Electronic Journal of Qualitative Theory of Differential Equations
2015, No. **90**, 1–10; doi: 10.14232/ejqtde.2015.1.90 http://www.math.u-szeged.hu/ejqtde/

Partial observability of a wave-Petrovsky system with memory

Paola Loreti and **Daniela Sforza** ✉

Dipartimento di Scienze di Base e Applicate per l'Ingegneria, Sapienza Università di Roma, via A. Scarpa n. 16, Roma, I–00161, Italy

Received 15 May 2015, appeared 8 December 2015
Communicated by Vilmos Komornik

Abstract. Our goal is to show partial observability results for coupled systems with memory terms. To this end, by means of non-harmonic analysis techniques we prove Theorem 3.2 and Theorem 3.7 below.

Keywords: coupled systems, convolution kernels, Fourier series, Ingham estimates.

2010 Mathematics Subject Classification: 42A38, 45K05, 93C20.

1 Introduction

In this paper we consider a coupled system obtained by combining a wave equation with an integral relaxation term and a Petrovsky type equation, that is

$$\begin{cases} u_{1tt} - u_{1xx} + \beta \int_0^t e^{-\eta(t-s)} u_{1xx}(s,x)ds + Au_2 = 0 \qquad \text{in } (0,\infty)\times(0,\pi)\,, \\ u_{2tt} + u_{2xxxx} + Bu_1 = 0 \qquad \text{in } (0,\infty)\times(0,\pi)\,, \\ u_1(\cdot,0) = u_1(\cdot,\pi) = u_2(\cdot,0) = u_2(\cdot,\pi) = u_{2xx}(\cdot,0) = u_{2xx}(\cdot,\pi) = 0 \qquad \text{in } (0,\infty)\,, \\ u_1(0,\cdot) = u_{10}\,, \quad u_{1t}(0,\cdot) = u_{11}\,, \quad u_2(0,\cdot) = u_{20}\,, \quad u_{2t}(0,\cdot) = u_{21} \qquad \text{in } (0,\pi)\,, \end{cases} \tag{1.1}$$

where $0 < \beta < \eta$ and A, B are real constants.

In [14] we proved that the observation of the solution at a point of the boundary allows us to recognize the unknown initial data. In the following theorem we recall that result.

Theorem 1.1. *Let $\eta > 3\beta/2$ and $T > 2\pi$. For any $(u_{10}, u_{11}) \in H_0^1(0,\pi)\times L^2(0,\pi)$ and $(u_{20}, u_{21}) \in H_0^1(0,\pi)\times H^{-1}(0,\pi)$, if (u_1, u_2) is a solution of problem (1.1) we have*

$$\int_0^T |u_{1x}(t,\pi)|^2 + |u_{2x}(t,\pi)|^2\, dt \asymp \|u_{10}\|^2_{H_0^1(0,\pi)} + \|u_{11}\|^2 + \|u_{20}\|^2_{H_0^1(0,\pi)} + \|u_{21}\|^2_{H^{-1}(0,\pi)}\,. \tag{1.2}$$

✉ Corresponding author. Email: daniela.sforza@sbai.uniroma1.it

Throughout the paper, we will use the notation $\|\cdot\| = \|\cdot\|_{L^2(0,\pi)}$. Moreover, we will adopt the convention to write $f \asymp g$ if there exist two positive constants c_1 and c_2 such that $c_1 f \le g \le c_2 f$.

Our goal is to establish a partial observability result where we only observe u_1 or u_2 at the boundary. Indeed, we will show sufficient conditions guaranteeing the validity of estimates

$$\int_0^T |u_{1x}(t,\pi)|^2\, dt \asymp \|u_{10}\|^2_{H_0^1(0,\pi)} + \|u_{11}\|^2 + \|u_{20}\|^2_{H_0^1(0,\pi)} + \|u_{21}\|^2_{H^{-1}(0,\pi)}\,, \tag{1.3}$$

or

$$\int_0^T |u_{2x}(t,\pi)|^2\, dt \asymp \|u_{10}\|^2_{H_0^1(0,\pi)} + \|u_{11}\|^2 + \|u_{20}\|^2_{H_0^1(0,\pi)} + \|u_{21}\|^2_{H^{-1}(0,\pi)}\,. \tag{1.4}$$

It is evident that the direct inequality in (1.3) and in (1.4) follows from (1.2), and hence the key point is to prove the inverse inequalities. In fact, by writing the solution of system (1.1) as a Fourier series and using typical techniques of non-harmonic analysis, we are able to establish Theorem 3.2 and Theorem 3.7 below. It is noteworthy to observe that in Theorem 3.2 we have to assume that the initial data u_{20} and u_{21} are null, while the same condition on the initial data u_{10} and u_{11} is not required in Theorem 3.7.

For references related to integral equations and viscoelasticity theory see e.g [1–3,15,16]. It is worthwhile to mention a partial observability problem for a wave-Petrovsky system (without memory) analyzed in [7]. For a classical overview about exact controllability problems see [8–11,17].

2 The Fourier series expansion of the solution

Let $T > 0$. Fix two real numbers A, B different from zero. For any $(u_{10}, u_{11}) \in H_0^1(0,\pi) \times L^2(0,\pi)$ and $(u_{20}, u_{21}) \in H_0^1(0,\pi) \times H^{-1}(0,\pi)$ there exists a unique *weak* solution (u_1, u_2) with $u_1 \in C(\mathbb{R}_+; H_0^1(0,\pi)) \cap C^1(\mathbb{R}_+; L^2(0,\pi))$ and $u_2 \in C(\mathbb{R}_+; H_0^1(0,\pi)) \cap C^1(\mathbb{R}_+; H^{-1}(0,\pi))$ of the following coupled system

$$\begin{cases} u_{1tt} - u_{1xx} + \beta \displaystyle\int_0^t e^{-\eta(t-s)} u_{1xx}(s,x)ds + Au_2 = 0 \qquad \text{in } (0,\infty)\times(0,\pi)\,, \\ u_{2tt} + u_{2xxxx} + Bu_1 = 0 \qquad \text{in } (0,\infty)\times(0,\pi)\,, \\ u_1(\cdot,0) = u_1(\cdot,\pi) = u_2(\cdot,0) = u_2(\cdot,\pi) = u_{2xx}(\cdot,0) = u_{2xx}(\cdot,\pi) = 0 \qquad \text{in } (0,\infty)\,, \\ u_1(0,\cdot) = u_{10}\,, \quad u_{1t}(0,\cdot) = u_{11}\,, \quad u_2(0,\cdot) = u_{20}\,, \quad u_{2t}(0,\cdot) = u_{21} \qquad \text{in } (0,\pi)\,. \end{cases} \tag{2.1}$$

If we expand the initial data according to the eigenfunctions of the Laplacian $\sin(nx)$, $n \in \mathbb{N}$, then we obtain the expressions

$$\begin{aligned} u_{10}(x) &= \sum_{n=1}^{\infty} \alpha_{1n} \sin(nx), & \|u_{10}\|^2_{H_0^1(0,\pi)} &= \frac{\pi}{2}\sum_{n=1}^{\infty} \alpha_{1n}^2 n^2, \\ u_{11}(x) &= \sum_{n=1}^{\infty} \chi_{1n} \sin(nx), & \|u_{11}\|^2 &= \frac{\pi}{2}\sum_{n=1}^{\infty} \chi_{1n}^2, \end{aligned} \tag{2.2}$$

$$\begin{aligned} u_{20}(x) &= \sum_{n=1}^{\infty} \alpha_{2n} \sin(nx)\,, & \|u_{20}\|^2_{H_0^1(0,\pi)} &= \frac{\pi}{2}\sum_{n=1}^{\infty} \alpha_{2n}^2 n^2, \\ u_{21}(x) &= \sum_{n=1}^{\infty} \chi_{2n} \sin(nx)\,, & \|u_{21}\|^2_{H^{-1}(0,\pi)} &= \frac{\pi}{2}\sum_{n=1}^{\infty} \frac{\chi_{2n}^2}{n^2}\,. \end{aligned} \tag{2.3}$$

By applying the spectral analysis developed in Hilbert spaces, see [14, Section 4], we are able to write the solution (u_1, u_2) of problem (2.1) as Fourier series.

Theorem 2.1. *For any $(u_{10}, u_{11}) \in H_0^1(0,\pi) \times L^2(0,\pi)$ and $(u_{20}, u_{21}) \in H_0^1(0,\pi) \times H^{-1}(0,\pi)$, the weak solution (u_1, u_2) of problem (2.1) is given by*

$$\begin{aligned} u_1(t,x) &= \sum_{n=1}^{\infty} \left(R_n e^{r_n t} + C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega_n} t} + D_n e^{ip_n t} + \overline{D_n} e^{-i\overline{p_n} t} \right) \sin(nx) \\ u_2(t,x) &= \sum_{n=1}^{\infty} \left(d_n D_n e^{ip_n t} + \overline{d_n D_n} e^{-i\overline{p_n} t} - \frac{2\beta}{A} n^2 \Re \frac{D_n}{\eta + ip_n} e^{-\eta t} \right) \sin(nx)\,, \end{aligned} \tag{2.4}$$

for $t \geq 0$ and $x \in (0,\pi)$, where $r_n\,, R_n \in \mathbb{R}$ and $\omega_n\,, C_n\,, p_n\,, D_n\,, d_n \in \mathbb{C}$ are defined by

$$\begin{aligned} r_n &= \beta - \eta + O\left(\frac{1}{n^2}\right), \\ R_n &= \frac{\beta}{n^2}(\alpha_{1n}(\beta-\eta) + \chi_{1n}) + (\alpha_{1n} + \chi_{1n}) O\left(\frac{1}{n^4}\right), \\ \omega_n &= n + \frac{\beta}{2}\left(\frac{3}{4}\beta - \eta\right)\frac{1}{n} + i\frac{\beta}{2} + O\left(\frac{1}{n^2}\right), \\ C_n &= \frac{\alpha_{1n}}{2} - \frac{i}{4n}(\beta\alpha_{1n} + 2\chi_{1n}) + (\alpha_{1n} + \chi_{1n}) O\left(\frac{1}{n^2}\right), \\ p_n &= n^2 + O\left(\frac{1}{n^6}\right), \qquad (2.5) \\ D_n &= \frac{A\alpha_{2n}}{2n^4} + (\alpha_{2n} - i\chi_{2n})\frac{A}{2n^6} + (\alpha_{2n} + \chi_{2n}) O\left(\frac{1}{n^7}\right), \qquad (2.6) \\ d_n &= \frac{1}{A}\left(p_n^2 - n^2 + \frac{\beta n^2}{\eta + ip_n}\right). \end{aligned}$$

Moreover, for any $n \in \mathbb{N}$ one has

$$|d_n| \asymp |p_n|^2\,, \tag{2.7}$$

$$n^2 |C_n|^2 \asymp \alpha_{1n}^2 n^2 + \chi_{1n}^2\,, \tag{2.8}$$

$$n^2 |p_n|^4 |D_n|^2 \asymp \alpha_{2n}^2 n^2 + \frac{\chi_{2n}^2}{n^2}\,. \tag{2.9}$$

3 Partial observability results

To establish the result concerning the observation of the first component of the solution of problem (2.1), we need an inverse estimate of Ingham type (see [5]), involving only the terms $R_n e^{r_n t}$ and $C_n e^{i\omega_n t}$, see (2.4). For the reader's convenience, we recall a known theorem.

Theorem 3.1 ([12, 13])**.** *Let $\omega_n \in \mathbb{C}$ and $r_n \in \mathbb{R}$ be sequences of pairwise distinct numbers such that $r_n \neq i\omega_m$ for any n, $m \in \mathbb{N}$. Assume*

$$\liminf_{n\to\infty} (\Re\omega_{n+1} - \Re\omega_n) = \gamma > 0\,,$$

$$\lim_{n\to\infty} \Im\omega_n = \alpha\,, \qquad r_n \leq -\Im\omega_n \qquad \forall\, n \geq n'\,,$$

$$|R_n| \leq \frac{\mu}{n^\nu}|C_n| \quad \forall\, n \geq n'\,, \qquad |R_n| \leq \mu |C_n| \quad \forall\, n \leq n'\,,$$

for some $n' \in \mathbb{N}$, $\alpha \in \mathbb{R}$, $\mu > 0$ *and* $\nu > 1/2$.

Then, for any $T > \frac{2\pi}{\gamma}$ *there exists* $c(T) > 0$ *such that*

$$\int_0^T \Big| \sum_{n=1}^{\infty} R_n e^{r_n t} + C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega_n} t} \Big|^2 \, dt \geq c(T) \sum_{n=1}^{\infty} (1 + e^{-2(\Im\omega_n - \alpha)T}) |C_n|^2 \,. \tag{3.1}$$

Now, we are in condition to show our first result.

Theorem 3.2. *Let* $\eta > 3\beta/2$ *and* $T > 2\pi$. *If* (u_1, u_2) *is a solution of problem* (2.1) *with* $(u_{10}, u_{11}) \in H_0^1(0,\pi) \times L^2(0,\pi)$ *and*

$$u_{20} = u_{21} = 0 \,, \tag{3.2}$$

then we have

$$\int_0^T |u_{1x}(t,\pi)|^2 \, dt \geq c(T) \big(\|u_{10}\|^2_{H_0^1(0,\pi)} + \|u_{11}\|^2 \big) \,, \tag{3.3}$$

where $c(T)$ *is a positive constant.*

Proof. If we bear in mind formulas (2.3), from (3.2) it follows

$$\alpha_{2n} = \chi_{2n} = 0 \qquad \text{for any } n \in \mathbb{N} \,,$$

whence, in virtue of (2.6) we get

$$D_n = 0 \qquad \text{for any } n \in \mathbb{N} \,.$$

Therefore, from (2.4) it follows

$$u_{1x}(t,\pi) = \sum_{n=1}^{\infty} (-1)^n n \big(R_n e^{r_n t} + C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega_n} t} \big) \,.$$

Now, we can employ Theorem 3.1 ($\gamma = \nu = 1$, $\alpha = \beta/2$) for dealing with the previous sum. Indeed, applying formula (3.1) to $u_{1x}(t,\pi)$ we obtain

$$\int_0^T \big| u_{1x}(t,\pi) \big|^2 \, dt \geq c(T) \sum_{n=1}^{\infty} n^2 |C_n|^2 \,,$$

whence, in virtue of (2.8) and (2.2) our statement follows. □

We note that, in the above result, we have to assume the condition (3.2) just as in the non-integral case, see [7, Theorem 1.2].

Before studying the observation of the second component, we have to show an inverse estimate regarding only the second component of the solution of problem (2.1), see (2.4).

Proposition 3.3. *Let* $\{p_n\}_{n\in\mathbb{N}}$ *be a sequence of pairwise distinct nonzero complex numbers, satisfying*

$$\lim_{n\to\infty} (\Re p_{n+1} - \Re p_n) = +\infty \,, \qquad \lim_{n\to\infty} \Im p_n = 0 \,.$$

Then, for any $T > 0$ *there exists a positive constant* $c(T)$ *such that*

$$\int_0^T \left| \sum_{n=1}^{\infty} \Big(d_n D_n e^{i p_n t} + \overline{d_n D_n} e^{-i\overline{p_n} t} \Big) - \frac{2\beta}{A} e^{-\eta t} \sum_{n=1}^{\infty} n^2 \Re \frac{D_n}{\eta + i p_n} \right|^2 dt$$
$$\geq c(T) \left(\sum_{n=1}^{\infty} |p_n|^4 |D_n|^2 + \left| \sum_{n=1}^{\infty} n^2 \Re \frac{D_n}{\eta + i p_n} \right|^2 \right) . \tag{3.4}$$

To prove that inverse estimate, we need some preliminary results. The first step is to state inverse and direct inequalities for Fourier series without a finite number of terms. The following result follows from [14, Propositions 5.8–5.9].

Lemma 3.4. *There exist $n_0 \in \mathbb{N}$ such that for any sequence $\{E_n\}$ of complex numbers, with $\sum_{n=1}^{\infty} |E_n|^2 < +\infty$ and $E_n = 0$ for any $n < n_0$, we have*

$$\int_0^T \left| \sum_{n=n_0}^{\infty} E_n e^{ip_n t} + \overline{E_n} e^{-i\overline{p_n} t} \right|^2 dt \asymp \sum_{n=n_0}^{\infty} |E_n|^2 . \tag{3.5}$$

The second step is to recover the finite number of terms in the series. To this end, we need to establish a so-called Haraux type estimate.

For the sake of completeness, we introduce a family of integral operators which annihilate a finite number of terms in the Fourier series. We begin by recalling the definition of operators, which was given in [12] and is slightly different from that introduced in [4] and [6].

Given $\delta > 0$ and $z \in \mathbb{C}$ arbitrarily, the symbol $I_{\delta,z}$ denotes the linear operator defined as follows: for every continuous function $u : \mathbb{R} \to \mathbb{C}$ the function $I_{\delta,z}u : \mathbb{R} \to \mathbb{C}$ is given by the formula

$$I_{\delta,z}u(t) := u(t) - \frac{1}{\delta} \int_0^{\delta} e^{-izs} u(t+s)\, ds , \qquad t \in \mathbb{R} . \tag{3.6}$$

For the reader's convenience, we list some known properties verified by the operators $I_{\delta,z}$, see e.g. [4, 6, 12].

Lemma 3.5. *For any $\delta > 0$ and $z \in \mathbb{C}$ the following statements hold true.*

(i) $I_{\delta,z}(e^{izt}) = 0$.

(ii) *For any $z' \in \mathbb{C}$, $z' \neq z$, we have*

$$I_{\delta,z}(e^{iz't}) = \left(1 - \frac{e^{i(z'-z)\delta} - 1}{i(z'-z)\delta}\right) e^{iz't} .$$

(iii) *The linear operators $I_{\delta,z}$ commute: for any $\delta' > 0$ and $z' \in \mathbb{C}$ we have*

$$I_{\delta,z} \circ I_{\delta',z'} = I_{\delta',z'} \circ I_{\delta,z} ,$$

where the symbol $\circ$ denotes the standard composition among operators.

(iv) *For any $T > 0$ and continuous function $u : \mathbb{R} \to \mathbb{C}$ we have*

$$\int_0^T |I_{\delta,z}u(t)|^2 \, dt \leq 2(1 + e^{2|\Im z|\delta}) \int_0^{T+\delta} |u(t)|^2 \, dt . \tag{3.7}$$

The following result is similar to [6, Prop. 1.9], but due to the presence of another term (see inequality (3.11) below), we prefer to prove it, to make also the paper as self-contained as possible.

Proposition 3.6. *Let $\{p_n\}_{n\in\mathbb{N}}$ be a sequence of pairwise distinct nonzero complex numbers such that $p_n \neq i\eta$, for any $n \in \mathbb{N}$,*

$$\lim_{n\to\infty} |p_n| = +\infty , \quad \text{the sequence } \{\Im p_n\} \text{ is bounded} . \tag{3.8}$$

Assume that there exists $n_0 \in \mathbb{N}$ such that for any sequence $\{E_n\}$ the estimates

$$\int_0^T \left| \sum_{n=n_0}^{\infty} E_n e^{ip_n t} + \overline{E_n} e^{-i\overline{p_n} t} \right|^2 dt \geq c_1 \sum_{n=n_0}^{\infty} |E_n|^2 , \tag{3.9}$$

$$\int_0^T \left| \sum_{n=n_0}^{\infty} E_n e^{ip_n t} + \overline{E_n} e^{-i\overline{p_n} t} \right|^2 dt \leq c_2 \sum_{n=n_0}^{\infty} |E_n|^2 , \tag{3.10}$$

hold for some constants $c_1, c_2 > 0$.

Then, there exists $C_1 > 0$ such that for any sequence $\{E_n\}$ and $\mathcal{D} \in \mathbb{R}$ the estimate

$$\int_0^T \left| \sum_{n=1}^{\infty} \left(E_n e^{ip_n t} + \overline{E_n} e^{-i\overline{p_n} t} \right) + \mathcal{D} e^{-\eta t} \right|^2 dt \geq C_1 \left(\sum_{n=1}^{\infty} |E_n|^2 + |\mathcal{D}|^2 \right) \tag{3.11}$$

is true.

Proof. To begin with, we will transform

$$u(t) = \sum_{n=1}^{\infty} \left(E_n e^{ip_n t} + \overline{E_n} e^{-i\overline{p_n} t} \right) + \mathcal{D} e^{-\eta t}$$

into a function without those terms corresponding to indices $n = 1, \ldots, n_0 - 1$ and without the term $\mathcal{D} e^{-\eta t}$, so we can apply the assumptions (3.9) and (3.10).

To this end, we fix $\varepsilon > 0$ and choose $\delta \in \left(0, \frac{\varepsilon}{2n_0}\right)$, where n_0 is the integer for which the estimates (3.9) and (3.10) hold. Let us denote by $\mathbb{I}$ the composition of $I_{\delta,i\eta}$ and all linear operators $I_{\delta,p_j} \circ I_{\delta,-\overline{p_j}}$, $j = 1, \ldots, n_0 - 1$. We note that by Lemma 3.5 (iii) the definition of $\mathbb{I}$ does not depend on the order of the operators.

By using Lemma 3.5, we get

$$\mathbb{I}u(t) = \sum_{n=n_0}^{\infty} \left(E'_n e^{ip_n t} + \overline{E'_n} e^{-i\overline{p_n} t} \right)$$

where

$$E'_n := \left(1 - \frac{e^{i(p_n - i\eta)\delta} - 1}{i(p_n - i\eta)\delta} \right) \prod_{j=1}^{n_0 - 1} \prod_{z \in \{p_j, -\overline{p_j}\}} \left(1 - \frac{e^{i(p_n - z)\delta} - 1}{i(p_n - z)\delta} \right) E_n .$$

Therefore, estimate (3.9) holds for function $\mathbb{I}u(t)$, that is

$$\int_0^T |\mathbb{I}u(t)|^2 \, dt \geq c_1 \sum_{n=n_0}^{\infty} |E'_n|^2 . \tag{3.12}$$

Next, we observe that we can choose $\delta \in \left(0, \frac{\varepsilon}{2n_0}\right)$ such that for any $n \geq n_0$ none of the products

$$\left(1 - \frac{e^{i(p_n - i\eta)\delta} - 1}{i(p_n - i\eta)\delta} \right) \prod_{j=1}^{n_0 - 1} \prod_{z \in \{p_j, -\overline{p_j}\}} \left(1 - \frac{e^{i(p_n - z)\delta} - 1}{i(p_n - z)\delta} \right) \tag{3.13}$$

vanishes. Indeed, that is possible because the analytic function

$$w \longmapsto 1 - \frac{e^w - 1}{w}$$

does not vanish identically, and hence, keeping in mind that every number $p_n - z$ with $z \in \{i\eta, p_j, -\overline{p_j} : j = 1, \dots, n_0 - 1\}$ is different from zero, we have to exclude only a countable set of values of δ.

Then, we note that there exists a constant $c' > 0$ such that for any $n \geq n_0$

$$\left| \left(1 - \frac{e^{i(p_n - i\eta)\delta} - 1}{i(p_n - i\eta)\delta}\right) \prod_{j=1}^{n_0-1} \prod_{z \in \{p_j, -\overline{p_j}\}} \left(1 - \frac{e^{i(p_n - z)\delta} - 1}{i(p_n - z)\delta}\right) \right|^2 \geq c' \,. \tag{3.14}$$

Actually, it is sufficient to observe that for $z \in \{i\eta, p_j, -\overline{p_j}\}$, we have

$$\left| \frac{e^{i(p_n - z)\delta} - 1}{i(p_n - z)\delta} \right| \leq \frac{e^{-\Im(p_n - z)\delta} + 1}{|p_n - z|\delta} \to 0 \qquad \text{as } n \to \infty \,,$$

thanks to (3.8). As a result, the product in (3.13) tends to 1 as $n \to \infty$ and hence, for example, we can take it greater than $1/2$ for n large enough. Therefore, (3.12) and (3.14) yield

$$\int_0^T |\mathbb{I}u(t)|^2 \, dt \geq c' c_1 \sum_{n=n_0}^{\infty} |E_n|^2 \,. \tag{3.15}$$

On the other hand, applying (3.7) repeatedly with $z = i\eta$, $z = p_j$ and $z = -\overline{p_j}$, $j = 1, \dots, n_0 - 1$, we have

$$\int_0^T |\mathbb{I}u(t)|^2 \, dt \leq 2^{2n_0 - 1}(1 + e^{2|\eta|\delta}) \prod_{j=1}^{n_0-1} (1 + e^{2|\Im p_j|\delta})^2 \int_0^{T + (2n_0 - 1)\delta} |u(t)|^2 dt \,.$$

From the above inequality, by using (3.15) and $2n_0\delta < \varepsilon$, it follows

$$\sum_{n=n_0}^{\infty} |E_n|^2 \leq \frac{2^{2n_0-1}}{c' c_1}(1 + e^{\eta\varepsilon/n_0}) \prod_{j=1}^{n_0-1} (1 + e^{|\Im p_j|\varepsilon/n_0})^2 \int_0^{T+\varepsilon} |u(t)|^2 dt \,,$$

whence, passing to the limit as $\varepsilon \to 0^+$, we have

$$\sum_{n=n_0}^{\infty} |E_n|^2 \leq \frac{2^{4n_0-2}}{c' c_1} \int_0^T |u(t)|^2 \, dt \,. \tag{3.16}$$

Moreover, thanks to the triangle inequality, we get

$$\begin{aligned} \int_0^T \left| \sum_{n=1}^{n_0-1} \left(E_n e^{ip_n t} + \overline{E_n} e^{-i\overline{p_n} t}\right) + \mathcal{D} e^{-\eta t} \right|^2 dt &= \int_0^T \left| u(t) - \sum_{n=n_0}^{\infty} \left(E_n e^{ip_n t} + \overline{E_n} e^{-i\overline{p_n} t}\right) \right|^2 dt \\ &\leq 2 \int_0^T |u(t)|^2 dt + 2 \int_0^T \left| \sum_{n=n_0}^{\infty} E_n e^{ip_n t} + \overline{E_n} e^{-i\overline{p_n} t} \right|^2 dt \,. \end{aligned}$$

By using (3.10) and (3.16) in the previous inequality, we have

$$\begin{aligned} \int_0^T \left| \sum_{n=1}^{n_0-1} \left(E_n e^{ip_n t} + \overline{E_n} e^{-i\overline{p_n} t}\right) + \mathcal{D} e^{-\eta t} \right|^2 dt &\leq 2 \int_0^T |u(t)|^2 \, dt + 2c_2 \sum_{n=n_0}^{\infty} |E_n|^2 \\ &\leq 2 \left(1 + c_2 \frac{2^{4n_0-2}}{c' c_1}\right) \int_0^T |u(t)|^2 \, dt \,. \end{aligned} \tag{3.17}$$

Let us note that the expression

$$\int_0^T \left| \sum_{n=1}^{n_0-1} \left(E_n e^{ip_n t} + \overline{E_n} e^{-i\overline{p_n} t} \right) + \mathcal{D} e^{-\eta t} \right|^2 dt$$

is a positive semidefinite quadratic form of the variable $\left(\{E_n\}_{n<n_0}, \mathcal{D} \right) \in \mathbb{C}^{n_0-1} \times \mathbb{R}$. Moreover, it is positive *definite*, because the functions $e^{-\eta t}$, $e^{ip_n t}$, $n < n_0$, are linearly independent. Hence, there exists a constant $c'' > 0$ such that

$$\int_0^T \left| \sum_{n=1}^{n_0-1} \left(E_n e^{ip_n t} + \overline{E_n} e^{-i\overline{p_n} t} \right) + \mathcal{D} e^{-\eta t} \right|^2 dt \geq c'' \left(\sum_{n=1}^{n_0-1} |E_n|^2 + |\mathcal{D}|^2 \right).$$

So, from (3.17) and the above inequality we deduce that

$$\sum_{n=1}^{n_0-1} |E_n|^2 + |\mathcal{D}|^2 \leq \frac{2}{c''} \left(1 + c_2 \frac{2^{4n_0-2}}{c' c_1} \right) \int_0^T |u(t)|^2 \, dt \,.$$

Finally, the above estimate and (3.16) yield the required inequality (3.11). □

Finally, we are able to prove Proposition 3.3

Proof of Proposition 3.3. If we consider the sequence $\{d_n D_n\}$, thanks to (2.7) and (2.9) we have $\sum_{n=1}^{\infty} |d_n D_n|^2 < +\infty$. Therefore, we can apply Lemma 3.4 to $E_n = d_n D_n$: for a suitable integer n_0 one has

$$\int_0^T \left| \sum_{n=n_0}^{\infty} d_n D_n e^{ip_n t} + \overline{d_n D_n} e^{-i\overline{p_n} t} \right|^2 dt \asymp \sum_{n=n_0}^{\infty} |d_n D_n|^2 \,, \tag{3.18}$$

that is, the estimates (3.9) and (3.10) hold for the sequence $\{d_n D_n\}$. Moreover, we note that the sum

$$\sum_{n=1}^{\infty} n^2 \Re \frac{D_n}{\eta + ip_n}$$

is a real number. Indeed, in view of the inequality

$$\sum_{n=1}^{\infty} n^2 \left| \Re \frac{D_n}{\eta + ip_n} \right| \leq \sum_{n=1}^{\infty} n^2 \frac{|D_n|}{|\eta + ip_n|} \,,$$

we get

$$\begin{aligned}
\left| \sum_{n=1}^{\infty} n^2 \left| \Re \frac{D_n}{\eta + ip_n} \right| \right|^2 &\leq \sum_{n,m=1}^{\infty} \frac{n^2 |D_n|}{|\eta + ip_m|} \frac{m^2 |D_m|}{|\eta + ip_n|} \\
&\leq \frac{1}{2} \sum_{n=1}^{\infty} n^4 |D_n|^2 \sum_{m=1}^{\infty} \frac{1}{(\eta - \Im p_m)^2 + \Re p_m^2} \\
&\quad + \frac{1}{2} \sum_{m=1}^{\infty} m^4 |D_m|^2 \sum_{n=1}^{\infty} \frac{1}{(\eta - \Im p_n)^2 + \Re p_n^2} \\
&= \sum_{n=1}^{\infty} \frac{1}{(\eta - \Im p_n)^2 + \Re p_n^2} \sum_{n=1}^{\infty} n^4 |D_n|^2 < +\infty \,,
\end{aligned}$$

thanks to (2.5) and (2.9).

At last, we are in condition to apply Proposition 3.6: the estimate (3.11) holds when $E_n = d_n D_n$ and $\mathcal{D} = -\frac{2\beta}{A} \sum_{n=1}^{\infty} n^2 \Re \frac{D_n}{\eta + ip_n}$; in consequence, thanks also to (2.7), it follows (3.4). □

Finally, we are able to show a partial observability result for the second component. We note that, unlike Theorem 3.2, we do not need to assume that the initial data u_{10} and u_{11} are null.

Theorem 3.7. *Let $T > 0$. For any $(u_{10}, u_{11}) \in H_0^1(0,\pi) \times L^2(0,\pi)$ and $(u_{20}, u_{21}) \in H_0^1(0,\pi) \times H^{-1}(0,\pi)$, if (u_1, u_2) is a solution of problem (2.1), then we have*

$$\int_0^T |u_{2x}(t,\pi)|^2 \, dt \geq c(T)\left(\|u_{20}\|^2_{H_0^1(0,\pi)} + \|u_{21}\|^2_{H^{-1}(0,\pi)}\right), \tag{3.19}$$

where $c(T)$ is a positive constant.

Proof. From (2.4) it follows

$$u_{2x}(t,\pi) = \sum_{n=1}^{\infty} (-1)^n n \left(d_n D_n e^{ip_n t} + \overline{d_n}\,\overline{D_n} e^{-i\overline{p_n} t} - \frac{2\beta}{A} n^2 \Re \frac{D_n}{\eta + ip_n} e^{-\eta t} \right).$$

We can employ Proposition 3.3 to treat the previous sum. Indeed, applying formula (3.4) to $u_{2x}(t,\pi)$ we obtain

$$\int_0^T \left| u_{2x}(t,\pi) \right|^2 dt \geq c(T) \sum_{n=1}^{\infty} n^2 |p_n|^4 |D_n|^2,$$

whence, in virtue of (2.9) and (2.3), our statement follows. □

In conclusion, the partial observability of the first component has been established in Theorem 3.2, while by Theorem 3.7 and assuming $u_{10} = u_{11} = 0$ the partial observability of the second component follows.

References

[1] C. M. Dafermos, Asymptotic stability in viscoelasticity, *Arch. Rational Mech. Anal.* **37**(1970), 297–308. MR0281400

[2] C. M. Dafermos, An abstract Volterra equation with applications to linear viscoelasticity, *J. Differential Equations* **7**(1970), 554–569. MR0259670

[3] G. Gripenberg, S. O. Londen, O. J. Staffans, *Volterra integral and functional equations*, Encyclopedia of Mathematics and its Applications, Vol. 34, Cambridge University Press, Cambridge, 1990. MR1050319; url

[4] A. Haraux, Séries lacunaires et contrôle semi-interne des vibrations d'une plaque rectangulaire, *J. Math. Pures Appl. (9)* **68**(1989), 457–465. MR1046761

[5] A. E. Ingham, Some trigonometrical inequalities with applications to the theory of series, *Math. Z.* **41**(1936), 367–379. MR1545625; url

[6] V. Komornik, P. Loreti, Ingham type theorems for vector-valued functions and observability of coupled linear system, *SIAM J. Control Optim.* **37**(1998), 461–485. MR1655862; url

[7] V. Komornik, P. Loreti, Partial observability of coupled linear systems, *Acta Math. Hungar.* **86**(2000), 49–74. MR1728589; url

[8] V. Komornik, P. Loreti, *Fourier series in control theory*, Springer Monographs in Mathematics, Springer-Verlag, New York, 2005. MR2114325

[9] J. E. Lagnese, J.-L. Lions, *Modelling analysis and control of thin plates*, Recherches en Mathématiques Appliquées, Vol. 6, Masson, Paris, 1988. MR953313

[10] J.-L. Lions, *Contrôlabilité exacte, perturbations et stabilisation de systèmes Distribués. Tome 1. Contrôlabilité exacte* (in French), with appendices by E. Zuazua, C. Bardos, G. Lebeau and J. Rauch, Recherches en Mathématiques Appliquées, Vol. 8, Masson, Paris, 1988. MR953547

[11] J.-L. Lions, *Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués. Tome 2. Perturbations* (in French), Recherches en Mathématiques Appliquées, Vol. 9, Masson, Paris, 1988. MR963060

[12] P. Loreti, D. Sforza, Reachability problems for a class of integro-differential equations, *J. Differential Equations* **248**(2010), 1711–1755. MR2593605; url

[13] P. Loreti, D. Sforza, Multidimensional controllability problems with memory, in: *Modern aspects of the theory of partial differential equations (eds. M. Ruzhansky and J. Wirth)*, Operator Theory: Advances and Applications, Vol. 216, Birkhäuser/Springer, Basel, 2011, 261–274. MR2858874; url

[14] P. Loreti, D. Sforza, Control problems for weakly coupled systems with memory, *J. Differential Equations* **257**(2014), 1879–1938. MR3227286; url

[15] J. Prüss, *Evolutionary integral equations and applications*, Monographs in Mathematics, Vol. 87, Birkhäuser Verlag, Basel, 1993. MR1238939; url

[16] M. Renardy, W. J. Hrusa, J. A. Nohel, *Mathematical problems in viscoelasticity*, Pitman Monographs and Surveys in Pure and Applied Mathematics, Vol. 35, Longman Scientific & Technical, Harlow; John Wiley & Sons, Inc., New York, 1988. MR919738

[17] D. L. Russell, Controllability and stabilizability theory for linear partial differential equations: recent progress and open questions, *SIAM Rev.* **20**(1978), 639–739. MR508380; url