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ON NONCONTINUABLE SOLUTIONS OF DIFFERENTIAL EQUATIONS WITH DELAY

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Honoring the Career of John Graef on the Occasion of His Sixty-Seventh Birthday

Abstract

The author considers the n-th order nonlinear differential equation with delays. He presents sufficient conditions for this equation to have (not to have) noncontinuable solutions. The Cauchy problem and a boundary value problem are investigated.

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1 Introduction

In the paper we study the problem of continuability of solutions of the n-th order differential equation with delays of the form

$$y^{(n)} = f\big(t, y(\tau_0), \dots, y^{(n-1)}(\tau_{n-1})\big) \tag{1}$$

where $n \geq 2$, f is a continuous function defined on $\mathbb{R}_+ \times \mathbb{R}^n$, $\mathbb{R}_+ = [0, \infty)$, $\mathbb{R} = (-\infty, \infty)$, $\tau_i \in C^0(\mathbb{R}_+)$ and

$$\tau_i(t) \leq t \quad \text{for} \quad t \in \mathbb{R}_+ \quad \text{and} \quad i = 0, 1, \dots, n-1\,.$$

We will suppose for the simplicity that $\inf\limits_{t \in \mathbb{R}_+} \tau_i(t) > -\infty$ for $i = 0, 1, \dots, n-1$. Note, that $C^s(I)$, $s \in \{0, 1, \dots\}$, $I \subset \mathbb{R}_+$ is the set of continuous functions on I that have continuous derivatives up to the order s.

A special case of equation (1) is the equation without delays,

$$y^{(n)} = f(t, y, y', \dots, y^{(n-1)})\,. \tag{2}$$

Definition 1.1. *Let $T \in (0, \infty]$, $T_1 \in [0, T)$, $\sigma_i = \inf_{T_1 \le t < T} \tau_i(t)$ and $\phi_i \in C^0[\sigma_i, T_1]$ for $i = 0, 1, \dots, n-1$. It is said that a function y is a solution of* (1) *on $[T_1, T)$ (with the initial conditions $\{\phi_i\}_{i=0}^{n-1}$) if $y \in C^{(n)}[T_1, T)$,* (1) *holds on $[T_1, T)$, $y^{(i)}(t) = \phi_i(t)$ on $[\sigma_i, T_1]$ and $y^{(i)} \in C^0[\sigma_i, T)$ for $i = 0, 1, \dots, n-1$.*

We will study solutions on their maximal interval of existence to the right.

Definition 1.2. *Let y be defined on $[T_1, T) \subset [0, \infty)$. Then y is called noncontinuable if $T < \infty$ and $\sup_{T_1 \le t < T} |y^{(n-1)}(t)| = \infty$ (i.e. if y cannot be defined at $t = T$). A solution y is defined to be continuable if $T = \infty$.*

Definition 1.3. *Let y be a noncontinuable solution of* (1) *on $[T_1, T)$. y is called oscillatory if there exists a sequence of its zeros tending to T. Otherwise, y is called nonoscillatory.*

In the two last decades the existence and properties of noncontinuable solutions are investigated mainly for equation (2). It is important to study the existence/nonexistence of such solutions. Properties and different types of continuable solutions of (2) are studied intensively now; it is necessary to know if solutions are continuable or not. Furthermore, noncontinuable solutions appear e.g. in water flow problems in one space dimension, see e.g. [14] (flood waves, a flow in sewerage systems).

Sometimes, the noncontinuability is very important in a definition of some problems. For example, the limit-circle/limit-point problem for (2) has an old history, see e.g. the monograph [7]. The first mathematicians who studied it for the nonlinear equation (2) (with $n = 2$) were Spikes and Graef [18, 19, 28].

Definition 1.4. *Let $\alpha \in \{-1, 1\}$ and $\alpha f(t, x_0, \dots, x_{n-1}) x_0 \ge 0$ on $\mathbb{R}_+ \times \mathbb{R}^n$. Equation* (2) *is said to be of the nonlinear limit-circle type if for any solution y defined on $\mathbb{R}_+$,*

$$\int_0^\infty y(t)\, f(t, y(t), \dots, y^{(n-1)}(t))\, dt < \infty$$

holds. Equation (2) *is said to be of the nonlinear limit-point type if there exists a solution y of* (2) *defined on $\mathbb{R}_+$ such that*

$$\int_0^\infty y(t)\, f(t, y(t), \dots, y^{(n-1)}(t))\, dt = \infty. \tag{3}$$

According to Definition 1.4 it is necessary to know if a solution y defined on $\mathbb{R}_+$ and satisfying (3) exists. The following example is very instructive.

Example 1.1. *Consider the differential equation*

$$y'' = t^\alpha |y|^\lambda \operatorname{sgn} y \tag{4}$$

with $\lambda > 1$ and $\alpha \in \mathbb{R}$.

(i) Then $\varepsilon > 0$ exists such that every solution y of (4) *with Cauchy initial conditions $|y(0)| \le \varepsilon$, $|y'(0)| \le \varepsilon$ is continuable if and only if $\alpha < -\lambda - 1$ (see [12]). Hence, if $\alpha < -\lambda - 1$ then* (4) *is of the nonlinear limit-point type.*

(ii) If $\alpha \ge -\lambda - 1$, then every solution y of (4) *satisfying $y(T)y'(T) > 0$ at some $T \in \mathbb{R}_+$ is noncontinuable (see Lemma 5 in [9]) and* (4) *is of the nonlinear limit-circle type (Theorem 4 in [9]).*

The first results for the nonexistence of noncontinuable solutions of (2) (or its special cases) are given by Wintner, see [21] or [25] ($n = 2$); other results are obtained in [3, 4, 10, 12, 13, 17, 18, 19, 23, 24]. In particular, noncontinuable solutions do not exist if either f is sublinear in neighbourhoods of $\pm\infty$ or

$$\left|f(t, x_0, \dots, x_{n-1})\right| \le r(t) \sum_{i=0}^{n-1} |x_i| \quad \text{for } t \in \mathbb{R}_+ \text{ and } |x| \text{ large}$$

with $r \in C^0(\mathbb{R}_+)$ is positive.

Hence, the existence of noncontinuable solutions may be studied mainly for superlinear equations in neighbourhoods of $\pm\infty$. The first results for (2) ($n = 2$) are obtained in [23] and they are generalized e.g. in [15, 17, 24, 25]. The typical assumptions for the existence of a nonoscillatory noncontinuable solution are $\alpha \in \{-1, 1\}$, $i \in \{0, 1, \dots, n-1\}$, $\lambda > 1$ and

$$\alpha f(t, x_0, \dots, x_{n-1}) \ge r(t)|x_i|^\lambda$$

for $\alpha x_i > 0$, $i = 0, \dots, n-1$ large enough and $r \in C^0(\mathbb{R}_+)$, $r > 0$.

Note, that noncontinuable oscillatory solutions are studied only in [11], [15] and [16].

The authors of the papers [8, 9, 10, 11, 27] stated sets of Cauchy initial conditions for which solutions are noncontinuable; the results are used mainly for solving the nonlinear limit-circle/limit-point problem.

On the other hand, in [10, 12, 26] a set of initial conditions is described for which solutions are continuable even in the superlinear case.

In the last decade, the problem of the existence of noncontinuable solutions with prescribed asymptotics on the right-hand side point T of the definition interval is studied. More precisely, let $T \in (0, \infty)$. In [1, 2, 4, 5, 6], necessary and sufficient conditions for the existence of a solution y satisfying boundary value problem

$$\begin{gathered} T \in (0, \infty)\,, \quad l \in \{-1, 0, 1, \dots, n-2\}\,, \\ C_i \in \mathbb{R} \quad \text{for} \quad i = 0, 1, \dots, l\,, \\ \lim_{t \to T-} y^{(i)}(t) = C_i \quad \text{for} \quad i = 0, 1, \dots, l\,, \\ \lim_{t \to T-} |y^{(j)}(t)| = \infty \quad \text{for} \quad j = l+1, \dots, n-1 \end{gathered} \tag{5}$$

(for $l = -1$, C_i and the first equality is missing); the solution y is defined in a left neighbourhood of T. It has to be stressed that the first authors who studied the problem (5) were Jaroš and Kusano [22] (for $n = 2$).

As concerns equation (1), see the monography [20]. In [12] the continuability of solutions of a functional-differential system is investigated; note that (1) cannot be transformed into this system.

The goal of the paper is to generalize some results given above to equation (1). The following example shows that noncontinuable solutions exist.

Example 1.2. *Consider the equation*

$$y'' = \frac{2(\lambda+1)}{(\lambda-1)^2}(1+t)^{\frac{2\lambda}{\lambda-1}}\, f\big(y(\tau(t))\big)$$

with $\lambda > 1$, $f \in C^0(\mathbb{R}_+)$, $f(x) = x^\lambda$ *on* $[0,1]$ *and* $\tau(t) = t^2$ *on* $[0,1]$, $\tau(t) = t$ *for* $t > 1$. *Then* $y(t) = (1-t)^{-\frac{2}{\lambda-1}}$, $t \in [0,1)$ *is the noncontinuable solution of the above given equation.*

2 Nonexistence Results

In this section nonexistence results for noncontinuable solutions will be given. The first theorem is very simple but important.

Theorem 2.1. *Let* $\tau_i(t) < t$ *on* $(0,\infty)$ *for* $i = 0,1,\dots,n-1$. *Then all solutions of* (1) *are continuable.*

Proof. Let, contrarily, y be a noncontinuable solution of (1) defined on $[T_1, T)$, $T > T_1$. If we define $\tau = \max\limits_{0\le i<n} \tau_i(T)$, the assumptions of the theorem yield $\tau < T$. Let $0 < \varepsilon < T - T_1$ and $\bar\tau = \min\limits_{0\le i<n} \min\limits_{t\in J} \tau_i(t)$ with $J = [T-\varepsilon, T]$. Then $\tau_i(t) \in [\bar\tau, \tau]$ for $i = 0,1,\dots,n-1$ and $t \in J$, and due to $\tau < T$ a constant N exists such that

$$\big|y^{(i)}(\tau_i(t))\big| \le N \quad \text{on} \quad [T-\varepsilon, T) \quad \text{for} \quad i = 0,1,\dots,n-1\,. \tag{6}$$

Note, that y is a solution of the equation $z^{(n)} = b(t)$ with $b(t) = f\big(t, y(\tau_0(t)),\dots, y^{(n-1)}(\tau_{n-1}(t))\big)$, $t \in [T-\varepsilon, T)$. From this and from (6), y can be defined at $t = T$; that contradicts the noncontinuability of y. $\square$

The following lemma is very useful.

Lemma 2.1 ([26] Lemma 2.1)**.** *Let* $\lambda > 1$, $K > 0$, Q *be a continuous nonnegative function on* $\mathbb{R}_+$, *and* u *be continuous and nonnegative on* $\mathbb{R}_+$ *satisfying*

$$u(t) \le K + \int_0^t Q(s)\, u^\lambda(s)\, ds \quad on \quad [0,\tau),\ \tau \le \infty\,.$$

If $(\lambda-1)K^{\lambda-1}\int\limits_0^\infty Q(s)\,ds<1$ *then*

$$u(t)\le K\Big[1-(\lambda-1)K^{\lambda-1}\int_0^t Q(s)\,ds\Big]^{1/(1-\lambda)}, \qquad t\in[0,\tau)\,.$$

The next theorem is devoted to the sublinearity of f.

Theorem 2.2. *Let $r\in C^0(\mathbb{R}_+)$, $r\ge 0$ on $\mathbb{R}_+$ and let*

$$\big|f(t,x_0,\dots,x_{n-1})\big|\le r(t)\Big[1+\sum_{i=0}^{n-1}|x_i|\Big]$$

on $\mathbb{R}_+\times\mathbb{R}^n$. Then every solution y of (1) is continuable.

Proof. Let y be a solution of (1) defined on $J\overset{\text{def}}{=}[0,T)\subset\mathbb{R}_+$ and let ϕ_i and σ_i, $i=0,1,\dots,n-1$, be given by Definition 1.1. If y is defined on $[T_1,T)$ with $T_1>0$ the proof is similar. We prove that $T=\infty$. Let $T<\infty$ and y be noncontinuable. Then

$$\limsup_{t\to T-}u(t)=\infty \tag{7}$$

where

$$u(t)=\max_{\sigma_{n-1}\le s\le t}\big|y^{(n-1)}(s)\big|+1\,, \qquad t\in J\,. \tag{8}$$

The Taylor series theorem implies

$$\begin{aligned}\big|y^{(i)}(t)\big| &\le \sum_{j=i}^{n-2}\frac{|y^{(j)}(0)|}{(j-i)!}t^{j-i}+\int_0^t\frac{(t-s)^{n-i-2}}{(n-i-2)!}\big|y^{(n-1)}(s)\big|\,ds\\ &\le M+M_1u(t)\le (M+M_1)u(t)\end{aligned}$$

on J with $i=0,1,\dots,n-2$, $M=\max\limits_{0\le i\le n-2}\sum\limits_{j=i}^{n-2}\frac{|y^{(j)}(0)|}{(j-i)!}T^{j-i}$ and $M_1=\max\limits_{0\le i\le n-2}\frac{T^{n-i-1}}{(n-i-1)!}$. From this and from $u\ge 1$,

$$\big|y^{(i)}(\tau_i(t))\big|\le\max_{\sigma_i\le s\le 0}\big|\phi_i(s)\big|+(M+M_1)u(t)\le M_2u(t)$$

on J for $i=0,1,\dots,n-2$ with $M_2=\max\limits_{0\le i<n}\max\limits_{\sigma_i\le s\le 0}\big|\phi_i(s)\big|+M+M_1$ and, hence,

$$\big|y^{(i)}(\tau_i(t))\big|\le (M_2+1)u(t)\,, \qquad i=0,1,\dots,n-1\,. \tag{9}$$

Furthermore, (1), (8) and (9) imply

$$\begin{aligned}\big|y^{(n-1)}(t)\big| &\le \big|y^{(n-1)}(0)\big|+\int_0^t\Big|f\big(s,y(\tau_0(s)),\dots,y^{(n-1)}(\tau_{n-1}(s))\big)\Big|\,ds\\ &\le \big|y^{(n-1)}(0)\big|+\int_0^t|r(s)|\Big[1+\sum_{i=0}^{n-1}\big|y^{(i)}(\tau_i(s))\big|\Big]\,ds\\ &\le \big|y^{(n-1)}(0)\big|+\int_0^t Q(s)\,u(s)\,ds\end{aligned}$$

on J with $Q(t) = |r(t)|\big[1 + n + M_2 n\big]$. Hence,

$$u(t) \le K + \int_0^t Q(s)\, u(s)\, ds\,, \qquad t \in J \tag{10}$$

with $K = 1 + \big|y^{(n-1)}(0)\big| + \max_{\sigma_{n-1} \le s \le 0} \big|\phi_{n-1}(s)\big|$. Then according to Gronwall's inequality, applied to (10), u is bounded on J, and that contradicts (7). $\square$

The following theorem describes a set of Cauchy initial conditions whose corresponding solutions are defined on $\mathbb{R}_+$, even in the superlinear case.

Theorem 2.3. *Let $\lambda > 1$, $r \in C^0(\mathbb{R}_+)$, $r \ge 0$ on $\mathbb{R}_+$,*

$$\big|f(t, x_0, \dots, x_{n-1})\big| \le r(t) \sum_{i=0}^{n-1} |x_i|^\lambda \tag{11}$$

on $\mathbb{R}_+ \times \mathbb{R}^n$ and let

$$\int_0^\infty t^{\lambda(n-1)} r(t)\, dt < \infty\,. \tag{12}$$

Then there exists $\varepsilon > 0$ such that a solution y of (1) defined in a right neighbourhood of $t = 0$ with Cauchy initial conditions

$$\big|\phi_i(t)\big| \le \varepsilon \quad \text{on} \quad [\sigma_i 0],\ \ i = 0, 1, \dots, n-1\,, \tag{13}$$

is continuable.

Proof. Consider a solution y of (1 with the initial conditions satisfying (13) where $\varepsilon > 0$ is such that

$$(\lambda - 1) K^{\lambda - 1} \int_0^\infty Q(s)\, ds < 1 \tag{14}$$

where

$$Q(t) = (n+1)^\lambda\, r(t) \sum_{i=0}^{n-1} t^{\lambda(n-i-1)} \tag{15}$$

and

$$K = 2\varepsilon + \varepsilon^\lambda (n+1)^\lambda \int_0^\infty r(s) \Big[n + \sum_{i=0}^{n-1} \sum_{j=i}^{n-2} s^{\lambda(j-i)}\Big]\, ds\,. \tag{16}$$

Note, that according to (12), $K < \infty$ and ε exist.

We prove that y is defined on $\mathbb{R}_+$. Assume for the sake of contradiction that y is noncontinuable on $J \overset{\text{def}}{=} [0, T)$, $T < \infty$. Then

$$\limsup_{t\to T-} u(t) = \infty \tag{17}$$

where

$$u(t) = \max_{\sigma_{n-1}\le s\le t} \left|y^{(n-1)}(s)\right|, \qquad t\in[0,T)\,. \tag{18}$$

The Taylor series theorem and (13) imply

$$\left|y^{(i)}(t)\right| \le \varepsilon \sum_{j=i}^{n-2} \frac{t^{j-i}}{(j-i)!} + \int_0^t \frac{(t-\sigma)^{n-i-2}}{(n-i-2)!}\left|y^{(n-1)}(\sigma)\right| d\sigma \tag{19}$$

on J for $i = 0,1,\dots,n-2$. Hence (13) and (19) yield

$$\left|y^{(i)}(\tau_i(t))\right| \le \max_{\sigma_i\le s\le 0} |\phi_i(s)| + \max_{0\le s\le t}\left|y^{(i)}(s)\right| \le \varepsilon\Big(1+\sum_{j=i}^{n-2} t^{j-i}\Big) + t^{n-i-1}u(t) \tag{20}$$

on J for $i = 0,1,\dots,n-2$. If we define $\sum_{i=k}^{l} = 0$ for $k > l$, (20) is valid for $i = n-1$, too.

The following inequality is very useful. If $\lambda > 0$, $m \in \{1,2,\dots\}$ and $a_i \in (0,\infty)$ for $i = 1,2,\dots m$, then

$$(a_1+\cdots+a_m)^\lambda \le m^\lambda(a_1^\lambda+\cdots+a_m^\lambda)\,. \tag{21}$$

According to (11), (13), (15), (20) and (21)

$$\begin{aligned}
\left|y^{(n-1)}(t)\right| &\le \left|y^{(n-1)}(0)\right| + \int_0^t \left|y^{(n)}(s)\right| ds \\
&\le \varepsilon + \int_0^t r(s)\sum_{i=0}^{n-1}\left|y^{(i)}(\tau_i(s))\right|^\lambda ds \\
&\le \varepsilon + \int_0^t r(s)\sum_{i=0}^{n-1}\Big(\varepsilon+\varepsilon\sum_{j=i}^{n-2}s^{j-i}+s^{n-i-1}u(s)\Big)^\lambda ds \\
&\le \varepsilon + \int_0^t r(s)\sum_{i=0}^{n-1}(n-i+1)^\lambda \\
&\qquad \times\Big[\varepsilon^\lambda+\varepsilon^\lambda\sum_{j=i}^{n-2}s^{\lambda(j-i)}+s^{\lambda(n-i-1)}u^\lambda(s)\Big]\,ds \\
&\le K_1 + \int_0^t Q(s)u^\lambda(s)\,ds
\end{aligned}$$

on J with

$$K_1 = \varepsilon + \varepsilon^\lambda (n+1)^\lambda \int_0^\infty r(s) \Big[n + \sum_{i=0}^{n-1} \sum_{j=i}^{n-2} s^{\lambda(j-i)} \Big] .$$

From this and from (18)

$$u(t) \le K + \int_0^t Q(s)\, u^\lambda(s)\, ds\,, \qquad t \in J\,. \tag{22}$$

Then (14)–(16) and (22) imply all assumptions of Lemma 2.1 are satisfied (with $\tau = T$) and u is bounded on J; that contradicts (17). □

Remark 2.1. *The method of the proof of Theorem 2.3 was used in [12] for a different type of the differential equation.*

Next, we look for sufficient conditions under which such solutions do not exist for Equation (1). So we study the nonexistence of noncontinuable solutions of (1) defined in a left neighbourhood of $T \in (0,\infty)$ and satisfying either

$$\lim_{t\to T-} y^{(i)}(t) = \infty\,, \qquad i = 0,1,\dots,n-1 \tag{23}$$

or $l \in \{0,1,\dots,n-2\}$, $C_i > 0$ for $i = 0,1,\dots,l$,

$$\begin{aligned} &\lim_{t\to T-} y^{(i)}(t) = C_i \qquad \text{for} \quad i = 0,1,\dots,l \quad \text{and} \\ &\lim_{t\to T-} y^{(j)}(t) = \infty \qquad \text{for} \quad j = l+1,\dots,n-1\,. \end{aligned} \tag{24}$$

Needed results for equations (2) and

$$z^{(n)} = K (z^{(k)})^\lambda \tag{25}$$

and a comparison theorem will be given in the following two lemmas.

Lemma 2.2. *Let $T \in (0,\infty)$, $k \in \{0,1,\dots,n-1\}$ and $K > 0$.*

(i) *If $k > 0$, $\lambda > 1 + \frac{1}{k}$, then (25) has no solution y satisfying (23).*

(ii) *Let $l \in \{0,1,\dots,n-2\}$ and let either $l < k$ and $1 < \lambda < 1 + \frac{n-k}{k-l}$ or $l < k-1$ and $\lambda \ge 1 + \frac{n-k}{k-l-1}$. Then (25) has no solution y satisfying (24).*

Proof. It follows from Theorem 2 in [4] (Theorems 3.1 and 3.2 in [6] or [3]) in case (i) (case (ii)). □

Lemma 2.3. *Let $T_1 > 0$, $K > 0$, $M \geq 0$, $k \in \{0, 1, \ldots, n-1\}$, $\lambda > 1$ and*

$$0 \leq f(t, x_0, x_1, \ldots, x_{n-1}) \leq K\, x_k^\lambda \tag{26}$$

for $t \in [0, T_1]$ and $x_i \geq M$, $i = 0, 1, \ldots, n-1$. Let y be a solution of (1) *defined on $[0, T)$ satisfying Cauchy initial conditions $\phi_i(t) \geq M$, $t \in [\sigma_i, 0]$, $i = 0, 1, \ldots, n-1$. Let z be a solution of* (25) *satisfying Cauchy initial conditions*

$$z^{(i)}(0) > \max_{\sigma_i \leq t \leq 0} \phi_i(t)\,, \qquad i = 0, 1, \ldots, n-1$$

defined on $[0, T_1)$. Then

$$y^{(i)}(t) \leq z^{(i)}(t)\,, \qquad t \in [0, T_1),\ i = 0, 1, \ldots, n-1\,.$$

Proof. Note that solutions of Cauchy problems for (25) are unique. Moreover, $y^{(i)}$ and $z^{(i)}$ are nondecreasing on $[0, T_1)$ and so, (26) implies

$$\begin{aligned} y^{(n)}(t) &= f\big(t, y(\tau_0(t)), \ldots, y^{(n-1)}(\tau_{n-1}(t))\big) \leq K\big(y^k(\tau_k(t))\big)^\lambda \\ &\leq K(y^k(t))^\lambda\,, \qquad t \in [0, T)\,. \end{aligned}$$

From this and from theorems on differential inequalities (see [21]) applied on (1) and (25) (transformed into the system) we obtain the result of the lemma. □

Theorem 2.4. *Let $T \in (0, \infty)$, $\lambda > 1$ and $M \geq 0$.*

(i) *Suppose $K > 0$, $k \in \{1, \ldots, n-1\}$, $\lambda > 1 + \frac{1}{k}$,*

$$0 \leq f(t, x_0, x_1, \ldots, x_{n-1}) \leq K\, x_k^\lambda \tag{27}$$

for $t \in [0, T]$ and $x_i \geq M$, $i = 0, 1, \ldots, n-1$. Then there exists no solution y of (1) *defined on $[0, T)$ with Cauchy initial conditions*

$$M \leq \phi_i(t)\,, \qquad t \in [\sigma_i, 0],\ i = 0, 1, \ldots, n-1\,, \tag{28}$$

satisfying (23).

(ii) *Let*

$$f(t, x_0, \ldots, x_{n-1}) \leq 0 \tag{29}$$

for $t \in [0, T]$ and $x_i \geq M$, $i = 0, 1 \ldots, n-1$. Then there exists no solution y of (1) *defined on $[0, T)$ with* (28) *satisfying* (23).

Proof. (i) Assume for the sake of contradiction that y be a solution of (1) with (28) and (23). Let us consider (25) with Cauchy initial conditions

$$z^{(i)}(0) > \max_{\sigma_i \leq s \leq 0} \phi_i(s)\,, \qquad i = 0, 1, \ldots, n-1\,. \tag{30}$$

Note, that $y^{(i)}$ and $z^{(i)}$, $i = 0, 1, \dots, n-1$, are nonnegative and nondecreasing functions on their definition intervals. As according to (27)

$$f\big(t, y(\tau_0(t)), \dots, y^{(n-1)}(\tau_{n-1}(t))\big) \le K\big(y^{(k)}(\tau_k(t))\big)^{\lambda} \le K\big(y^{(k)}(t)\big)^{\lambda}$$

on $[0, T)$, Lemma 2.3 applied on (1) and (25), it follows that $z^{(i)}(t) \ge y^{(i)}(t)$, $i = 0, 1, \dots, n-1$, for all t where y and z are defined. It follows from this and from (23) that $T_1 \in (0, T]$ exists with $\lim_{t \to T_1-} z(t) = \infty$ and, hence, $\lim_{t \to T_1-} z^{(i)}(t) = \infty$ for $i = 0, 1, \dots, n-1$. The contradiction with Lemma 2.2(i) $(T = T_1)$ proves the statement.

(ii) Let y be a noncontinuable solution on $[0, T)$ with (29), (28) and (23). Then (23) implies

$$\limsup_{t \to T-} y^{(n)}(t) = \infty\,. \tag{31}$$

Furthermore, (1) and (29) imply $y^{(n)}$ is bounded in a left neighbourhood of T. The contradiction with (31) proves the statement. □

Theorem 2.5. *Let $T \in (0, \infty)$, $\lambda > 1$, $k \in \{1, 2, \dots, n-1\}$, $l \in \{0, 1, \dots, n-1\}$, $M \ge 0$, $C_i \ge M$ for $i = 0, 1, \dots, l$ and let either $l \ge k$ and $\lambda > 1 + \frac{1}{k}$ or*

$$l < k < \frac{n-1}{2} \quad \text{and} \quad 1 + \frac{1}{k} < \lambda < 1 + \frac{n-k}{k} \tag{32}$$

or $l < k - 1$ and $\lambda \ge 1 + \frac{n-k}{k-l-1}$. Suppose

$$0 \le f(t, x_0, x_1, \dots, x_{n-1}) \le K\, x_k^{\lambda} \tag{33}$$

for $t \in [0, T]$ and $x_i \ge M$, $i = 0, 1, \dots, n-1$. Then there exists no solution y of (1) *defined on $[0, T)$ with Cauchy initial conditions*

$$M \le \phi_i(t)\,, \qquad t \in [\sigma_i, 0],\ i = 0, 1, \dots, n-1 \tag{34}$$

satisfying (24).

Proof. For the sake of contradiction, let y be a solution of (1) with (33) and (24). Let us consider Equation (25) with (30). We prove similarly as in the proof of Theorem 2.4 that $z^{(i)}(t) \ge y^{(i)}(t)$, $i = 0, 1, \dots, n-1$, on the intersection of the definition intervals of y and z. Note, that $z^{(i)}$ and $y^{(i)}$ are nonnegative and nondecreasing for $i = 0, 1, \dots, n-1$. Hence, $T_1 \in (0, T]$ exists such that

$$\lim_{t \to T_1-} z^{(j)}(t) = \infty \quad \text{for} \quad j = l+1, \dots, n-1\,.$$

As $z^{(i)}$ is nondecreasing on $[0, T_1)$ for $i = 0, 1, \dots, n-1$, $\lim_{t \to T-} z^{(i)}(t) = \bar{C}_i$, $i = 0, 1, \dots, n-1$, with $\bar{C}_i \in [M, \infty) \cup \{\infty\}$. From this either

$$\lim_{t \to T_1-} z^{(i)}(t) = \infty\,, \qquad i = 0, 1, \dots, n-1\,, \tag{35}$$

or $\bar{l} \in \{0, 1, \dots, l\}$ exists such that

$$\lim_{t \to T_1-} z^{(i)}(t) = \bar{C}_i\,, \quad \text{for} \quad i = 0, 1, \dots, l_0\,, \tag{36}$$

and

$$\lim_{t \to T_1-} z^{(j)}(t) = \infty\,, \quad \text{for} \quad j = l_0 + 1, l_0 + 2, \dots, n - 1\,. \tag{37}$$

Furthermore, the definition of λ implies $k > 0$ and $\lambda > 1 + \frac{1}{k}$ and according to Lemma 2.1(i) the relation (35) is not valid. Let l_0 exist such that (36) and (37) holds.

Let $k > l$. We apply Lemma 2.2 (ii) with $T = T_1$ and $l = l_0$. Then according to the definition of λ we have either $l_0 < k$ and $1 < \lambda < 1 + \frac{n-k}{k-l_0}$ or $l_0 < k - 1$ and $\lambda \geq 1 + \frac{n-k}{k-l_0-1}$. Hence, the assumptions of Lemma 2.2(ii) hold and there exists no solution satisfying (36) and (37). Note, that $k < \frac{n-1}{2}$ in (32) is necessary for the second assumptions in (32) to be nonempty.

Suppose the last case $k \leq l$. Then (1), (33) and (34) imply

$$\begin{aligned} y^{(n)}(t) &= f\big(t, y(\tau_0(t)), \dots, y^{(n-1)}(\tau_{n-1}(t))\big) \leq K\big(y^{(k)}(\tau_k(t))\big)^{\lambda} \\ &\leq K\big(y^{(k)}(t)\big)^{\lambda} \leq K\,C_k < \infty\,. \end{aligned}$$

Hence, $y^{(n-1)}$ is bounded on $[0, T)$ and the contradiction with $\lim\limits_{t \to T-} y^{(n-1)}(t) = \infty$ (see (24)) proves the statement in this case. $\square$

Remark 2.2. *In Theorems 2.4 and 2.5 positive solutions of* (1) *are studied. Negative solutions satisfying either* (23) *or* (24) *with* $-\infty$ *and* $C_i < 0$ *for* $i = 0, 1, \dots, l$*, can be studied similarly; it is possible to use the tranformation* $y = -Y$*.*

Example 1.2 shows that a solution satisfying (23) exists. Similarly, according to the following example, solutions satisfying (24) exist.

Example 2.1. *Consider the equation* $y'' = \frac{1}{8}(1+t)^{3/2} f\big(y'(\tau(t))\big)$ *with* $f \in C^0(\mathbb{R}_+)$, $f(x) = x^3$ *on [0,1] and* $\tau(t) = t^2$ *on* $[0,1]$, $\tau(t) = t$ *for* $t > 1$*. Then* $y(t) = -4(1-t)^{1/2} + 5$ *is noncontinuable on* $[0, 1)$, $\lim\limits_{t \to 1-} y(t) = 5$, $\lim\limits_{t \to 1-} y'(t) = \infty$*.*

3 Existence Theorems

In this section existence results will be derived. We need the following main lemma that investigates properties of the equation

$$z^{(n)} = K\big(z^{(k)}(\tau(t))\big)^{\lambda} \tag{38}$$

for a special τ.

Lemma 3.1. *Let $T > 0$, $\lambda > 1$, $K > 0$, $K_1 > 0$, $k \in \{0, 1, \dots, n-1\}$ $\bar\tau \ge 1$ and $\tau(t) = \bar\tau t + T(1-\bar\tau)$ on $(-\infty, T]$. If either $k = 0$ or $\lambda \ne \frac{n-m}{k-m}$ for $m = 0, 1, \dots, k-1$, then* (38) *has a noncontinuable solution*

$$z(t) = M(T-\tau(t))^{-s} + K_1(T+t)^{k-1}\operatorname{sgn} k \tag{39}$$

on $(-\infty, T)$ where $s = \frac{n-k\lambda}{\lambda-1}$, and

$$M = \bar\tau^{\frac{n+\lambda(s+k)}{\lambda-1}} K^{-\frac{1}{\lambda-1}} \prod_{j=0}^{n-k-1} \Big(\frac{n-k}{\lambda-1} + j\Big)^{1/(\lambda-1)} \prod_{j=0}^{k-1} \Big(\frac{n-k\lambda}{\lambda-1} + j\Big)^{-1}.$$

Proof. Note, that $\tau(T) = T$ and τ is the delay in $(-\infty, T]$. The fact that (39) is a solution of (38) can be obtained by the direct computation. This solution is defined if and only if $M \in \mathbb{R}$; i.e., if and only if $s = \frac{n-k\lambda}{\lambda-1} \notin \{0, -1, -2, \dots, -k+1\}$. This is valid if either $k = 0$ or $\lambda \ne \frac{n-m}{k-m}$ for $m = 0, 1, \dots, k-1$. As

$$\begin{aligned} z^{(n-1)}(t) &= M\bar\tau^{n-1} s(s+1)\dots(s+n-2)(T-\tau(t))^{-s-n+1} \\ &= C(T-\tau(t))^{-s-n+1} \end{aligned}$$

with

$$C = \bar\tau^{\frac{n+\lambda(s+k)}{\lambda-1}+n-1} K^{-\frac{1}{\lambda-1}} \prod_{j=0}^{n-k-1} \Big(\frac{n-k}{\lambda-1} + j\Big)^{1/(\lambda-1)} \prod_{j=0}^{n-k-2} \Big(\frac{n-k}{\lambda-1} + j\Big) > 0\,,$$

it follows that z is noncontinuable on $(-\infty, T)$. ☐

Lemma 3.2. *Let $0 \le T_1 < T < \infty$, $K > 0$, $k \in \{0, 1, \dots, n-1\}$, $\lambda > 1$, $M \ge 0$, $\tau \in C^0[T_1, T)$, $\tau(t) \le \tau_k(t)$ on $[T_1, T]$. Let $f(t, x_0, \dots, x_{n-1}) \ge K\, x_k^\lambda$ for $t \in [T_1, T]$ and $x_i \ge M$, $i = 0, 1, \dots, n-1$. Suppose z is a solution of* (38) *defined on $[T_1, T)$ with the initial conditions $\bar\phi_i(t) \ge M$, $t \in [\bar\sigma_i, T_1]$, $i = 0, 1, \dots, n-1$. If y is a solution of* (1)*, defined on $[T_1, T)$ with the initial conditions $\phi_i(t) > \max\limits_{\bar\sigma_i \le s \le T_1} \bar\phi_i(s)$, $t \in [\sigma_i, T_1]$, $i = 0, 1, \dots, n-1$, then*

$$z^{(i)}(t) \le y^{(i)}(t)\,, \qquad t \in [T_1, T),\ i = 0, 1, \dots, n-1\,.$$

Here σ_i and $\bar\sigma_i$ are numbers from Definition 1.1, applied on (38) *and* (1)*, respectively.*

Proof. It can be proved by the same way as well known theorems on differential inequalities without delay, see e.g. [21]. ☐

The next theorem addresses a boundary value problem and provides sufficient conditions for the existence of noncontinuable solution of (1) in a left neighbourhood of a given number $T > 0$.

Theorem 3.1. *Let* $0 \le T_1 < T < \infty$, $r \in C^0([T_1, T])$, $k \in \{0, 1, \dots, n-1\}$, $M \ge 0$, $\bar\tau \ge 1$, $\lambda > 1$,

$$t - \tau_i(t) > 0 \quad for \quad i = 0, 1, \dots, n-1 \quad on \quad [T_1, T)\,, \tag{40}$$
$$r(t) > 0 \quad and \quad t - \tau_k(t) \le (\bar\tau - 1)(T - t) \quad on \quad [T_1, T]\,, \tag{41}$$

and

$$f(t, x_0, x_1, \dots, x_{n-1}) \ge r(t)\, x_k^\lambda \tag{42}$$

for $T_1 \le t \le T$, $x_i \ge M$, $i = 0, 1, \dots, n-1$. *Then there exists* $\delta > 0$ *such that every solution of* (1) *with the initial conditions*

$$\phi_i(t) > \delta \quad for \quad \sigma_i \le t \le T_1\,, \quad \sigma_i = \min_{T_1 \le t \le T} \tau_i(t)\,, \quad i = 0, 1, \dots, n-1, \tag{43}$$

is defined on $[T_1, T)$ *and it is noncontinuable.*

Proof. Let $K = \min\limits_{T_1 \le t \le T} r(t) > 0$ and $\tau(t) = \bar\tau t + T(1 - \bar\tau)$; hence $t - \tau(t) = (\bar\tau - 1)(T - t)$. Let $\lambda \ne \frac{n-m}{k-m}$ for $m = 0, 1, \dots, k-1$ and let z be a solution of (38) given by Lemma 3.1 with K_1 to be such that

$$z^{(i)}(t) \ge M \quad \text{for} \quad i = 0, 1, \dots, k-1 \quad \text{and} \quad \tau(T_1) \le t \le T_1\,.$$

Then (40) and (41) imply

$$t > \tau_k(t) \ge \tau(t)\,, \qquad t \in [T_1, T)\,, \tag{44}$$

$z^{(j)} \ge 0$ are nondecreasing on $\big[\tau(T_1), T\big)$ for $i = 0, 1, \dots, n-1$, and z is noncontinuable on $[T_1, T)$. Define δ by

$$\delta > \max_{0 \le j < n} z^{(j)}(T_1)\,. \tag{45}$$

Let y be a solution of (1) satisfying (43); due to Theorem 2.1 and (40) it is defined on $[T_1, T)$. Then Lemma 3.2, (42), (44) and (45) imply $y^{(j)}(t) \ge z^{(j)}(t)$, $t \in [T_1, T)$ and y is noncontinuable due to z having this property.

Suppose $m \in \{0, 1, \dots, k-1\}$ exists such that $\lambda = \frac{n-m}{k-m}$, $0 < \varepsilon < \frac{n-k}{(k-m)(k-m-1)}$ and let z_1 be the solution of (38) given by (39) with $\lambda = \lambda + \varepsilon$. The condition posed on ε ensures that $\lambda + \varepsilon \ne \frac{n-\sigma}{k-\sigma}$ for $\sigma = 0, 1, \dots, k-1$, and Lemma 3.2 may be applied. K_1 is given such that $z^{(i)}(T_1) > 0$ for $i \ne k$ and $z^{(k)}(t) > 0$ for $\tau(T_1) \le t \le T_1$. Then (39) implies $z_1^{(j)}(t) \ge 0$, $z_1^{(j)}$ is nondecreasing on $\tau(T_1) \le t < T$, $j = 0, 1, \dots, n-1$, and $\lim\limits_{t \to T-} z_1^{(k)}(t) = \infty$. Let $T_0 \in [T_1, T)$ be such that

$$z_1^{(k)}(t) \ge \max(1, M) \quad \text{on} \quad [T_0, T)\,. \tag{46}$$

Let z be a solution of (38), defined on $[T_0, \bar{T}) \subset [T_0, T)$ satisfying the initial conditions

$$\bar{\phi}_i(t) = \delta \stackrel{\text{def}}{=} \max_{0 \le j \le n-1} z_1^{(j)}(T_0) + 1\,, \qquad i = 0, 1, \dots, n-1\,, \tag{47}$$

and $t = T_1$ for $i \neq k$ and $\tau(T_0) \le t \le T_0$ in case $i = k$. From this, (40), (46), from Lemma 3.2 (applied on (38) and on (38) with $\lambda = \lambda + \varepsilon$) and Theorem 2.1, it follows that $z_1^{(i)}(t) \le z^{(i)}(t)$ on $[T_0, T)$, $i = 0, 1, \dots, n-1$. Let y be a solution of (1) defined on $[T_1, T)$ satisfying (43); note that due to (40) and Theorem 2.1, y is defined on $[T_1, T)$. Then Lemma 3.2, (42), (44) and (47) imply $y^{(j)}(t) \ge z^{(j)}(t)$, $t \in [T_0, T)$ and y is noncontinuable as z has this property. □

The next theorem solves the problem of existence of noncontinuable solutions for the initial problem.

Theorem 3.2. *Let $T > 0$, $\tau_k(T) = T$, $r \in C^0[0, T]$, $k \in \{0, 1, \dots, n-1\}$, $M > 0$, $0 \le T_1 < T$, $\lambda > 1$, $\bar{\tau} \ge 1$, $r(t) > 0$ on $[0, T]$,*

$$0 \le t - \tau_k(t) \le (\bar{\tau} - 1)(T - t) \quad on \quad [T_1, T]$$

and (42) holds for $T_1 \le t \le T$, $x_i \ge M$, $i = 0, 1, \dots, n-1$. Then there exists $\delta > 0$ such that every solution y on (1) with the initial conditions

$$\phi_i(t) > \delta \quad for \quad \sigma_i \le t \le 0\,, \quad \sigma_i = \inf_{0 \le t < \infty} \tau_i(t)\,, \quad i = 0, 1, \dots, n-1\,, \tag{48}$$

is defined only on a finite interval $[0, t_0)$; i.e., y is noncontinuable on $[0, t_0)$ (t_0 depends on y).

Proof. If y is noncontinuable on $[0, t_0) \subset [0, T)$ then the statement is valid. Let y be defined on $[0, T]$, then we can prove by the same way as in the proof of Theorem 3.1 that y is noncontinuable; it is a contradiction. Note, that, as $y^{(i)}$ is nondecreasing, (48) implies $y^{(i)} > \delta$ in a left neighbourhood of T_1 given by the proof of Theorem 3.1. Moreover, assumption (40) is not supposed as we only need to show that t_0 exists. □

Remark 3.1. *Theorem 3.2 is proved in [25] for special case of (1) without delays. Theorem 3.1 extends results given in [4], [6] for (1) without delays.*

Remark 3.2. *Similar results as in Theorems 3.1 and 3.2 can be proved under an assumption*

$$f(t, x_0, \dots, x_{n-1}) \le -K|x_k|^\lambda$$

for $t \in [T_1, T]$ and $x_i \le -M$, $i = 0, 1, \dots, n-1$. The transformation $y = -Y$ may be used, too.

Corollary 3.1. *Consider* $y^{(n)} = r(t)\big|y(\tau(t))\big|^\lambda \operatorname{sgn} y(\tau(t))$ *with $\lambda > 1$, $\tau \in C^0(\mathbb{R}_+)$. Let $C > 0$ and $0 \le T_1 < T < \infty$ be such that $r > 0$ and $0 \le t - \tau(t) \le C(T - t)$ on $[T_1, T]$. Then the differential equation has a noncontinuable solution on $[0, T)$.*

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