Electronic Journal of Qualitative Theory of Differential Equations
Spec. Ed. I, 2009 No. 1, 1–20; **http://www.math.u-szeged.hu/ejqtde/**

EXISTENCE OF PERIODIC SOLUTIONS IN TOTALLY NONLINEAR DELAY DYNAMIC EQUATIONS

Murat Adıvar [1] and Youssef N. Raffoul [2]

[1] Department of Mathematics, Izmir University of Economics
35530 Balcova, Izmir, Turkey
e-mail: murat.adivar@ieu.edu.tr,
[2] Department of Mathematics, University of Dayton, Dayton,
OH 45469-2316, USA
e-mail: youssef.raffoul@notes.udayton.edu

Honoring the Career of John Graef on the Occasion of His Sixty-Seventh Birthday

Abstract

By means of a fixed point theorem we offer sufficient conditions for the existence of periodic solutions of totally nonlinear delay dynamic equations, where the solution maps a periodic time scale into another time scale.

Key words and phrases: Fixed point, large contraction, periodic solutions, time scales, totally nonlinear delay dynamic equations
AMS (MOS) Subject Classifications: Primary 34K13, 34A34; Secondary 34K30, 34L30

1 Introduction

We begin with asking the following questions:

- *When does a nonlinear delay difference equation*

$$\Delta x(n) = -a(n)h(x(n)) + G(n, x(n-\tau)), \quad n \in \mathbb{Z}_+, \tau \in \mathbb{Z}_+, \tag{1}$$

  have an integer valued periodic solution?

- *When does a nonlinear delay differential equation*

$$x'(t) = -a(t)h(x(t)) + G(t, x(t-r(t))), \quad t \in \mathbb{R}_+, \quad 0 < r(t) < t, \tag{2}$$

  have a non-zero positive valued periodic solution?

Time scale theory has given mathematicians a general perspective of the understanding on how to combine and unify the theories of difference and differential equations under the umbrella of dynamic equations on time scales. Hence, it is natural to ask the following question which is much more general than the ones above:

- *When does a totally nonlinear delay dynamic equation*

$$x^{\Delta}(t) = -a(t)h(x(t)) + G(t, x(\delta(t))), \quad t \in \mathbb{T}_1, \tag{3}$$

have a non-zero periodic solution which maps a periodic time scale $\mathbb{T}_1$ into another time scale $\mathbb{T}_2$?

The expression totally nonlinear implies that the functions h and G of (3) are nonlinear in x. In earlier time scale papers (e.g., [2], [3], [7]) concerning the existence of periodic solutions of dynamic equations on a time scale $\mathbb{T}$, sufficient conditions are given only for the existence of real valued periodic solutions in $C(\mathbb{T}, \mathbb{R})$. By doing so, existence is shown but the existence of positive periodic solutions is handled in a totally different manner. On the other hand, most of the studies of difference equations show the existence of real valued solutions $x : \mathbb{Z} \to \mathbb{R}$. However, when we study some problem from biology, physics or any other applicable science described by difference equations, we should have integer valued solutions (see for instance [8], [9]). This, in return, requires showing the existence of integer valued solutions. The advantage of handling a problem on two time scales $\mathbb{T}_1$ and $\mathbb{T}_2$ instead of on a periodic time scale $\mathbb{T}$ and $\mathbb{R}$ not only fills this gap but also helps us to understand positivity of solutions. By this approach it is enough to set the problem on $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{Z}$ to obtain existence of integer valued solutions of a difference equation and it is easy to obtain positivity of solutions by taking the positive part of the time scale $\mathbb{T}_2$ as the range of functions. It is worth mentioning that existence of periodic solutions of the equation (3) has not been studied before even for the particular case when $\mathbb{T}_1$ is a periodic time scale and $\mathbb{T}_2 = \mathbb{R}$.

For clarity, we restate the following definitions, introductory examples, and lemmas which can be found in [3] and [7].

Definition 1.1 *A time scale $\mathbb{T}$ is said to be* periodic *if there exists $P > 0$ such that $t \pm P \in \mathbb{T}$ for all $t \in \mathbb{T}$. If $\mathbb{T} \neq \mathbb{R}$, the smallest positive P is called the* period *of the time scale.*

Example 1.1 *The following time scales are periodic.*

i. $\mathbb{T} = \mathbb{Z}$ *has period* $P = 1$;

ii. $\mathbb{T} = h\mathbb{Z}$ *has period* $P = h$;

iii. $\mathbb{T} = \mathbb{R}$;

iv. $\mathbb{T} = \bigcup_{i=-\infty}^{\infty} [(2i-1)h, 2ih]$, $h > 0$ *has period* $P = 2h$;

v. $\mathbb{T} = \{t = k - q^m : k \in \mathbb{Z},\ m \in \mathbb{N}_0,\ q \in (0,1)\}$ *has period* $P = 1$.

Remark 1.1 *All periodic time scales are unbounded above and below.*

Definition 1.2 *Let* $\mathbb{T} \neq \mathbb{R}$ *be a periodic time scale with period* P. *We say that the function* $f\colon \mathbb{T} \to \mathbb{R}$ *is periodic with period* T *if there exists a natural number* n *such that* $T = nP$, $f(t \pm T) = f(t)$ *for all* $t \in \mathbb{T}$ *and* T *is the smallest number such that* $f(t \pm T) = f(t)$. *If* $\mathbb{T} = \mathbb{R}$, *we say that* f *is periodic with period* $T > 0$ *if* T *is the smallest positive number such that* $f(t \pm T) = f(t)$ *for all* $t \in \mathbb{T}$.

Let $\mathbb{T}_1$ be a periodic time scale and $\mathbb{T}_2$ a time scale that is closed under addition, i.e., $u + v \in \mathbb{T}_2$ for all $u, v \in \mathbb{T}_2$. In this paper, using the concept of large contraction, we study existence of periodic solutions $x : \mathbb{T}_1 \to \mathbb{T}_2$ of totally nonlinear dynamic equations

$$x^{\Delta}(t) = -a(t)h(x(t)) + G(t, x(\delta(t))), \quad t \in \mathbb{T}_1, \tag{4}$$

where $a : \mathbb{T}_1 \to \mathbb{R}$, $h : \mathbb{T}_2 \to \mathbb{R}$, $G : \mathbb{T}_1 \times \mathbb{T}_2 \to \mathbb{R}$ and $\delta : \mathbb{T}_1 \to \mathbb{T}_1$ is a strictly increasing delay function satisfying

$$\delta(t) < t \text{ and } \delta \circ \sigma = \sigma \circ \delta. \tag{5}$$

In the following, we give some particular time scales with corresponding delay functions.

Time scale	**Delay function**
$\mathbb{T} = \mathbb{R}$	$\delta(t) = t - \tau, \quad \tau \in \mathbb{R}_+$
$\mathbb{T} = h\mathbb{Z} = \{hk : k \in \mathbb{Z}\}, \quad h > 0$	$\delta(t) = t - h\tau, \quad \tau \in \mathbb{Z}_+$
$\mathbb{T} = \{t = k - q^m : k \in \mathbb{Z},\ m \in \mathbb{N}_0,\ q \in (0,1)\}$	$\delta(t) = t - \tau, \quad \tau \in \mathbb{Z}_+$

Throughout the paper we suppose that the functions a, h, and G are continuous in their respective domains and that for at least $T > 0$

$$a(t+T) = a(t), \quad \delta(t+T) = \delta(t) + T, \quad G(t,.) = G(t+T,.), \quad t \in \mathbb{T}_1. \tag{6}$$

To avoid obtaining the zero solution we also suppose that $G(t, 0) \neq a(t)h(0)$ for some $t \in \mathbb{T}_1$.

In the analysis, we employ a fixed point theorem in which the notion of a large contraction is required as one of the sufficient conditions. First, we give the following definition which can be found in [6].

Definition 1.3 (Large Contraction) *Let* $(\mathcal{M},d)$ *be a metric space and* $B\colon \mathcal{M} \to \mathcal{M}$. B *is said to be a large contraction if* $\phi, \varphi \in \mathcal{M}$, *with* $\phi \neq \varphi$ *then* $d(B\phi, B\varphi) \leq d(\phi, \varphi)$ *and if for all* $\varepsilon > 0$, *there exists a* $\delta < 1$ *such that*

$$[\phi, \varphi \in \mathcal{M}, d(\phi, \varphi) \geq \varepsilon] \Rightarrow d(B\phi, B\varphi) \leq \delta d(\phi, \varphi).$$

The next theorem, which constitutes a basis for our main result, is a reformulated version of Krasosel'kii's fixed point theorem.

Theorem 1.1 *[6] Let $\mathcal{M}$ be a bounded convex nonempty subset of a Banach space $(\mathbb{B}, \|\cdot\|)$. Suppose that A and B map $\mathcal{M}$ into $\mathbb{B}$ such that*

i. $x, y \in \mathcal{M}$, implies $Ax + By \in \mathcal{M}$;

ii. A is compact and continuous;

iii. B is a large contraction mapping.

Then there exists $z \in \mathcal{M}$ with $z = Az + Bz$.

Define the *forward jump operator* σ by

$$\sigma(t) = \inf\{s > t : s \in \mathbb{T}\},$$

and the *graininess function* μ by $\mu(t) = \sigma(t) - t$. A point t in a time scale is called right scattered if $\sigma(t) > t$. Hereafter, we denote by x^σ the composite function $x \circ \sigma$. Note that in a periodic time scale $\mathbb{T}$ with period P the inequality $0 \le \mu(t) \le P$ holds for all $t \in \mathbb{T}$.

Remark 1.2 *If $\mathbb{T}$ is a periodic time scale with period ω, then $\sigma(t \pm n\omega) = \sigma(t) \pm n\omega$. Consequently, the graininess function μ satisfies $\mu(t \pm n\omega) = \sigma(t \pm n\omega) - (t \pm n\omega) = \sigma(t) - t = \mu(t)$ and so, μ is a periodic function with period ω.*

Definition 1.4 *A function $p : \mathbb{T} \to \mathbb{R}$ is said to be* regressive *provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^\kappa$, where*

$$\mathbb{T}^\kappa = \begin{cases} \mathbb{T}\backslash(\rho(\sup\mathbb{T}), \sup\mathbb{T}] & \text{if } \sup\mathbb{T} < \infty, \\ \mathbb{T}, & \text{if } \sup\mathbb{T} = \infty. \end{cases}$$

The set of all regressive rd-continuous functions $p : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{R}$ while the set $\mathcal{R}^+$ is given by $\mathcal{R}^+ = \{p \in \mathcal{R} : 1 + \mu(t)p(t) > 0 \text{ for all } t \in \mathbb{T}\}$.

Let $p \in \mathcal{R}$ and $\mu(t) > 0$ for all $t \in \mathbb{T}$. The *exponential function* on $\mathbb{T}$ is defined by

$$e_p(t, s) = \exp\left(\int_s^t \frac{1}{\mu(z)} \mathrm{Log}(1 + \mu(z)p(z))\,\Delta z\right). \tag{7}$$

The exponential function $y(t) = e_p(t, s)$ is the solution of the initial value problem $y^\Delta = p(t)y$, $y(s) = 1$. Other properties of the exponential function are given by the following.

Lemma 1.1 *[3, Lemma 2.7.] If $p, q \in \mathcal{R}$, then*

$$e_p(t,t) = 1, \quad e_p(t,s) = 1/e_p(s,t), \quad e_p(t,u)e_p(u,s) = e_p(t,s),$$
$$e_p(\sigma(t),s)(1+\mu(t)p(t))e_p(t,s), \quad e_p(s,\sigma(t)) = \frac{e_p(s,t)}{1+\mu(t)p(t)},$$
$$e_p^{\Delta}(.,s) = pe_p(.,s), \quad e_p^{\Delta}(s,.) = (\ominus p)e_p(s,.),$$
$$e_{p\oplus q} = e_p e_q, \quad e_{p\ominus q} = \frac{e_p}{e_q}.$$

Theorem 1.2 *[3, Theorem 2.1.] Let $\mathbb{T}$ be a time scale with period $\omega > 0$. If $p \in C_{rd}(\mathbb{T})$ is a periodic function with the period $T = n\omega$, then*

$$\int_{a+T}^{b+T} p(t)\Delta t = \int_a^b p(t)\Delta t, \quad e_p(b,a) = e_p(b+T, a+T) \quad \textit{if} \quad p \in \mathcal{R},$$
$$c_p := 1 - e_p(t+T,t) \text{ is independent of } t \in \mathbb{T} \text{ whenever } p \in \mathcal{R}.$$

Definition 1.5 *[5, Definition 1.62] A continuous function $f : \mathbb{T} \to \mathbb{R}$ is called pre-differentiable with (region of differentiation) D, provided $D \subset \mathbb{T}^{\kappa}$, $\mathbb{T}^{\kappa}\backslash D$ is countable and contains no right scattered elements of $\mathbb{T}$, and f is differentiable at each $t \in D$.*

We will resort to the next theorem at several occasions in our further work.

Theorem 1.3 *[4, Theorem 1.67-Corollary 1.68] Let f and g be real-valued functions defined on $\mathbb{T}$, both pre-differentiable with $D \subset \mathbb{T}$. Then*

1.
$$\left|f^{\Delta}(t)\right| \leq g^{\Delta}(t) \text{ for all } t \in D$$
implies
$$|f(s) - f(r)| \leq g(s) - g(r) \text{ for all } r, s \in \mathbb{T}, \quad r \leq s. \tag{8}$$

2. If U is a compact interval with endpoints $r, s \in \mathbb{T}$, then
$$|f(s) - f(r)| \leq \sup_{t \in U^{\kappa} \cap D} \left|f^{\Delta}(t)\right| |s - r|. \tag{9}$$

2 Existence of periodic solution

Suppose that $\mathbb{T}_1$ is a periodic time scale and that $\mathbb{T}_2$ is an arbitrary time scale that is closed under addition. The space $P_T(\mathbb{T}_1, \mathbb{T}_2)$, given by

$$P_T(\mathbb{T}_1, \mathbb{T}_2) = \{\varphi \in C(\mathbb{T}_1, \mathbb{T}_2) : \varphi(t+T) = \varphi(t)\},$$

is a Banach space when it is endowed with the supremum norm

$$\|x\| = \sup_{t \in [0,T] \cap \mathbb{T}_1} |x(t)| = \sup_{t \in \mathbb{T}_1} |x(t)|.$$

Determine $\alpha \in (0, \infty)$ to be a fixed real number such that

$$U_\alpha = [-\alpha, \alpha] \cap \mathbb{T}_2 \neq \varnothing. \tag{10}$$

We ask for the condition (10) since we need to guarantee that the set $\mathbb{M}_\alpha$ given by

$$\mathbb{M}_\alpha = \{\phi \in P_T(\mathbb{T}_1, \mathbb{T}_2) : \|\phi\| \leq \alpha\} \tag{11}$$

(which will be shown to include a solution of (4)) is nonempty. Note that if $t \in \mathbb{T}_1$ and $\phi \in \mathbb{M}_\alpha$, then $\phi(t) \in \mathbb{T}_2$ and $-\alpha \leq \phi(t) \leq \alpha$; i.e., $\phi(t) \in U_\alpha$. Moreover, $\mathbb{M}_\alpha$ is a closed, bounded, and convex subset of the Banach space $P_T(\mathbb{T}_1, \mathbb{T}_2)$.

Hereafter, we use the notation $\gamma = -a$ and assume that $\gamma \in \mathcal{R}^+$.

Lemma 2.1 *If $x \in P_T(\mathbb{T}_1, \mathbb{T}_2)$, then x is a solution of equation (4) if, and only if,*

$$x(t) = k_\gamma \int_t^{t+T} \{a(s)H(x(s)) + G(s, x(\delta(s)))\} e_\gamma(t+T, \sigma(s)) \Delta s, \tag{12}$$

where $k_\gamma = (1 - e_\gamma(t+T, t))^{-1}$, $\gamma(t) = -a(t)$ and

$$H(x(t)) = x(t) - h(x(t)). \tag{13}$$

Proof. Let $x \in P_T(\mathbb{T}_1, \mathbb{T}_2)$ be a solution of (4). The equation (4) can be expressed as

$$x^\Delta(t) - \gamma(t)x(t) = a(t)H(x(t)) + G(t, x(\delta(t))). \tag{14}$$

Multiplying both sides of (14) by $e_{\ominus\gamma}(\sigma(t), t_0)$ we get

$$\left\{x^\Delta(t) - \gamma(t)x(t)\right\} e_{\ominus\gamma}(\sigma(t), t_0) = \{a(t)H(x(t)) + G(t, x(\delta(t)))\} e_{\ominus\gamma}(\sigma(t), t_0).$$

Since $e_{\ominus\gamma}(t, t_0)^\Delta = -\gamma(t)e_{\ominus\gamma}(\sigma(t), t_0)$ we find

$$[x(t) e_{\ominus\gamma}(t, t_0)]^\Delta = \{a(t)H(x(t)) + G(t, x(\delta(t)))\} e_{\ominus\gamma}(\sigma(t), t_0).$$

Taking the integral from $t - T$ to t, we arrive at

$$\begin{aligned} x(t+T) e_{\ominus\gamma}(t+T, t_0) - x(t) e_{\ominus\gamma}(t, t_0) = \int_t^{t+T} &\{a(s)H(x(s)) \\ &+ G(s, x(\delta(s)))\} e_{\ominus\gamma}(\sigma(s), t_0) \Delta s. \end{aligned}$$

Using $x(t+T) = x(t)$ for $x \in P_T(\mathbb{T}_1, \mathbb{T}_2)$, and

$$\frac{e_{\ominus\gamma}(t, t_0)}{e_{\ominus\gamma}(t+T, t_0)} = e_\gamma(t+T, t), \frac{e_{\ominus\gamma}(\sigma(s), t_0)}{e_{\ominus\gamma}(t+T, t_0)} = e_\gamma(t+T, \sigma(s)),$$

we obtain (12). Since each step in the above work is reversible, the proof is complete.

Lemma 2.2 *If $p \in \mathcal{R}^+$, then*

$$0 < e_p(t,s) \leq \exp\left(\int_s^t p(r)\Delta r\right) \tag{15}$$

for all $t \in \mathbb{T}$.

Proof. If $p \in \mathcal{R}^+$, then

$$\mathrm{Log}(1+\mu(t)p(t)) = \log(1+\mu(t)p(t)) \in \mathbb{R}.$$

It follows from (7) that $e_p(t,s) > 0$. On the other hand, since we have $\exp(u) \geq 1+u$, and therefore, $u \geq \log(1+u)$ for all $u \in (-1,\infty)$, we find

$$\begin{aligned} e_p(t,s) &= \exp\left(\int_s^t \frac{1}{\mu(r)}\log(1+\mu(r)p(r))\,\Delta r\right) \\ &\leq \exp\left(\int_s^t p(r)\,\Delta r\right). \end{aligned}$$

This completes the proof.

We derive the next result from (15).

Corollary 2.1 *If $p \in \mathcal{R}^+$ and $p(t) < 0$ for all $t \in \mathbb{T}$, then for all $s \in \mathbb{T}$ with $s \leq t$ we have*

$$0 < e_p(t,s) \leq \exp\left(\int_s^t p(r)\Delta r\right) < 1. \tag{16}$$

As a consequence of Lemma 2.2 we note that for $\gamma \in \mathcal{R}^+$, $t \in [0,T] \cap \mathbb{T}$ and $s \in [t, t+T) \cap \mathbb{T}$,

$$\begin{aligned} \frac{e_\gamma(t+T,\sigma(s))}{1-e_\gamma(t+T,t)} &\leq \frac{\exp\left(\int_{\sigma(s)}^{t+T}\gamma(r)\Delta r\right)}{|1-e_\gamma(t+T,t)|} \\ &\leq \frac{\exp\left(\int_0^{2T}|\gamma(r)|\,\Delta r\right)}{|1-e_\gamma(T,0)|} := K. \end{aligned} \tag{17}$$

Let the maps A and B be defined by

$$(A\varphi)(t) = \int_t^{t+T} G(s,\varphi(\delta(s)))\frac{e_\gamma(t+T,\sigma(s))}{1-e_\gamma(t+T,t)}\Delta s,\ t \in \mathbb{T}_1, \tag{18}$$

and

$$(B\psi)(t) = \int_t^{t+T} a(s)H(\psi(s))\frac{e_\gamma(t+T,\sigma(s))}{1-e_\gamma(t+T,t)}\Delta s,\ t \in \mathbb{T}_1, \tag{19}$$

respectively. It is clear from (6) that the maps A and B are T periodic. To make sure $A : \mathbb{M}_\alpha \to P_T(\mathbb{T}_1, \mathbb{T}_2)$ and $B : \mathbb{M}_\alpha \to P_T(\mathbb{T}_1, \mathbb{T}_2)$ we also need to ask for the following condition:

$$A\phi(t), B\varphi(t) \in \mathbb{T}_2 \text{ for all } t \in \mathbb{T}_1 \text{ and } \phi, \varphi \in \mathbb{M}_\alpha. \tag{20}$$

In the proof of the next result, we use a time scale version of the Lebesgue dominated convergence theorem. For a detailed study on Δ-Riemann and Lebesgue integrals on time scales we refer the reader to [5].

Lemma 2.3 *Suppose that there exists a positive valued function $\xi : \mathbb{T}_1 \to (0, \infty)$ which is continuous on $[0, T) \cap \mathbb{T}_1$ such that*

$$|G(t, \varphi(\delta(t)))| \leq \xi(t) \text{ for all } t \in \mathbb{T}_1 \text{ and } \varphi \in \mathbb{M}_\alpha. \tag{21}$$

Then the mapping A, defined by (18), is continuous on $\mathbb{M}_\alpha$.

Proof. To see that A is a continuous mapping, let $\{\varphi_i\}_{i\in\mathbb{N}}$ be a sequence of functions in $\mathbb{M}_\alpha$ such that $\varphi_i \to \varphi$ as $i \to \infty$. Since (21) holds, the continuity of G, and the dominated convergence theorem yield

$$\begin{aligned}
\lim_{i\to\infty} \left\{ \sup_{t\in[0,T]\cap\mathbb{T}_1} \left| A\varphi_i(t) - A\varphi(t) \right| \right\} &\leq K \lim_{i\to\infty} \int_0^T \left| G(s, \varphi_i(\delta(s))) - G(s, \varphi(\delta(s))) \right| \Delta s \\
&\leq K \int_0^T \lim_{i\to\infty} \left| G(s, \varphi_j(\delta(s))) - G(s, \varphi(\delta(s))) \right| \Delta s \\
&= 0,
\end{aligned}$$

where K is defined as in (17). This shows continuity of the mapping A. The proof is complete.

One may illustrate with the following example what kind of functions ξ, satisfying (21), can be chosen to show the continuity of A.

Example 2.1 *Assume that $G(t, x)$ satisfies a Lipschitz condition in x; i.e., there is a positive constant k such that*

$$|G(t, z) - G(t, w)| \leq k\|z - w\|, \text{ for } z, w \in P_T. \tag{22}$$

Then for $\varphi \in \mathbb{M}_\alpha$,

$$\begin{aligned}
|G(t, \varphi(\delta(t)))| &= |G(t, \varphi(\delta(t))) - G(t, 0) + G(t, 0)| \\
&\leq |G(t, \varphi(\delta(t))) - G(t, 0)| + |G(t, 0)| \\
&\leq k\alpha + |G(t, 0)|.
\end{aligned}$$

In this case we may choose ξ as

$$\xi(t) = k\alpha + |G(t, 0)|. \tag{23}$$

Another possible ξ satisfying (21) is the following

$$\xi(t) = |g(t)| + |p(t)|\,|y(t)|^n\,, \tag{24}$$

where n is a positive integer and g and p are continuous functions on $\mathbb{T}_1$, and $y \in \mathbb{M}_\alpha$.

Remark 2.1 *Condition (22) is strong since it requires the function G to be globally Lipschitz. A lesser condition is (21) in which ξ can be directly chosen as in (23) or (24).*

In next two results we assume that for all $t \in \mathbb{T}_1$ and $\psi \in \mathbb{M}_\alpha$,

$$J := \int_t^{t+T} \{|a(s)||H(\psi(s))| + \xi(s)\} \frac{e_\gamma\,(t+T, \sigma\,(s))}{1 - e_\gamma\,(t+T, t)} \Delta s \le \alpha, \tag{25}$$

where ξ is defined by (21).

Lemma 2.4 *In addition to the assumptions of Lemma 2.3, suppose also that (20) and (25) hold. Then A is continuous in $\varphi \in \mathbb{M}_\alpha$ and maps $\mathbb{M}_\alpha$ into a compact subset of $\mathbb{M}_\alpha$.*

Proof. Let $\varphi \in \mathbb{M}_\alpha$. Continuity of A in $\varphi \in \mathbb{M}_\alpha$ follows from Lemma 2.3. Now, by (18), (21), and (25) we have $|(A\varphi)(t)| < \alpha$. Thus, from (20), $A\varphi \in \mathbb{M}_\alpha$. Let $\varphi_i \in \mathbb{M}_\alpha,\ i = 1, 2, ...$ Then from the above discussion we conclude that

$$||A\varphi_j|| \le \alpha.$$

This shows $A(\mathbb{M}_\alpha)$ is uniformly bounded. It is left to show that $A(\mathbb{M}_\alpha)$ is equicontinuous. Since ξ is continuous and T-periodic, by (21) and differentiation of (18) with respect to $t \in \mathbb{T}_1$ (for the differentiation rule see [1, Lemma 1]) we arrive at

$$\begin{aligned} \left|(A\varphi_j)^\Delta(t)\right| &= |G(t, \varphi_j(\delta(t))) - a(t)(A\varphi_i)(t)| \\ &\le \xi(t) + |a(t)|\ |(A\varphi_i)(t)| \\ &\le \xi(t) + ||a||\ ||A\varphi_i|| \le L, \text{ for } t \in [0, T]_{\mathbb{T}_1}\,, \end{aligned}$$

where L is a positive constant. Thus, the estimation on $|(A\varphi_i)^\Delta(t)|$ and (9) imply that $A(\mathbb{M}_\alpha)$ is equicontinuous. Then the Arzela–Ascoli theorem yields compactness of the mapping A. The proof is complete.

$\mathbb{T}_2$ is closed under addition and so (20) implies

$$(A\phi + B\varphi)(t) \in \mathbb{T}_2 \text{ for all } t \in \mathbb{T}_1 \text{ and } \phi, \varphi \in \mathbb{M}_\alpha. \tag{26}$$

Theorem 2.1 *Suppose all assumptions of Lemma 2.4. If B is a large contraction on $\mathbb{M}_\alpha$, then (4) has a periodic solution in $\mathbb{M}_\alpha$.*

Proof. Let A and B be defined by (18) and (19), respectively. By Lemma 2.4, the mapping A is compact and continuous. Then using (25), (26), and the periodicity of A and B, we have

$$A\varphi + B\psi : \mathbb{M}_\alpha \to \mathbb{M}_\alpha \quad \text{for} \quad \varphi, \psi \in \mathbb{M}_\alpha.$$

Hence an application of Theorem 1.1 implies the existence of a periodic solution in $\mathbb{M}_\alpha$. This completes the proof.

The next result gives a relationship between the mappings H and B in the sense of large contraction.

Lemma 2.5 *Let a be a positive valued function. If H is a large contraction on $\mathbb{M}_\alpha$, then so is the mapping B.*

Proof. If H is a large contraction on $\mathbb{M}_\alpha$, then for $x, y \in \mathbb{M}_\alpha$, with $x \neq y$, we have $||Hx - Hy|| \leq ||x - y||$. Since $\gamma = -a \in \mathcal{R}^+$ and a is positive valued, $\gamma(t) < 0$ for all $t \in \mathbb{T}$. Thus, it follows from the equality

$$a(s)e_\gamma\left(t+T, \sigma\left(s\right)\right) = \left[e_\gamma\left(t+T, s\right)\right]^{\Delta_s},$$

where Δ_s indicates the delta derivative with respect to s, and (16), that

$$\begin{aligned} |Bx(t) - By(t)| &\leq \int_t^{t+T} \frac{e_\gamma\left(t+T, \sigma\left(s\right)\right)}{1 - e_\gamma\left(t, t+T\right)} a(s)|H(x)(s) - H(y)(s)|\Delta s \\ &\leq \frac{||x-y||}{1 - e_\gamma\left(t+T, t\right)} \int_t^{t+T} a(s)e_\gamma\left(t+T, \sigma\left(s\right)\right)\Delta s \\ &= ||x - y||. \end{aligned}$$

Taking the supremum over the set $[0, T] \cap \mathbb{T}_1$, we get that $\|Bx - By\| \leq ||x - y||$. One may also show in a similar way that

$$\|Bx - By\| \leq \delta||x - y||$$

holds if we know the existence of a $0 < \delta < 1$ such that for all $\varepsilon > 0$

$$[x, y \in \mathbb{M}_\alpha, ||x - y|| \geq \varepsilon] \Rightarrow ||Hx - Hy|| \leq \delta||x - y||.$$

The proof is complete.

From Theorem 2.1 and Lemma 2.5, we deduce the following result.

Corollary 2.2 *In addition to the assumptions of Theorem 2.1, suppose also that a is a positive valued function. If H is a large contraction on $\mathbb{M}_\alpha$, then (4) has a periodic solution in $\mathbb{M}_\alpha$.*

3 Classification and applications

We derive the next result by making use of Theorem 1.3.

Lemma 3.1 *Suppose $g : \mathbb{T} \to \mathbb{R}$ is pre-differentiable with D. Suppose U is a compact interval with endpoints $r, s \in \mathbb{T}$ and $g^\Delta(t) \geq 0$ for all $t \in U^\kappa \cap D$. Then we have*

$$g(s) - g(r) \geq |s - r| \left\{ \inf_{t \in U^\kappa \cap D} g^\Delta(t) \right\}. \tag{27}$$

Proof. Let the function $f : \mathbb{T} \to \mathbb{R}$ be defined by

$$f(t) = (t - r) \left\{ \inf_{t \in U^\kappa \cap D} g^\Delta(t) \right\} \text{ for } t \in \mathbb{T}.$$

Evidently, f is pre-differentiable with D and

$$\left| f^\Delta(t) \right| = f^\Delta(t) = \left\{ \inf_{t \in U^\kappa \cap D} g^\Delta(t) \right\} \leq g^\Delta(t).$$

From (8), we derive

$$g(s) - g(r) \geq |f(s) - f(r)| = |s - r| \left\{ \inf_{t \in U^\kappa \cap D} g^\Delta(t) \right\}$$

as desired. The proof is complete.

Corollary 3.1 *Suppose $g : \mathbb{T} \to \mathbb{R}$ is pre-differentiable with D. Suppose U is a compact interval with endpoints $r, s \in \mathbb{T}$. $g^\Delta(t) \geq 0$ for all $t \in U^\kappa \cap D$ if and only if g is non-decreasing on U.*

Proof. If $g^\Delta(t) \geq 0$ for all $t \in U^\kappa \cap D$, then from (27), we have

$$g(s) - g(r) \geq (s - r) \left\{ \inf_{t \in U^\kappa \cap D} g^\Delta(t) \right\} \geq 0$$

for $s, r \in U$ with $s \geq r$. Conversely, let g be non-decreasing on U. For a $t \in U^\kappa \cap D$, there are two possible cases:

$$\mu(t) = 0 \text{ or } \mu(t) > 0.$$

If $\mu(t) = \sigma(t) - t > 0$, then by [4, Theorem 1.16, (ii)] we have

$$g^\Delta(t) = \frac{g(\sigma(t)) - g(t)}{\mu(t)} > 0.$$

If $\mu(t) = 0$, then from [4, Theorem 1.16, (iii), Exercise 1.17] we find

$$g^{\Delta}(t) = \lim_{s \to t} \frac{g(t) - g(s)}{s - t} \geq 0.$$

This completes the proof.

Corollary 2.2 shows that having a large contraction on a class of periodic functions plays a substantial role in proving existence of periodic solutions. We deduce by the next theorem that

H.1. $h : \mathbb{T}_2 \to \mathbb{R}$ is continuous on U_α and differentiable on U_α^κ,

H.2. h is strictly increasing on U_α,

H.3. $\sup\limits_{s \in U_\alpha^\kappa} h^{\Delta}(s) \leq 1$

are the conditions implying that the mapping H in (13) is a large contraction on the set $\mathbb{M}_\alpha$.

Theorem 3.1 *Let $h : \mathbb{T}_2 \to \mathbb{T}_2$ be a function satisfying (H.1-H.3). Then the mapping H is a large contraction on the set $\mathbb{M}_\alpha$.*

Proof. It is obvious that the function h satisfies the assumptions of Lemma 3.1 on the compact interval $U_\alpha = [-\alpha, \alpha] \cap \mathbb{T}_2$. Thus, (9) and (27) give

$$(s - r) \left\{ \sup_{t \in U_\alpha^\kappa} h^{\Delta}(t) \right\} \geq h(s) - h(r) \geq (s - r) \left\{ \inf_{t \in U_\alpha^\kappa} h^{\Delta}(t) \right\} \geq 0 \tag{28}$$

for $s, r \in U_\alpha$ with $s \geq r$. Let $\phi, \varphi \in \mathbb{M}_\alpha$ with $\phi \neq \varphi$. Then $\phi(t) \neq \varphi(t)$ for some $t \in \mathbb{T}$. Let us introduce the set

$$D(\phi, \varphi) = \{t \in \mathbb{T}_2 : \phi(t) \neq \varphi(t)\}.$$

Note that $\varphi(t) \in U_\alpha$ for all $t \in \mathbb{T}_1$ whenever $\varphi \in \mathbb{M}_\alpha$. Since h is strictly increasing

$$\frac{h(\varphi(t)) - h(\phi(t))}{\varphi(t) - \phi(t)} = \frac{h(\phi(t)) - h(\varphi(t))}{\phi(t) - \varphi(t)} > 0 \tag{29}$$

holds for all $t \in D(\phi, \varphi)$. By (H.3), we have

$$1 \geq \sup_{t \in U_\alpha^\kappa} h^{\Delta}(t) \geq \inf_{s \in U_\alpha^\kappa} h^{\Delta}(s) \geq 0. \tag{30}$$

Define the set $U_t \subset U_\alpha$ by

$$U_t = \begin{cases} [\varphi(t), \phi(t)] \cap U_\alpha, & \text{if } \phi(t) > \varphi(t), \\ [\phi(t), \varphi(t)] \cap U_\alpha, & \text{if } \phi(t) < \varphi(t), \end{cases} \quad \text{for } t \in D(\phi, \varphi).$$

Hence, for a fixed $t_0 \in D(\phi, \varphi)$ we get by (28) and (29) that

$$\sup\left\{h^{\Delta}(u) : u \in U_{t_0}^{\kappa}\right\} \geq \frac{h(\phi(t_0)) - h(\varphi(t_0))}{\phi(t_0) - \varphi(t_0)} \geq \inf\left\{h^{\Delta}(u) : u \in U_{t_0}^{\kappa}\right\}.$$

Since $U_t \subset U_\alpha$ for every $t \in D(\phi, \varphi)$, we find

$$\sup_{u \in U_\alpha^\kappa} h^{\Delta}(u) \geq \sup\left\{h^{\Delta}(u) : u \in U_{t_0}^{\kappa}\right\} \geq \inf\left\{h^{\Delta}(u) : u \in U_{t_0}^{\kappa}\right\} \geq \inf_{u \in U_\alpha^\kappa} h^{\Delta}(u),$$

and therefore,

$$1 \geq \sup_{u \in U_\alpha^\kappa} h^{\Delta}(u) \geq \frac{h(\varphi(t)) - h(\phi(t))}{\varphi(t) - \phi(t)} \geq \inf_{u \in U_\alpha^\kappa} h^{\Delta}(u) \geq 0 \tag{31}$$

for all $t \in D(\phi, \varphi)$. So, (31) yields

$$\begin{aligned} |H\phi(t) - H\varphi(t)| &= |\phi(t) - h(\phi(t)) - \varphi(t) + h(\varphi(t))| \\ &= |\phi(t) - \varphi(t)| \left| 1 - \left(\frac{h(\phi(t)) - h(\varphi(t))}{\phi(t) - \varphi(t)} \right) \right| \\ &\leq |\phi(t) - \varphi(t)| \left(1 - \inf_{u \in U_\alpha^\kappa} h^{\Delta}(u) \right) \end{aligned} \tag{32}$$

for all $t \in D(\phi, \varphi)$. Thus, (31) and (32) imply that H a large contraction in the supremum norm.

To see this, choose a fixed $\varepsilon \in (0,1)$ and assume that ϕ and φ are two functions in $\mathbb{M}_\alpha$ satisfying

$$\|\phi - \varphi\| = \sup_{t \in [-a, \alpha]} |\phi(t) - \varphi(t)| \geq \varepsilon.$$

If $|\phi(t) - \varphi(t)| \leq \frac{\varepsilon}{2}$ for some $t \in D(\phi, \varphi)$, then from (32)

$$|H(\phi(t)) - H(\varphi(t))| \leq |\phi(t) - \varphi(t)| \leq \frac{1}{2} \|\phi - \varphi\|. \tag{33}$$

Since h is continuous and strictly increasing, the function $h\left(u + \frac{\varepsilon}{2}\right) - h(u)$ attains its minimum on the closed and bounded interval $[-\alpha, \alpha]$. Thus, if $\frac{\varepsilon}{2} < |\phi(t) - \varphi(t)|$ for some $t \in D(\phi, \varphi)$, then from (31) and (H.3) we conclude that

$$1 \geq \frac{h(\phi(t)) - h(\varphi(t))}{\phi(t) - \varphi(t)} > \lambda,$$

and therefore,

$$\begin{aligned} |H\phi(t) - H\varphi(t)| &\leq |\phi(t) - \varphi(t)| \left\{ 1 - \left(\frac{h(\phi(t)) - h(\varphi(t))}{\phi(t) - \varphi(t)} \right) \right\} \\ &\leq (1 - \lambda) \|\phi(t) - \varphi(t)\|, \end{aligned} \tag{34}$$

where

$$\lambda := \frac{1}{2\alpha}\min\left\{h\left(u+\frac{\varepsilon}{2}\right)-h(u) : u\in[-\alpha,\alpha]\right\}>0.$$

Consequently, it follows from (33) and (34) that

$$|H\phi(t)-H\varphi(t)|\le\delta\,\|\phi-\varphi\|,$$

where

$$\delta=\max\left\{\frac{1}{2},1-\lambda\right\}<1.$$

The proof is complete.

If $\mathbb{T}_2$ is a time scale such that the interval $U_\alpha=[-\alpha,\alpha]\cap\mathbb{T}_2$ contains negative reals, then functions of type $h_1(t)=\frac{1}{2n(\alpha+1)}t^{2n}$, $n\in\mathbb{N}$, do not satisfy (H.2) since h_1 is decreasing on $U_\alpha^-=[-\alpha,0]\cap\mathbb{T}_2$. But functions of type $h_2(t)=\frac{1}{(2n+1)(\alpha+1)}t^{2n+1}$, $n\in\mathbb{Z}_+$, obey the conditions (H.1-H.3). To show that (H.3) is satisfied for h_2 we need to calculate $[t^{2n}]^\Delta$. For the particular time scale $\mathbb{T}=\mathbb{R}$, it is easy to see by the chain rule,

$$\left\{f^{2n+1}(t)\right\}^\Delta=(2n+1)f^{2n}(t)f'(t),$$

that

$$h_2^\Delta(t)=\left(\frac{t}{\alpha+1}\right)^{2n}\le\left(\frac{\alpha}{\alpha+1}\right)^{2n}<1 \text{ for } t\in U_\alpha.$$

But for an arbitrary time scale $\mathbb{T}$, it is not that easy since the rule for $\{f^{n+1}(t)\}^\Delta$ is changed to

$$\left\{f^{2n+1}(t)\right\}^\Delta=\left\{\sum_{k=0}^{2n}f(t)^k f(\sigma(t))^{2n-k}\right\}f^\Delta(t). \tag{35}$$

(see [4, Exercise 1.23]). Throughout the discussion below we shall always assume that $f:\mathbb{T}\to\mathbb{R}$ is a nondecreasing differentiable function. If $f(t)f(\sigma(t))=0$ for some $t\in\mathbb{T}$, then (35) implies

$$\left[f(t)^{2n+1}\right]^\Delta=\left\{f(t)^{2n}+f(\sigma(t))^{2n}\right\}f^\Delta(t),$$

and therefore,

$$(2n+1)f(t)^{2n}f^\Delta(t)\le\left[f^{2n+1}(t)\right]^\Delta\le(2n+1)f(\sigma(t))^{2n}f^\Delta(t),\quad n\in\mathbb{Z}_+. \tag{36}$$

On the other hand, If $f(t)f(\sigma(t))\ne0$, there are three possibilities: (i) $0<f(t)\le f(\sigma(t))$, (ii) $f(t)\le f(\sigma(t))<0$, and (iii) $f(t)<0<f(\sigma(t))$. Let us separate these cases by defining the sets

$$\begin{aligned}S_+&=\{t\in D:0<f(t)\le f(\sigma(t))\},\\ S_-&=\{t\in D:f(t)\le f(\sigma(t))<0\},\\ S_0&=\{t\in D:f(t)<0<f(\sigma(t))\}.\end{aligned}$$

The next lemma gives the relationship between $[f^{2n+1}(t)]^\Delta$ and $(2n+1)f^{2n}(t)f^\Delta(t)$.

Lemma 3.2 *Let $f : \mathbb{T} \to \mathbb{R}$ be a differentiable function in D. If f is a non decreasing function, then*

$$(2n+1)f^{2n}(t)f^{\Delta}(t) \leq \left[f^{2n+1}(t)\right]^{\Delta} \leq (2n+1)f^{2n}(\sigma(t))f^{\Delta}(t) \tag{37}$$

for $t \in S_+$,

$$(2n+1)f^{2n}(\sigma(t))f^{\Delta}(t) \leq \left[f^{2n+1}(t)\right]^{\Delta} \leq (2n+1)f^{2n}(t)f^{\Delta}(t) \tag{38}$$

for $t \in S_-$, and

$$f(t)^{2n}f^{\Delta}(t) < \left\{f^{2n+1}(t)\right\}^{\Delta} < (n+1)f(\sigma(t))^{2n}f^{\Delta}(t) \tag{39}$$

for $t \in S_0$, where $n = 1, 2, ...$

Proof. We use the formula (35). (37) follows from (35) and the fact that

$$f^{2n}(t) \leq f(t)^k f(\sigma(t))^{2n-k} \leq f^{2n}(\sigma(t)) \text{ for } t \in S_+.$$

On the other hand, for all $t \in S_-$ we have

$$f^{2n}(t) = f(t)^{2s+1}f(t)^{2n-2s-1} \geq f(t)^{2s+1}f(\sigma(t))^{2n-2s-1}$$

and

$$f^{2n}(t) = f(t)^{2s}f(t)^{2n-2s} \geq f(t)^{2s}f(\sigma(t))^{2n-2s}$$

which imply

$$\begin{aligned}
\left\{f^{2n+1}(t)\right\}^{\Delta} &= \left\{\sum_{k=0}^{2n} f(t)^k f(\sigma(t))^{2n-k}\right\} f^{\Delta}(t) \\
&= \left\{\sum_{s=0}^{n} f(t)^{2s}f(\sigma(t))^{2n-2s} + \sum_{s=0}^{n-1} f(t)^{2s+1}f(\sigma(t))^{2n-2s-1}\right\} f^{\Delta}(t) \\
&\leq (2n+1)f^{2n}(t)f^{\Delta}(t).
\end{aligned}$$

Similarly, for $t \in S_-$

$$f^{2n}(\sigma(t)) = f(\sigma(t))^{2s+1}f(\sigma(t))^{2n-2s-1} \leq f(t)^{2s+1}f(\sigma(t))^{2n-2s-1}$$

and

$$f^{2n}(\sigma(t)) = f(\sigma(t))^{2s}f(\sigma(t))^{2n-2s} \leq f(t)^{2s}f(\sigma(t))^{2n-2s}.$$

So, we get that

$$\left\{f^{2n+1}(t)\right\}^{\Delta} \geq (2n+1)f^{2n}(\sigma(t))f^{\Delta}(t)$$

for all $t \in S_-$. If $t \in S_0$, we have

$$-f(\sigma(t)) < f(t) < 0 < f(\sigma(t)),$$

and hence,

$$-1 < \frac{f(t)}{f(\sigma(t))} < 0. \tag{40}$$

Since

$$\begin{aligned}\left\{f^{2n+1}(t)\right\}^{\Delta} &= \left\{\sum_{s=0}^{n} f(t)^{2s} f(\sigma(t))^{2n-2s} + \frac{f(t)}{f(\sigma(t))}\sum_{s=0}^{n-1} f(t)^{2s} f^{2n-2s}(\sigma(t))\right\} f^{\Delta}(t) \\ &= \left\{\sum_{s=0}^{n} f(t)^{2s} f(\sigma(t))^{2n-2s}\left[1+\frac{f(t)}{f(\sigma(t))}\right] - \frac{f(t)}{f(\sigma(t))} f(t)^{2n}\right\} f^{\Delta}(t)\end{aligned}$$

and

$$(n+1)f(t)^{2n} < \sum_{s=0}^{n} f(t)^{2s} f(\sigma(t))^{2n-2s} < (n+1)f(\sigma(t))^{2n},$$

we obtain

$$f(t)^{2n} f^{\Delta}(t) < \left\{f^{2n+1}(t)\right\}^{\Delta} < (n+1)f(\sigma(t))^{2n} f^{\Delta}(t).$$

The proof is complete.

From (36)-(39) we derive the following result.

Corollary 3.2 *Let $U = [a, b] \cap \mathbb{T}$ be an arbitrary interval. Suppose all assumptions of Lemma 3.2. If*

$$\sup_{t\in U^{\kappa}} \left|f^{\Delta}(t)\right| \le 1 \tag{41}$$

holds and there exists a positive integer $n \in \mathbb{Z}_+$ such that

$$\sup_{t\in U} |f(t)| \le (2n+1)^{-1/2n}, \tag{42}$$

then we have

$$0 \le \sup_{t\in U^{\kappa}} \left\{\left[f^{2n+1}(t)\right]^{\Delta}\right\} \le 1. \tag{43}$$

Proof. If f is nondecreasing, then so is f^{2n+1}. Thus, $[f^{2n+1}(t)]^{\Delta} \ge 0$ for all $t \in \mathbb{T}^{\kappa}$. From (36)-(39) we find

$$0 \le \left[f^{2n+1}(t)\right]^{\Delta} \le (2n+1)\zeta(t) f^{\Delta}(t), \tag{44}$$

where

$$\zeta(t) = \begin{cases} f^{2n}(\sigma(t)) & \text{for } t \in S_+ \cup S_0 \\ f^{2n}(t) & \text{for } t \in S_- \end{cases}.$$

Since $\sigma(t) \in U$ for all $t \in U^{\kappa}$, taking the supremum over the set U^{κ} we get by (44) that

$$0 \le \sup_{t\in U^{\kappa}} \left[f^{2n+1}(t)\right]^{\Delta} \le \frac{2n+1}{2n+1} = 1.$$

The proof is complete.

Corollary 3.3 *Let $f : \mathbb{T}_2 \to \mathbb{T}_2$ be a strictly increasing function satisfying (41) and (42). If the function $h : \mathbb{T}_2 \to \mathbb{T}_2$ be defined by*

$$h(u) = f^{2n+1}(u) \text{ for } n \in \mathbb{Z}_+, \tag{45}$$

then the mapping $Hx = x - h \circ x$ defines a large contraction on the set

$$\mathbb{M}_{\beta(n)} = \{\phi \in P(\mathbb{T}_1, \mathbb{T}_2) : \|\phi\| \le \beta(n)\}, \tag{46}$$

where

$$\beta(n) := (2n+1)^{-1/2n}. \tag{47}$$

Proof. We proceed by Theorem 3.1. It is obvious that (H.1) and (H.2) hold whenever the function h is defined as in (45). Since f satisfies all assumptions of Corollary 3.2, we get by (43) that

$$\sup_{u \in U_\alpha^\kappa} \left| h^\Delta(u) \right| \le 1.$$

From Theorem 3.1 the proof is completed.

In [6], it was shown by an example that the function $Hx(t) = x(t) - x^3(t)$ defines a large contraction on the set

$$\mathcal{M}_{1/\sqrt{3}} = \left\{ x \in P(\mathbb{R}_+, \mathbb{R}) : \sup_{t \in \mathbb{R}_+} |x(t)| \le 3^{-1/2} \right\}.$$

Also, in [2], the authors constructed a large contraction $Hx(t) = x(t) - x^5(t)$ on the set

$$\mathcal{M}_{1/\sqrt[4]{5}} = \left\{ x \in P(\mathbb{R}, \mathbb{R}) : \sup_{t \in \mathbb{R}_+} |x(t)| \le 5^{-1/4} \right\}.$$

Evidently, these results can be derived from Corollary 3.3 by choosing the time scales $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{R}$, and taking $n = 1$ and $n = 3$ in (47), respectively.

Theorem 3.2 *Let $\mathbb{T}_1$ be a periodic time scale. For a fixed $n \in \mathbb{Z}_+$, set*

$$G(u, x(\delta(u))) = b(u) x^{2n+1}(\delta(t)) + c(u)$$

and

$$h(u) = u^{2n+1} \quad \text{for } u \in \mathbb{T}_2,$$

and define the mappings A and B as in (18) and (19), respectively. Suppose that the time scale $\mathbb{T}_2$ is closed under addition and that (20) and $U_{\beta(n)} = [-\beta(n), \beta(n)] \cap \mathbb{T}_2 \neq \varnothing, n \in \mathbb{Z}_+$, hold. Define the set $\mathbb{M}_{\beta(n)}$ and the function h by (46) and (45), respectively. If a is a positive valued T periodic function and $c \neq 0 \in P_T(\mathbb{T}_1, \mathbb{R})$, then

$$2n\beta^{2n+1}(n) + \frac{1}{1 - e_\gamma(t+T, t)} \int_t^{t+T} \xi(s) e_\gamma(t+T, \sigma(s)) \Delta s \le \beta(n), \tag{48}$$

implies the existence of non-zero periodic solution $x \in \mathbb{M}_{\beta(n)}$ of the equation

$$x^\Delta(t) = -a(t) x^{2n+1}(t) + b(t) x^{2n+1}(\delta(t)) + c(t).$$

Proof. First, it follows from Corollary 3.3 that the mapping H given by (13) is a large contraction on $\mathbb{M}_{\beta(n)}$. Also for $x \in \mathbb{M}_{\beta(n)}$, we have

$$|x(t)|^{2n+1} \leq \beta^{2n+1}(n),$$

and therefore,

$$G(u, x(\delta(u))) \leq \beta^{2n+1}(n)|b(u)| + |c(u)| := \xi(t); \tag{49}$$

i.e., (21) holds. Using standard techniques of calculus one may verify that

$$\begin{aligned} |H(x(t))| &= \left|x(t) - x^{2n+1}(t)\right| \\ &\leq 2n(2n+1)^{-(2n+1)/2n} = 2n\beta^{2n+1}(n). \end{aligned}$$

Since $a(t) > 0$ and for all $x \in \mathbb{M}_\alpha$, we get by (48) that

$$\begin{aligned} J &\leq \frac{2n\beta^{2n+1}(n)}{1 - e_\gamma(t+T, t)} \int_t^{t+T} a(s) e_\gamma(t+T, \sigma(s)) \Delta s \\ &+ \frac{1}{1 - e_\gamma(t+T, t)} \int_t^{t+T} \xi(s) e_\gamma(t+T, \sigma(s)) \Delta s \\ &= 2n\beta^{2n+1}(n) + \frac{1}{1 - e_\gamma(t+T, t)} \int_t^{t+T} \xi(s) e_\gamma(t+T, \sigma(s)) \Delta s \leq \beta(n). \end{aligned}$$

Thus, (25) is satisfied. The proof is completed by making use of Corollary 2.2.

Note that in [2] it has been shown that

$$4(5^{-5/4}) + \eta \int_t^{t+T} \left(5^{-5/4}|b(u)| + |c(u)|\right) e^{-\int_u^{t+T} a(s)ds} du \leq 5^{-1/4} \tag{50}$$

is the condition guaranteeing that the totally nonlinear delay differential equation

$$x'(t) = -a(t)x(t)^5 + b(t)x(t - r(t))^5 + c(t) \tag{51}$$

has a T-periodic solution in $\mathcal{M}_{1/\sqrt[4]{5}}$, where $a(t) > 0$ for all $t \in \mathbb{R}$,

$$\eta := \left|(1 - e^{-\int_0^T a(s)ds})^{-1}\right|,$$

and $c \neq 0 \in P_T(\mathbb{R}, \mathbb{R})$. Observe that Theorem 3.2 not only contains this result but also offers a sufficient condition for positivity of existing periodic solution.

Remark 3.1 *Despite the fact that Theorem 3.2 can be used to obtain existence of positive valued non-zero periodic solutions (to see this choose $\mathbb{T}_2$ as a time scale consisting of positive real numbers) of a difference equation of type (1), it cannot be employed to conclude the existence of integer valued solutions of the difference equation (1) since $U_{\beta(n)} = \varnothing$ whenever $\mathbb{T}_2 = \mathbb{Z}$. To get over this difficulty we choose α to be a sufficiently large positive real number such that $U_\alpha = [-\alpha, \alpha] \cap \mathbb{T}_2 \neq \varnothing$ and define the set $\mathbb{M}_\alpha$ as in (11).*

Similar to Corollary 3.2 and Corollary 3.3 one may prove the following results:

Corollary 3.4 *In addition to all assumptions of Lemma 3.2 suppose also that*

$$\sup_{t\in U^\kappa}\left|f^\Delta(t)\right|\le 1$$

for $U=[a,b]\cap\mathbb{T}$. *If there exists a constant* $\beta>0$ *such that* $\sup_{t\in U^\kappa}|f(t)|\le\beta$, *then*

$$0\le\sup_{t\in U^\kappa}\left\{\left[\frac{f^{2n+1}(t)}{(2n+1)(\beta+1)^{2n}}\right]^\Delta\right\}\le 1,\ n=1,2,....$$

Example 3.1 *Let* $h:\mathbb{T}_2\to\mathbb{R}$ *be defined by*

$$h(s)=\frac{s^{2n+1}}{(2n+1)(\alpha+1)^{2n}},\quad \text{for } s\in\mathbb{T}_2,\ n=1,2,.... \tag{52}$$

Then the mapping H *defined by (13) defines a large contraction on the set* $\mathbb{M}_\alpha$.

By choosing $\mathbb{T}_2=\mathbb{Z}$ in the next result one obtains sufficient conditions for the existence of integer valued periodic solution of the nonlinear difference equation (1).

Corollary 3.5 *Let* $\mathbb{T}_1$ *be a periodic time scale. For a fixed* $n\in\mathbb{Z}_+$ *define the function* h *by (52) and set*

$$G(t,x(\delta(t)))=\frac{b(t)}{(2n+1)(\alpha+1)^{2n}}x^{2n+1}(\delta(t))+c(t).$$

Let the mappings A *and* B *be given by (18) and (19), respectively. Suppose that the time scale* $\mathbb{T}_2$ *is closed under addition and that the condition (20) holds. If* a *is positive valued* T *periodic function and* $c\neq 0\in P_T(\mathbb{T}_1,\mathbb{R})$, *then*

$$(\alpha+1)^{\frac{1}{2n}}\left(1-\frac{1}{2n+1}\right)+\frac{1}{1-e_\gamma(t+T,t)}\int_t^{t+T}\xi(s)e_\gamma(t+T,\sigma(s))\,\Delta s\le\alpha,$$

implies the existence of a non-zero periodic solution $x\in\mathbb{M}_\alpha$ *of the equations*

$$x^\Delta(t)=-\frac{a(t)}{(2n+1)(\alpha+1)^{2n}}x^{2n+1}(t)+\frac{b(t)}{(2n+1)(\alpha+1)^{2n}}x^{2n+1}(\delta(t))+c(t).$$

It is worth mentioning that Theorem 3.5 is given for arbitrary time scales $\mathbb{T}_1$ and $\mathbb{T}_2$, where $\mathbb{T}_1$ is assumed to be periodic and $\mathbb{T}_2$ is a time scale, closed under addition, such that (20) holds. One may apply this theorem to the particular time scales including the following cases:

1. $\mathbb{T}_1=\mathbb{T}_2=\mathbb{R}$;

2. $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{Z}$;
3. $\mathbb{T}_1 = \mathbb{Z}$ and $\mathbb{T}_2 = \mathbb{R}$ (or $h\mathbb{Z}$);
4. $\mathbb{T}_1 = \{t = k - q^m : k \in \mathbb{Z},\ m \in \mathbb{N}_0\}$ and $\mathbb{T}_2 = \mathbb{R}$ (or $\mathbb{Z}$).

Note that the first case has been handled by [2] in which some sufficient conditions are offered for the existence of periodic solutions of totally nonlinear differential equations

$$x'(t) = -a(t)h(x(t)) + G(t, x(t - r(t))), \quad t \in \mathbb{R}.$$

However, the acquired results in this paper are not known for the last three particular cases.

References

[1] M. Adıvar and Y. N. Raffoul, Existence results for periodic solutions of integro-dynamic equations on time scales, *Annali Di Matematica Pura ed Applicata*, DOI 10.1007/s10231-008-0088-z (to appear).

[2] M. Adıvar, M. N. Islam, and Y.N. Raffoul, Existence of periodic and positive periodic solutions in totally nonlinear delay differential equations (preprint).

[3] L. Bi, M. Bohner, and M. Fan, Periodic solutions of functional dynamic equations with infinite delay, *Nonlinear Anal.* **68** (2008), 1226–1245.

[4] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales, An Introduction with Applications*, Birkhäuser Boston Inc., Boston, MA, 2001.

[5] M. Bohner and G. Sh. Guseinov, Riemann and Lebesgue integration, *Advances in Dynamic Equations on Time Scales*, 117–163, Birkhäuser Boston Inc., Boston, MA, 2003.

[6] T. A. Burton, *Stability by Fixed Point Theory for Functional Differential Equations*, Dover, New York, 2006.

[7] E. R. Kaufmann and Y. N. Raffoul, Periodicity and stability in neutral nonlinear dynamic equations with functional delay on a time scale, *Electron. J. Differential Equations*, (2007), No. 27, 12 pp. (electronic).

[8] J. Popenda and E. Schmeidel, On the asymptotic behaviour of solutions of integer valued difference equations, *Proceedings of the Second World Congress of Nonlinear Analysts, Part 2 (Athens, 1996), Nonlinear Anal.* **30** (1997), 1119–1124.

[9] V. G. Tkachev, Algebraic structure of quasiradial solutions to the γ-harmonic equation, *Pacific J. Math.* **226** (2006), 179–200.