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EXISTENCE OF PERIODIC SOLUTIONS FOR A CLASS OF EVEN ORDER DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENT

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Honoring the Career of John Graef on the Occasion of His Sixty-Seventh Birthday

Abstract

Using Mawhin's continuation theorem we establish the existence of periodic solutions for a class of even order differential equations with deviating argument.

Key words and phrases: Even order differential equation, deviating argument, Mawhin's continuation theorem, Green's function.
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1 Introduction

In this paper, we discuss the even order differential equation with deviating argument of the form

$$x^{(2n)}(t) + \sum_{i=0}^{2n-2} a_i(t)x^{(i)}(t) + g(x(t-\tau(t))) = p(t), \tag{1}$$

where $\tau(t)$, $a_i(t)$ $(i = 0, 1, 2, \cdots, n)$, $p(t)$ are real continuous functions defined on $\mathbf{R}$ with positive period T and $a_{2k-2}(t) > 0$ $(k = 1, 2, \cdots, n)$ for $t \in \mathbf{R}$, and $g(x)$ is a real continuous function defined on $\mathbf{R}$.

Periodic solutions for differential equations were studied in [2-12] and we note that most of the results in the literatue concern lower order problems. There are only a few papers [1,13,14] which discuss higher order problems.

For the sake of completeness, we first state Mawhin's continuation theorem [3]. Let X and Y be two Banach space and $L : DomL \subset X \longrightarrow Y$ is a linear mapping and

$N : X \longrightarrow Y$ is a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $dimKerL = codimImL < +\infty$, and ImL is closed in Y. If L is a Fredholm mapping of index zero, there exist continuous projectors $P : X \longrightarrow X$ and $Q : Y \longrightarrow Y$ such that $ImP = KerL$ and $ImL = KerQ = Im(I - Q)$. It follows that $L|_{DomL\cap KerP} : (I - P)X \longrightarrow ImL$ has an inverse which will be denoted by K_P. If Ω is an open and bounded subset of X, the mapping N will be called $L-$compact on Ω if $QN(\overline{\Omega})$ is bounded and $\overline{K_P(I - Q)N(\overline{\Omega})}$ is compact. Since ImQ is isomorphic to $KerL$, there exists an isomorphism $J : ImQ \longrightarrow KerL$. The following theorem is called Mawhin's continuation theorem (see [3]).

Theorem 1.1 *Let L be a Fredholm mapping of index zero, and let N be $L-$compact on $\overline{\Omega}$. Suppose*

(1) for each $\lambda \in (0, 1)$ and $x \in \partial\Omega, Lx \neq \lambda Nx$, and

(2) for each $x \in \partial\Omega \cap Ker(L), QNx \neq 0$ and $deg(QN, \Omega \cap Ker(L), 0) \neq 0$.

Then the equation $Lx = Nx$ has at least one solution in $\overline{\Omega} \cap D(L)$.

2 Main Result

Now we make the following assumptions on $a_i(t)$:

(i) $M_{2k-2} = \max_{t\in[0,T]} a_{2k-2}(t) \geq a_{2k-2}(t) \geq m_{2k-2} = \min_{t\in[0,T]} a_{2k-2}(t) > 0, (k = 1, 2, \cdots, n)$ for each $t \in [0, T]$;

(ii) $M_{2n-2} < (\frac{\pi}{T})^2$ and $\frac{M_{2n-2i}}{M_{2n-2i+2}} < (\frac{\pi}{T})^2 \quad (i = 2, 3, \cdots, n)$;

(iii) There exists a positive constant r with $m_0 > r$, such that with $A - \frac{2M_0+m_0+r}{2(m_0-r)} B > 0$ and $1 - A^* > 0$, where $A = 1 - A^*$,

$$B = M_1(\tfrac{T}{2})^{2n-2} + (M_2 - m_2)(\tfrac{T}{2})^{2n-3} + M_3(\tfrac{T}{2})^{2n-4} + (M_4 - m_4)(\tfrac{T}{2})^{2n-5}$$
$$+ \cdots + M_{2n-3}(\tfrac{T}{2})^2 + (M_{2n-2} - m_{2n-2})\tfrac{T}{2},$$
$$A^* = [M_{2n-2}(\tfrac{T}{2})^2 + M_{2n-3}(\tfrac{T}{2})^3 + M_{2n-4}(\tfrac{T}{2})^4 + \cdots + M_2(\tfrac{T}{2})^{2n-2} + M_1(\tfrac{T}{2})^{2n-1}]$$

and $M_{2k-1} = \max_{t\in[0,T]} |a_{2k-1}(t)| \quad (k = 1, 2, \cdots, n - 1)$.

Our main result is the following theorem.

Theorem 2.1 *Under the assumptions (i), (ii) and (iii), if*

$$\lim_{|x|\to\infty} sup|\frac{g(x)}{x}| \leq r \tag{2}$$

and

$$\lim_{|x|\to\infty} sgn(x)g(x) = +\infty, \tag{3}$$

then Eq.(1) has at least one $T-$periodic solution.

In order to prove the main theorem we need some preliminaries. Set

$$X := \{x | x \in C^{2n-1}(\mathbf{R}, \mathbf{R}), x(t+T) = x(t), \forall t \in \mathbf{R}\}$$

and $x^{(0)}(t) = x(t)$, and define the norm on X by

$$||x|| = \max_{0 \le j \le 2n-1} \max_{t \in [0,T]} |x^{(j)}(t)|,$$

and set

$$Y := \{y | y \in C(\mathbf{R}, \mathbf{R}), y(t+T) = y(t), \forall t \in \mathbf{R}\}.$$

We define the norm on Y by $||y||_0 = \max_{t \in [0,T]} |y(t)|$. Thus both $(X, ||\cdot||)$ and $(Y, ||\cdot||_0)$ are Banach spaces.

Remark 2.1 *If $x \in X$, then it follows that $x^{(i)}(0) = x^{(i)}(T)$ $(i = 0, 1, 2, \cdots, 2n-1)$.*

Define the operators $L : X \longrightarrow Y$ and $N : X \longrightarrow Y$, respectively, by

$$Lx(t) = x^{(2n)}(t), \quad t \in \mathbf{R}, \tag{4}$$

and

$$Nx(t) = p(t) - \sum_{i=0}^{2n-2} a_i(t) x^{(i)}(t) - g(x(t - \tau(t))), t \in \mathbf{R}. \tag{5}$$

Clearly,

$$KerL = \{x \in X : x(t) = c \in \mathbf{R}\} \tag{6}$$

and

$$ImL = \{y \in Y : \textstyle\int_0^T y(t)dt = 0\} \tag{7}$$

is closed in Y. Thus L is a Fredholm mapping of index zero.

Let us define $P : X \to X$ and $Q : Y \to Y/Im(L)$, respectively, by

$$Px(t) = \tfrac{1}{T} \textstyle\int_0^T x(t)dt = x(0), \quad t \in \mathbf{R}, \tag{8}$$

for $x = x(t) \in X$ and

$$Qy(t) = \tfrac{1}{T} \textstyle\int_0^T y(t)dt, \quad t \in \mathbf{R} \tag{9}$$

for $y = y(t) \in Y$. It is easy to see that $ImP = KerL$ and $ImL = KerQ = Im(I - Q)$. It follows that $L|_{DomL \cap KerP} : (I - P)X \longrightarrow ImL$ has an inverse which will be denoted by K_P.

Furthermore for any $y = y(t) \in ImL$, if $n = 1$, it is well-known that

$$K_P y(t) = -\tfrac{t}{T} \textstyle\int_0^T du \int_0^u y(s)ds + \int_0^t du \int_0^u y(s)ds. \tag{10}$$

If $n > 1$, let $x(t) \in DomL \cap KerP$ be such that $K_P y(t) = x(t)$. Then $x^{(2n)}(t) = y(t)$,

$$x^{(2n-1)}(t) = x^{(2n-1)}(0) + \int_0^t x^{(2n)}(s)ds \tag{11}$$

and

$$x^{(2n-2)}(t) = x^{(2n-2)}(0) + x^{(2n-1)}(0)t + \int_0^t du \int_0^u x^{(2n)}(s)ds. \tag{12}$$

Since $x^{(2n-2)}(T) = x^{(2n-2)}(0)$, we have

$$x^{(2n-1)}(0)T + \int_0^T du \int_0^u x^{(2n)}(s)ds = 0$$

or

$$x^{(2n-1)}(0) = -\frac{1}{T}\int_0^T du \int_0^u x^{(2n)}(s)ds.$$

From (12), we have

$$x^{(2n-2)}(t) = x^{(2n-2)}(0) - \frac{t}{T}\int_0^T du \int_0^u x^{(2n)}(s)ds + \int_0^t du \int_0^u x^{(2n)}(s)ds. \tag{13}$$

Now since $\int_0^T x^{(2n-2)}(s)ds = 0$, from (13) we have

$$x^{(2n-2)}(0)T - \frac{T}{2}\int_0^T du \int_0^u x^{(2n)}(s)ds + \int_0^T dw \int_0^w du \int_0^u x^{(2n)}(s)ds = 0,$$

or

$$x^{(2n-2)}(0) = \frac{1}{2}\int_0^T du \int_0^u x^{(2n)}(s)ds - \frac{1}{T}\int_0^T dw \int_0^w du \int_0^u x^{(2n)}(s)ds. \tag{14}$$

From (13) and (14), we have

$$\begin{aligned} x^{(2n-2)}(t) &= \frac{1}{2}\int_0^T du \int_0^u x^{(2n)}(s)ds - \frac{1}{T}\int_0^T dw \int_0^w du \int_0^u x^{(2n)}(s)ds \\ &- \frac{t}{T}\int_0^T du \int_0^u x^{(2n)}(s)ds + \int_0^t du \int_0^u x^{(2n)}(s)ds \\ &= \left(\frac{1}{2} - \frac{t}{T}\right)\int_0^T du \int_0^u x^{(2n)}(s)ds + \int_0^t du \int_0^u x^{(2n)}(s)ds \\ &- \frac{1}{T}\int_0^T dw \int_0^w du \int_0^u x^{(2n)}(s)ds. \end{aligned} \tag{15}$$

Let $y_0(t) = y(t)$ and $y_1(t) = x^{(2n-2)}(t)$. Since $y(t) = x^{(2n)}(t)$, we have from (15) that

$$\begin{aligned} x^{(2n-2)}(t) = y_1(t) &= \left(\frac{1}{2} - \frac{t}{T}\right)\int_0^T du \int_0^u y_0(s)ds \\ &+ \int_0^t du \int_0^u y_0(s)ds - \frac{1}{T}\int_0^T dw \int_0^w du \int_0^u y_0(s)ds. \end{aligned} \tag{16}$$

From (16), we obtain

$$x^{(2n-3)}(t) = x^{(2n-3)}(0) + \int_0^t y_1(s)ds$$

and

$$x^{(2n-4)}(t) = x^{(2n-4)}(0) + x^{(2n-3)}(0)t + \int_0^t du \int_0^u y_1(s)ds. \tag{17}$$

Since $x^{(2n-4)}(T) = x^{(2n-4)}(0)$, we have from (17) that

$$x^{(2n-3)}(0) = -\frac{1}{T}\int_0^T du \int_0^u y_1(s)ds. \tag{18}$$

Since $\int_0^T x^{(2n-4)}(s)ds = 0$, we have from (17) that

$$x^{(2n-4)}(0) = \frac{1}{2}\int_0^T du \int_0^u y_1(s)ds - \frac{1}{T}\int_0^T dw \int_0^w du \int_0^u y_1(s)ds. \tag{19}$$

Let $y_2(t) = x^{(2n-4)}(t)$ and we have from (17)-(19) that

$$\begin{aligned} x^{(2n-4)}(t) = y_2(t) &= \left(\frac{1}{2} - \frac{t}{T}\right)\int_0^T du \int_0^u y_1(s)ds \\ &+ \int_0^t du \int_0^u y_1(s)ds - \frac{1}{T}\int_0^T dw \int_0^w du \int_0^u y_1(s)ds. \end{aligned}$$

Let $y_i(t) = x^{(2n-2i)}(t)$ $(i = 1, 2, \cdots, n-1)$ and as above it is easy to check that

$$\begin{aligned} x^{(2n-2i)}(t) = y_i(t) &= \left(\frac{1}{2} - \frac{t}{T}\right)\int_0^T du \int_0^u y_{i-1}(s)ds \\ &+ \int_0^t du \int_0^u y_{i-1}(s)ds - \frac{1}{T}\int_0^T dw \int_0^w du \int_0^u y_{i-1}(s)ds, \end{aligned}$$

for $(i = 1, 2, \cdots, n-1)$, and

$$y_n(t) = y_n(0) - \frac{t}{T}\int_0^T du \int_0^u y_{n-1}(s)ds + \int_0^t du \int_0^u y_{n-1}(s)ds.$$

Note that $y_n(t) = x(t) \in DomL \cap KerP$. Thus $y_n(0) = x(0) = 0$, and

$$K_P y(t) = -\frac{t}{T}\int_0^T du \int_0^u y_{n-1}(s)ds + \int_0^t du \int_0^u y_{n-1}(s)ds. \tag{20}$$

Let Ω be an open and bounded subset of X. In view of (5), (9) and (10) (or (20)), we can easily see that $QN(\overline{\Omega})$ is bounded and $\overline{K_P(I-Q)N(\overline{\Omega})}$ is compact. Thus the mapping N is $L-$compact on $\overline{\Omega}$. That is, we have the following result.

Lemma 2.1 *Let L, N, P and Q be defined by (4), (5), (8) and (9) respectively. Then L is a Fredholm mapping of index zero and N is $L-$compact on $\overline{\Omega}$, where Ω is any open and bounded subset of X.*

In order to prove our main result, we need the following Lemmas [6, 7]. The first result follows from [6 and Remark 2.1] and the second from [7].

Lemma 2.2 *Let $x(t) \in C^{(n)}(\mathbf{R}, \mathbf{R}) \cap C_T$. Then*

$$||x^{(i)}||_0 \le \frac{1}{2}\int_0^T |x^{(i+1)}(s)|ds, i = 1, 2, \cdots, n-1,$$

where $n \ge 2$ and $C_T := \{x | x \in C(R, R), x(t+T) = x(t), \forall t \in \mathbf{R}\}$.

Lemma 2.3 *Suppose that* M, λ *are positive numbers and satisfy* $0 < M < (\frac{\pi}{T})^2$ *and* $0 < \lambda < 1$ *, then for any function* φ *defined in* $[0, T]$, *the following problem*

$$\begin{cases} x^{''}(t) + \lambda M x(t) = \lambda\varphi(t), \\ x(0) = x(T), x^{'}(0) = x^{'}(T), \end{cases}$$

has a unique solution

$$x(t) = \int_0^T G(t,s)\lambda\varphi(s)ds,$$

where $\alpha = \sqrt{\lambda M}$, *and*

$$G(t,s) = \begin{cases} w(t-s), \quad (k-1)T \le s \le t \le kT, \\ w(T+t-s), \quad (k-1)T \le t \le s \le kT, \ (k \in \mathbf{N}), \end{cases}$$

with

$$w(t) = \frac{\cos\alpha(t-\frac{T}{2})}{2\alpha\sin\frac{\alpha T}{2}}.$$

Now, we consider the following auxiliary equation

$$x^{(2n)}(t) + \lambda\sum_{i=0}^{2n-2} a_i(t)x^{(i)}(t) + \lambda g(x(t-\tau(t))) = \lambda p(t), \tag{21}$$

where $0 < \lambda < 1$. We have

Lemma 2.4 *Suppose the conditions of Theorem 2.1 are satisfied. If* $x(t)$ *is a* $T-$*periodic solution of Eq.(21), then there are positive constants* D_i $(i = 0, 1, \cdots 2n-1)$, *which are independent of* λ, *such that*

$$||x^{(i)}||_0 \le D_i, \quad t \in [0,T] \ \textit{for}\ i = 0, 1, \cdots, 2n-1. \tag{22}$$

Proof. Suppose that $x(t)$ is a $T-$periodic solution of (21). By (2) of Theorem 2.1 we know that there exists a $\overline{M_1} > 0$, such that

$$|g(x(t))| \le r|x(t)|, \quad |x(t)| > \overline{M_1}, \quad t \in \mathbf{R}. \tag{23}$$

Set

$$E_1 = \{t : |x(t)| > \overline{M_1}, \quad t \in [0,T]\}, \tag{24}$$

$$E_2 = [0,T]\backslash E_1 \tag{25}$$

and

$$\rho = \max_{|x|\le\overline{M_1}} |g(x)|. \tag{26}$$

Let $\varepsilon = \frac{m_0 - r}{2}$. By (21), (23), (24), (25), (26) and Lemma 2.2, we obtain

$$\begin{aligned}
&||x^{(2n-1)}||_0 \le \tfrac{1}{2}\int_0^T |x^{(2n)}(s)|ds \\
&\le \tfrac{\lambda}{2}\int_0^T [|\sum_{i=0}^{2n-2} a_i(t)x^{(i)}(t)| + |g(x(t-\tau(t)))| + |p(t)|]dt \\
&\le \tfrac{\lambda T}{2}[M_{2n-2}||x^{(2n-2)}||_0 + M_{2n-3}||x^{(2n-3)}||_0 + \cdots + M_2||x^{(2)}||_0 + M_1||x^{(1)}||_0 \\
&+M_0||x||_0] + \tfrac{\lambda}{2}\int_0^T |g(x(t-\tau(t)))|dt + \tfrac{\lambda T}{2}||p||_0 \\
&\le \tfrac{T}{2}[M_{2n-2}\tfrac{T}{2} + M_{2n-3}(\tfrac{T}{2})^2 + \cdots + M_2(\tfrac{T}{2})^{2n-3} + M_1(\tfrac{T}{2})^{2n-2}]||x^{(2n-1)}||_0 + \\
&+\tfrac{T}{2}M_0||x||_0 + \tfrac{1}{2}[\int_{E_1}|g(x(t-\tau(t)))|dt + \int_{E_2}|g(x(t-\tau(t)))|dt] + \tfrac{T}{2}||p||_0 \\
&\le A^*||x^{(2n-1)}||_0 + \tfrac{T}{2}(M_0 + r + \varepsilon)||x||_0 + \tfrac{T}{2}C \qquad (27)\\
&= A^*x^{(2n-1)}||_0 + \tfrac{T}{4}(2M_0 + r + m_0)||x||_0 + \tfrac{T}{2}C,
\end{aligned}$$

where $C = (\rho + ||p||_0)$ and

$$\begin{aligned}
A^* &= [M_{2n-2}(\tfrac{T}{2})^2 + M_{2n-3}(\tfrac{T}{2})^3 + M_{2n-4}(\tfrac{T}{2})^4 \\
&+ \cdots + M_2(\tfrac{T}{2})^{2n-2} + M_1(\tfrac{T}{2})^{2n-1}].
\end{aligned}$$

Now from (27), we have

$$||x^{(2n-1)}||_0 \le (1 - A^*)^{-1}[\tfrac{T}{4}(2M_0 + r + m_0)||x||_0 + \tfrac{T}{2}C]. \qquad (28)$$

On the other hand, from (21) and Lemma 2.3, we get

$$\begin{aligned}
&x^{(2n-2)}(t) \\
&= \int_0^T G_1(t,t_1)\lambda[(M_{2n-2} - a_{2n-2}(t_1))x^{(2n-2)}(t_1) + p(t_1) \\
&-g(x(t-\tau(t_1)))]dt_1 - \lambda\int_0^T G_1(t,t_1)[\sum_{i=0}^{2n-3} a_i(t_1)x^{(i)}(t_1)]dt_1,
\end{aligned} \qquad (29)$$

where $\alpha_1 = \sqrt{\lambda M_{2n-2}}$, and

$$G_1(t,t_1) = \begin{cases} w_1(t-t_1), & (k-1)T \le t_1 \le t \le kT, \\ w_1(T+t-t_1), & (k-1)T \le t \le t_1 \le kT,\ (k \in \mathbf{N}), \end{cases} \qquad (30)$$

with

$$w_1(t) = \frac{\cos\alpha_1(t - \frac{T}{2})}{2\alpha_1 \sin\frac{\alpha_1 T}{2}} \qquad (31)$$

and

$$\int_0^T G_1(t,t_1)dt_1 = \frac{1}{\lambda M_{2n-2}}. \qquad (32)$$

From (29) and Lemma 2.3, we have

$$\begin{aligned}
&x^{(2n-4)}(t)\\
&= \lambda \int_0^T G_2(t,t_1) \int_0^T G_1(t_1,t_2)[p(t_2) - g(x(t-\tau(t_2)))]dt_2 dt_1\\
&+\lambda \int_0^T G_2(t,t_1) \int_0^T G_1(t_1,t_2)(M_{2n-2} - a_{2n-2}(t_2))x^{(2n-2)}(t_2)dt_2 dt_1\\
&+\int_0^T G_2(t,t_1)[\tfrac{M_{2n-4}}{M_{2n-2}} x^{(2n-4)}(t_1) - \lambda \int_0^T G_1(t_1,t_2)a_{2n-4}(t_2)x^{(2n-4)}(t_2)dt_2]dt_1\\
&-\lambda \int_0^T G_2(t,t_1) \int_0^T G_1(t_1,t_2)[\sum_{i=0}^{2n-5} a_i(t_1)x^{(i)}(t_2) + a_{2n-3}(t_2)x^{(2n-3)}(t_2)]dt_2 dt_1,
\end{aligned} \tag{33}$$

where $\alpha_2 = \sqrt{\frac{M_{2n-4}}{M_{2n-2}}}$, and

$$G_2(t,t_2) = \begin{cases} w_2(t-t_2), & (k-1)T \le t_2 \le t \le kT,\\ w_2(T+t-t_2), & (k-1)T \le t \le t_2 \le kT,\ (k \in \mathbf{N}),\end{cases} \tag{34}$$

with

$$w_2(t) = \frac{\cos \alpha_2 (t-\frac{T}{2})}{2\alpha_2 \sin \frac{\alpha_2 T}{2}} \tag{35}$$

and

$$\int_0^T G_2(t,t_2)dt_2 = \frac{M_{2n-2}}{M_{2n-4}}. \tag{36}$$

By induction, we have

$$\begin{aligned}
&x(t) = \lambda \int_0^T G_n(t,t_1) \cdots \int_0^T G_1(t_{n-1},t_n)[p(t_n) - g(x(t_n - \tau(t_n)))]dt_n \cdots dt_1\\
&+\lambda \int_0^T G_n(t,t_1) \cdots \int_0^T G_1(t_{n-1},t_n)(M_{2n-2} - a_{2n-2}(t_n))x^{(2n-2)}(t_n)dt_n \cdots dt_1\\
&+\int_0^T G_n(t,t_1) \cdots \int_0^T G_2(t_{n-2},t_{n-1})[\tfrac{M_{2n-4}}{M_{2n-2}} x^{(2n-4)}(t_{n-1}) -\\
&\lambda \int_0^T G_1(t_{n-1},t_n)a_{2n-4}x^{(2n-4)}(t_n)dt_n]dt_{n-1} \cdots dt_1\\
&+\int_0^T G_n(t,t_1) \cdots \int_0^T G_3(t_{n-3},t_{n-2})[\tfrac{M_{2n-6}}{M_{2n-4}} x^{(2n-6)}(t_{n-2}) -\\
&\lambda \int_0^T G_2(t_{n-2},t_{n-1}) \int_0^T G_1(t_{n-1},t_n)[a_{2n-6}x^{(2n-6)}(t_n)dt_n dt_{n-1}]dt_{n-2} \cdots dt_1\\
&+\cdots+\cdots\\
&+\int_0^T G_n(t,t_1)[\tfrac{M_0}{M_2} x(t_1) - \lambda \int_0^T G_{n-1}(t_1,t_2) \int_0^T G_{n-2}(t_2,t_3)\\
&\cdots \int_0^T G_1(t_{n-1},t_n)a_0(t_n)x(t_n)dt_n \cdots dt_2]dt_1\\
&-\lambda \int_0^T G_n(t,t_1) \cdots \int_0^T G_1(t_{n-1},t_n)[\sum_{k=1}^{n-1} a_{2k-1}(t_n)x^{(2k-1)}(t_n)]dt_n \cdots dt_1
\end{aligned} \tag{37}$$

where $\alpha_i = \sqrt{\frac{M_{2n-2i}}{M_{2n-2i+2}}} \quad (2 \le i \le n)$, and

$$G_i(t,t_i) = \begin{cases} w_i(t-t_i), & (k-1)T \le t_i \le t \le kT, \\ w_i(T+t-t_i), & (k-1)T \le t \le t_i \le kT, \ (k \in \mathbf{N}), \end{cases} \tag{38}$$

with

$$w_i(t) = \frac{\cos\alpha_i(t-\frac{T}{2})}{2\alpha_i \sin\frac{\alpha_i T}{2}} \tag{39}$$

and

$$\int_0^T G_i(t,t_i)dt_i = \frac{M_{2n-2i+2}}{M_{2n-2i}} \quad (2 \le i \le n). \tag{40}$$

From (32), (37), (40) and Lemma 2.2, we obtain

$$\begin{aligned}
&||x||_0 \\
&\le \max_{t\in[0,T]} \lambda \int_{E_1} |G_n(t,t_1)| \cdots \int_0^T |G_1(t_{n-1},t_n)||p(t_n) - g(x(t_n - \tau(t_n)))|dt_n \cdots dt_1 \\
&+ \max_{t\in[0,T]} \lambda \int_{E_2} |G_n(t,t_1)| \cdots \int_0^T |G_1(t_{n-1},t_n)||p(t_n) - g(x(t_n - \tau(t_n)))|dt_n \cdots dt_1 + \\
&\max_{t\in[0,T]} \lambda \int_0^T |G_n(t,t_1)| \cdots \int_0^T |G_1(t_{n-1},t_n)|(M_{n-1} - a_{n-1}(t_n))|x^{(2n-2)}(t_n)|dt_n \cdots dt_1 \\
&+ \max_{t\in[0,T]} \int_0^T |G_n(t,t_1)| \cdots \int_0^T |G_2(t_{n-2},t_{n-1})||\tfrac{M_{n-2}}{M_{n-1}} x^{(2n-4)}(t_{n-1}) - \\
&\lambda \int_0^T G_1(t_{n-1},t_n) a_{n-2} x^{(2n-4)}(t_n)dt_n|dt_{n-1} \cdots dt_1 \\
&+ \max_{t\in[0,T]} \int_0^T |G_n(t,t_1)| \cdots \int_0^T |G_3(t_{n-3},t_{n-2})||\tfrac{M_{n-3}}{M_{n-2}} x^{(2n-6)}(t_{n-2}) - \\
&\lambda \int_0^T G_2(t_{n-2},t_{n-1}) \int_0^T G_1(t_{n-1},t_n)[a_{n-3} x^{(2n-6)}(t_n)dt_n dt_{n-1}|dt_{n-2} \cdots dt_1 \\
&+ \cdots + \cdots \\
&+ \max_{t\in[0,T]} \int_0^T |G_n(t,t_1)||\tfrac{M_0}{M_1} x(t_1) - \lambda \int_0^T G_{n-1}(t_1,t_2) \int_0^T G_{n-2}(t_2,t_3) \\
&\cdots \int_0^T G_1(t_{n-1},t_n) a_0(t_n) x(t_n) dt_n \cdots dt_2|dt_1 \\
&+ \max_{t\in[0,T]} \lambda \int_0^T |G_n(t,t_1)| \cdots \int_0^T |G_1(t_{n-1},t_n)||[\sum_{k=1}^{n-1} a_{2k-1}(t_n) x^{(2k-1)}(t_n)]|dt_n \cdots dt_1 \\
&\quad \le \tfrac{1}{M_0}[||p||_0 + \rho + (r+\varepsilon)||x||_0] + \tfrac{M_0 - m_0}{M_0}||x||_0 + \tfrac{1}{M_0}[(M_{2n-2} - m_{2n-2})||x^{(2n-2)}||_0 \\
&\quad +(M_{2n-4} - m_{2n-4})||x^{(2n-4)}||_0 + \cdots + (M_2 - m_2)||x^{(2)}||_0] \\
&\quad +\tfrac{1}{M_0}[M_1||x^{(1)}||_0 + M_3||x^{(3)}||_0 + \cdots + M_{2n-3}||x^{(2n-3)}||_0] \\
&\quad \le \tfrac{1}{M_0}[C + (M_0 - m_0 + r + \varepsilon)||x||_0] + \tfrac{1}{M_0}[M_1(\tfrac{T}{2})^{2n-2} + (M_2 - m_2)(\tfrac{T}{2})^{2n-3} \\
&\quad +M_3(\tfrac{T}{2})^{2n-4} + (M_4 - m_4)(\tfrac{T}{2})^{2n-5} + \cdots + M_{2n-3}(\tfrac{T}{2})^2 \\
&\quad +(M_{2n-2} - m_{2n-2})\tfrac{T}{2}]||x^{(2n-1)}||_0.
\end{aligned} \tag{41}$$

Now (41) and $\varepsilon = \frac{m_0 - r}{2}$ give

$$||x||_0 \leq \frac{2(B||x^{(2n-1)}||_0 + C)}{(m_0 - r)}, \tag{42}$$

where

$$B = M_1(\tfrac{T}{2})^{2n-2} + (M_2 - m_2)(\tfrac{T}{2})^{2n-3} + M_3(\tfrac{T}{2})^{2n-4} + (M_4 - m_4)(\tfrac{T}{2})^{2n-5} + \cdots + \cdots + M_{2n-3}(\tfrac{T}{2})^2 + (M_{2n-2} - m_{2n-2})\tfrac{T}{2}$$

and $M_{2k-1} = \max_{t\in[0,T]} |a_{2k-1}(t)| \quad (k = 1, 2, \cdots n - 1)$.

Thus combining (27) and (42), we see that

$$\begin{aligned}(A - \tfrac{2M_0+m_0+r}{2(m_0-r)}B)||x^{(2n-1)}||_0 &\leq \tfrac{TC}{2}(\tfrac{2M_0+m_0+r}{(m_0-r)} + 1) \\ &= \tfrac{(M_0+m_0)TC}{(m_0-r)},\end{aligned} \tag{43}$$

where $A = 1 - A^*$.

From (42) and (43), we have

$$\begin{aligned}||x^{(2n-1)}||_0 &\leq \tfrac{(M_0+m_0)TC}{(m_0-r)}(A - \tfrac{2M_0+m_0+r}{2(m_0-r)}B)^{-1} \\ &= D_{2n-1}\end{aligned} \tag{44}$$

and

$$||x||_0 \leq \frac{2(BD_{2n-1} + C)}{(m_0 - r)} = D_0. \tag{45}$$

Finally from (44), (45) and Lemma 2.2, we get

$$||x^{(i)}||_0 \leq D_i \quad (1 \leq i \leq 2n - 2). \tag{46}$$

The proof of Lemma 2.4 is complete.

Proof of Theorem 2.1. Suppose that $x(t)$ is a T-periodic solution of Eq.(21). By Lemma 2.4, there exist positive constants D_i $(i = 0, 1, \cdots, 2n - 1)$ which are independent of λ such that (22) is true. By (3), we know that there exists a $M_2 > 0$, such that

$$sgn(x)g(x(t)) > 0, \quad |x(t)| > M_2, \quad t \in \mathbf{R}.$$

Consider any positive constant $\overline{D} > \max_{0\leq i\leq 2n-1}\{D_i\} + M_2$.

Set

$$\Omega := \{x \in X : ||x|| < \overline{D}\}.$$

We know that L is a Fredholm mapping of index zero and N is L-compact on $\overline{\Omega}$ (see [3]).

Recall

$$Ker(L) = \{x \in X : x(t) = c \in \mathbf{R}\}$$

and the norm on X is

$$||x|| = \max_{0 \le j \le 2n-1} \max_{t \in [0,T]} |x^{(j)}(t)|.$$

Then we have

$$x = \overline{D} \quad \text{or} \quad x = -\overline{D} \quad \text{for} \quad x \in \partial\Omega \cap Ker(L). \tag{47}$$

From (3) and (47), we have (if $\overline{D}$ is chosen large enough)

$$a_0(t)\overline{D} + g(\overline{D}) - \|p\|_0 > 0 \quad \text{for } t \in [0, T] \tag{48}$$

and

$$x^{(i)}(t) = 0, \quad \forall x \in \partial\Omega \cap Ker(L)(i = 1, 2, \cdots, 2n-1). \tag{49}$$

Finally from (5), (9) and (47)-(49), we have

$$\begin{aligned} (QNx) &= \frac{1}{T}\int_0^T [-\sum_{i=0}^{2n-2} a_i(t)x^{(i)}(t) - g(x(t-\tau(t))) + p(t)]dt \\ &= -\frac{1}{T}\int_0^T [a_0(t)x(t) + g(x(t-\tau(t))) - p(t)]dt \\ &\ne 0, \quad \forall x \in \partial\Omega \cap Ker(L). \end{aligned}$$

Then, for any $x \in KerL \cap \partial\Omega$ and $\eta \in [0, 1]$, we have

$$\begin{aligned} xH(x, \eta) &= -\eta x^2 - \frac{x}{T}(1-\eta)\int_0^T [\sum_{i=0}^{2n-2} a_i(t)x^{(i)}(t) + g(x(t-\tau(t))) - p(t)]dt \\ &\ne 0. \end{aligned}$$

Thus

$$\begin{aligned} deg\{QN, \Omega \cap Ker(L), 0\} &= deg\{-\frac{1}{T}\int_0^T [\sum_{i=0}^{2n-2} a_i(t)x^{(i)}(t) \\ &\quad + g(x(t-\tau(t))) - p(t)]dt, \Omega \cap Ker(L), 0\} \\ &= deg\{-x, \Omega \cap Ker(L), 0\} \\ &\ne 0. \end{aligned}$$

From Lemma 2.4 for any $x \in \partial\Omega \cap Dom(L)$ and $\lambda \in (0, 1)$ we have $Lx \ne \lambda Nx$. By Theorem 1.1, the equation $Lx = Nx$ has at least a solution in $Dom(L) \cap \overline{\Omega}$, so there exists a T-periodic solution of Eq.(1). The proof is complete.

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