \{0, u_\lambda^*\}, \qquad K_{e_\lambda^-} = \{v_\lambda^*, 0\}. \tag{4.25}$$

Claim: $u_\lambda^* \in \operatorname{int} C_+$ and $v_\lambda^* \in -\operatorname{int} C_+$ are local minimizers of e_λ.
From (4.24) it is clear that e_λ^+ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\tilde{u}_\lambda^* \in W^{1,p}(\Omega)$ such that

$$e_\lambda^+(\tilde{u}_\lambda^*) = \inf\{e_\lambda^+(u) : u \in W^{1,p}(\Omega)\}. \tag{4.26}$$

Using hypothesis $H(f)$ (iv) and choosing $\xi \in (0,1)$ small (take $0 < \xi \le \min_{\bar{\Omega}} u_\lambda^*$), we have $e_\lambda^+(\xi) < 0$, so that

$$e_\lambda^+(\tilde{u}_\lambda^*) < 0 = e_\lambda^+(0),$$

hence $\tilde{u}_\lambda^* \neq 0$. Since $\tilde{u}_\lambda^* \in K_{e_\lambda^+}$, from (4.25) it follows that $\tilde{u}_\lambda^* = u_\lambda^* \in \operatorname{int} C_+$. Note that $e_\lambda|_{C_+} = e_\lambda^+|_{C_+}$. Hence, $u_\lambda^* \in \operatorname{int} C_+$ is a local C^1-minimizer of e_λ. Invoking Proposition 2.7 we have that u_λ^* is a local $W^{1,p}(\Omega)$-minimizer of e_λ. Similarly for $v_\lambda \in -\operatorname{int} C_+$ using this time the functional e_λ^-. This proves the Claim.

Without any loss of generality, we may assume that $e_\lambda(v_\lambda^*) \le e_\lambda(u_\lambda^*)$ (the analysis is similar if the opposite inequality holds). Because of the Claim, we can find $\rho \in (0,1)$ small such that

$$e_\lambda(v_\lambda^*) \le e_\lambda(u_\lambda^*) < \inf\{e_\lambda(u) : \|u - u_\lambda^*\|_{1,p} = \rho\} = m_\lambda^*, \qquad \|v_\lambda^* - u_\lambda^*\|_{1,p} > \rho. \tag{4.27}$$

The functional e_λ is coercive (see (4.24)). So, it satisfies the C-condition. This fact and (4.27) permit the use of Theorem 2.1. So, we can find $y_\lambda \in W^{1,p}(\Omega)$ such that

$$y_\lambda \in K_{e_\lambda} \quad \text{and} \quad m_\lambda^* \leq e(y_\lambda). \tag{4.28}$$

From (4.24), (4.25), (4.27) and (4.28) we infer that $y_\lambda \notin \{u_\lambda^*, v_\lambda^*\}$ and solves problem (P_λ). Since y_λ is a critical point of mountain pass type for e_λ, we have

$$C_1(e_\lambda, y_\lambda) \neq 0. \tag{4.29}$$

From (4.24), we see that

$$e|_{[v_\lambda^*, u_\lambda^*]} = \varphi_\lambda|_{[v_\lambda^*, u_\lambda^*]}.$$

Because $v_\lambda^* \in -\operatorname{int} C_+$, $u_\lambda^* \in \operatorname{int} C_+$, from Palais [22] or equivalently from the homotopy invariance of critical groups and since $C^1(\bar{\Omega})$ is dense in $W^{1,p}(\Omega)$, we have

$$C_k(e_\lambda, 0) = C_k(\varphi_\lambda, 0) \quad \text{for all } k \geq 0,$$

hence, because of Proposition 4.4

$$C_k(e_\lambda, 0) = 0 \quad \text{for all } k \geq 0. \tag{4.30}$$

Comparing (4.29) and (4.30), we deduce that $y_\lambda \neq 0$. Since $y_\lambda \in [v_\lambda^*, u_\lambda^*]$ (see (4.25)), we infer that y_λ is nodal and the nonlinear regularity result of Lieberman [17, p. 320], implies $y_\lambda \in C^1(\bar{\Omega})$. □

Concluding this work, we can state the following multiplicity theorem for problem (P_λ).

Theorem 4.6. *If hypotheses $H(a)$ and $H(f)$ hold, then there exists $\lambda^* > 0$ such that for all $\lambda \in (0, \lambda^*)$ problem (P_λ) has at least five nontrivial solutions*

$$u_0, \hat{u} \in \operatorname{int} C_+, \quad v_0, \hat{v} \in -\operatorname{int} C_+ \quad \text{and} \quad y_\lambda \in C^1(\bar{\Omega}) \text{ nodal},$$

with $\hat{u}, \hat{v}$ local minimizer of the energy functional φ_λ and $\varphi(\hat{u}), \varphi(\hat{v}) < 0 < \varphi(u_0), \varphi(v_0)$; moreover, problem (P_λ) admits extremal constant sign solutions $u_\lambda^* \in \operatorname{int} C_+$, $v_\lambda^* \in -\operatorname{int} C_+$ and $y_\lambda \in [v_\lambda^*, u_\lambda^*] \cap C^1(\bar{\Omega})$.

Acknowledgements

The authors are very grateful to the anonymous referee for his/her knowledgeable report, which helped them improve their manuscript.

This work was performed under the auspices of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

References

[1] S. Aizicovici, N. S. Papageorgiou, V. Staicu, *Degree theory for operators of monotone type and nonlinear elliptic equations with inequality constraints*, Mem. Amer. Math. Soc., Vol. 196, No. 915, 2008. MR2459421; url

[2] A. Ambrosetti, H. Brezis, G. Cerami, Combined effects of concave and convex nonlinearities in some elliptic problems, *J. Funct. Anal.* **122**(1994), 519–543. MR1276168; url

[3] A. Ambrosetti, P. Rabinowitz, Dual variational methods in critical point theory and applications, *J. Functional Analysis* **14**(1973), 349–381. MR0370183

[4] H. Brezis, L. Nirenberg, H^1 versus C^1 minimizers, *C. R. Acad. Sci. Paris Sér. I Math.* **317**(1993), 476–472. MR1239032

[5] J. I. Diaz, J. E. Saa, Existence et unicité de solutions positives pour certaines equations elliptiques quasi-lineaires (in French) [Existence and uniqueness of positive solutions of some quasilinear elliptic equations], *C. R. Acad. Sci. Paris Sér. I Math.* **305**(1987), 521–524. MR916325

[6] J. Dugundji, *Topology*, Allyn and Bacon, Boston, 1966. MR0193606

[7] N. Dunford, J. Schwartz, *Linear operators I*, Wiley-Interscience, New York, 1958. MR0117523

[8] M. Filippakis, A. Kristály, N. S. Papageorgiou, Existence of five nonzero solutions with exact sign for a p-Laplacian equation, *Discrete Cont. Dyn. Syst.* **24**(2009), 405–440. MR2486583; url

[9] J. P. García Azorero, J. J. Manfredi, I. Peral Alonso, Sobolev versus Hölder local minimizers and global multiplicity for some quasilinear elliptic equations, *Commun. Contemp, Math.* **2**(2000), 385–404. MR1776988; url

[10] L. Gasiński, N. S. Papageorgiou, *Nonlinear analysis*, Chapman & Hall/CRC, Boca Raton, 2006. MR2168068

[11] L. Gasiński, N. S. Papageorgiou, Existence and multiplicity of solutions for Neumann p-Laplacian type equations, *Adv. Nonlinear Stud.* **8**(2008), 843–870. MR2454878

[12] A. Granas, J. Dugundji, *Fixed point theory*, Springer, New York, 2003. MR1987179; url

[13] Z. Guo, Z. Zhang, $W^{1,p}$ versus C^1 local minimizers and multiplicity results for quasilinear elliptic equations, *J. Math. Anal. Appl.* **286**(2003), 32–50. MR2009616; url

[14] S. Hu, N. S. Papageorgiou, Multiplicity of solutions for parametric p-Laplacian equations with nonlinearity concave near the origin, *Tohoku Math. J. (2)* **62**(2010), 137–162. MR2654306; url

[15] Q. Jiu, J. Su, Existence and multiplicity results for Dirichlet problems with p-Laplacian, *J. Math. Anal. Appl.* **281**(2003), 587–601. MR1982676; url

[16] G. Li, C. Yang, The existence of a nontrivial solution to a nonlinear elliptic boundary value problem of p-Laplacian type without the Ambrosetti–Rabinowitz condition, *Nonlinear Anal.* **72**(2010), 4602–4613. MR2639208; url

[17] G. Lieberman, The natural generalization of the natural conditions of Ladyzhenskaya and Uraltseva for elliptic equations, *Comm. Partial. Differential Equations* **16**(1991), 311–361. MR1104103; url

[18] S. Marano, N. S. Papageorgiou, Positive solutions to a Dirichlet problem with p-Laplacian and concave–convex nonlinearity depending on a parameter, *Commun. Pure Appl. Anal.* **12**(2013), 815–829. MR2982792; url

[19] O. H. Miyagaki, M. A. S. Souto, Superlinear problems without Ambrosetti and Rabinowitz growth condition, *J. Differential Equations* **245**(2008), 3628–3638. MR2462696; url

[20] V. Moroz, Solutions of superlinear at zero elliptic equations via Morse theory, *Topol. Methods Nonlinear Anal.* **10**(1997), 1–11. MR1634579

[21] D. Motreanu, N. S. Papageorgiou, Multiple solutions for nonlinear Neumann problems driven by a nonhomogeneous differential operator, *Proc. Amer. Math. Soc.* **139**(2011), 3527–3535. MR2813384; url

[22] R. Palais, Homotopy theory of infinite dimensional manifolds, *Topology* **5**(1968), 1–16. MR0189028

[23] N. S. Papageorgiou, V. D. Rădulescu, Qualitative phenomena for some classes of quasilinear elliptic equations with multiple resonance, *Appl. Math. Optim.* **69**(2014), 393–430. MR3197304; url

[24] N. S. Papageorgiou, G. Smyrlis, Positive solutions for nonlinear Neumann problems with concave and convex terms, *Positivity* **16**(2012), 271–296. MR2929091; url

[25] P. Pucci, J. Serrin, *The maximum principle*, Progress in Nonlinear Differential Equations and their Applications, Vol. 73, Birkhäuser, Basel, 2007. MR2356201

[26] P. Winkert, L^∞-estimates for nonlinear elliptic Neumann boundary value problems, *NoDEA Nonlinear Differential Equations Appl.* **17**(2010), 289–302. MR2652229; url