Electronic Journal of Qualitative Theory of Differential Equations
Spec. Ed. I, 2009 No. **16**, 1–11; **http://www.math.u-szeged.hu/ejqtde/**

A THIRD ORDER NONLOCAL BOUNDARY VALUE PROBLEM AT RESONANCE

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Honoring the Career of John Graef on the Occasion of His Sixty-Seventh Birthday

Abstract

We consider the third-order nonlocal boundary value problem

$$\begin{aligned}&u'''(t) = f(t, u(t)), \quad \text{a.e. in } (0, 1),\\ &u(0) = 0,\ u'(\rho) = 0,\\ &u''(1) = \lambda[u''],\end{aligned}$$

where $0 < \rho < 1$, the nonlinear term f satisfies Carathéodory conditions with respect to $L^1[0, T]$, $\lambda[v] = \int_0^1 v(t)\,\mathrm{d}\Lambda(\mathrm{t})$, and the functional λ satisfies the resonance condition $\lambda[1] = 1$. The existence of a solution is established via Mawhin's coincidence degree theory.

Key words and phrases: Coincidence degree theory, nonlocal boundary value problem, resonance.
AMS (MOS) Subject Classifications: 34B10, 34B15

1 Introduction

We study the third-order nonlocal boundary value problem

$$u'''(t) = f(t, u(t)), \quad \text{a.e. in } (0, 1), \tag{1}$$

$$u(0) = 0,\ u'(\rho) = 0, \tag{2}$$

$$u''(1) = \lambda[u''], \tag{3}$$

where $0 < \rho < 1$, the nonlinear term f satisfies Carathéodory conditions with respect to $L^1[0, T]$, and $\lambda[v]$ is a linear functional defined by the Riemann-Stieltjes integral $\lambda[v] = \int_0^1 v(t)\,\mathrm{d}\Lambda(\mathrm{t})$. Here Λ is a suitable monotonically increasing function on $[0, 1]$. We assume throughout that the functional λ satisfies

$$\lambda[1] = 1. \tag{4}$$

Due to the condition (4), the boundary value problem (1)-(3) can not be inverted, and as such, we say that the boundary value problem is at resonance. Recently, several authors have studied nonlocal boundary value problems at resonance, see for example [1, 2, 3, 9, 10, 11, 12, 13, 15, 16, 17, 18] and references therein. The literature is rich also with articles on nonlocal boundary value problems; see [4, 5, 6, 7, 8] and references therein. The primary motivation for this work is the article [8] due to Graef and Webb. In [8], the authors consider the existence of multiple positive solutions for the nonlocal boundary value problem

$$\begin{aligned} &u'''(t) = f(t, u(t)), \quad t \in (0,1),\\ &u(0) = 0,\, u'(\rho) = 0,\\ &u''(1) = \lambda[u''], \end{aligned}$$

where $\rho > 1/2$ and $\lambda[v] = \int_0^1 v(t)\, \mathrm{d}\Lambda(\mathrm{t})$, as well as a generalization of this problem. To ensure that the boundary value problem is invertible, the authors impose the condition that $\lambda[1] \neq 1$.

In Section 2 we give the background information from coincidence degree theory. So that the paper is self-contained, we state Mawhin's coincidence theorem, [14], in this section. We also define appropriate mappings and projectors that will be used in the sequel. We state and prove our main result in Section 3.

2 Background

Let X and Z be normed spaces. A linear mapping $L : \text{dom } L \subset X \to Z$ is called a *Fredholm mapping* if $\ker L$ has a finite dimension, and $\text{Im } L$ is closed and has finite codimension. The (Fredholm) *index* of a Fredholm mapping L is the integer, Ind L, given by $\text{Ind } L = \dim \ker L - \text{codim Im } L$.

For a Fredholm map of index zero, $L : \text{dom } L \subset X \to Z$, there exist continuous projectors $P : X \to X$ and $Q : Z \to Z$ such that

$$\text{Im } P = \ker L,\ \ker Q = \text{Im } L,\ X = \ker L \oplus \ker P,\ Z = \text{Im } L \oplus \text{Im } Q,$$

and the mapping

$$L|_{\text{dom } L \cap \ker P} : \text{dom } L \cap \ker P \to \text{Im } L$$

is invertible. The inverse of $L|_{\text{dom } L \cap \ker P}$ is denoted by

$$K_P : \text{Im } L \to \text{dom } L \cap \ker P.$$

The generalized inverse of L, denoted by $K_{P,Q} : Z \to \text{dom } L \cap \ker P$, is defined by $K_{P,Q} = K_P(I - Q)$.

If L is a Fredholm mapping of index zero, then for every isomorphism $J : \text{Im } Q \to \ker L$, the mapping $JQ + K_{P,Q} : Z \to \text{dom } L$ is an isomorphism and, for every $u \in \text{dom } L$,

$$(JQ + K_{P,Q})^{-1} u = (L + J^{-1}P)u.$$

Definition 2.1 *Let $L : \text{dom } L \subset X \to Z$ be a Fredholm mapping, E be a metric space, and $N : E \to Z$. We say that N is L-compact on E if $QN : E \to Z$ and $K_{P,Q}N : E \to X$ are compact on E. In addition, we say that N is L-completely continuous if it is L-compact on every bounded $E \subset X$.*

We will formulate the boundary value problem (1)-(3) as $Lu = Nu$, where L and N are appropriate operators. The existence of a solution to the boundary value problem will then be guaranteed by the following theorem due to Mawhin [14].

Theorem 2.1 *Let $\Omega \subset X$ be open and bounded. Let L be a Fredholm mapping of index zero and let N be L-compact on $\overline{\Omega}$. Assume that the following conditions are satisfied:*

(i) $Lu \neq \mu Nu$ *for every* $(u, \mu) \in ((\operatorname{dom} L \setminus \ker L) \cap \partial\Omega) \times (0, 1)$*;*

(ii) $Nu \notin \text{Im } L$ *for every* $u \in \ker L \cap \partial\Omega$*;*

(iii) $\deg_B (JQN|_{\ker L \cap \partial\Omega}, \Omega \cap \ker L, 0) \neq 0$*, with* $Q : Z \to Z$ *a continuous projector, such that* $\ker Q = \operatorname{Im} L$ *and* $J : \operatorname{Im} Q \to \ker L$ *is an isomorphism.*

Then the equation $Lu = Nu$ has at least one solution in $\text{dom } L \cap \overline{\Omega}$.

We say that the map $f : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ satisfies Carathéodory conditions with respect to $L^1[0, T]$ if the following conditions hold.

(i) For each $z \in \mathbb{R}^n$, the mapping $t \mapsto f(t, z)$ is Lebesgue measurable.

(ii) For a.e. $t \in [0, T]$, the mapping $z \mapsto f(t, z)$ is continuous on $\mathbb{R}^n$.

(iii) For each $r > 0$, there exists $\alpha_r \in L^1([0, T], \mathbb{R})$ such that for a.e. $t \in [0, T]$ and for all z such that $|z| < r$, we have $|f(t, z)| \leq \alpha_r(t)$.

Let $AC[0, 1]$ denote the space of absolutely continuous functions on the interval $[0, 1]$. Define $Z = L^1[0, 1]$ with norm $\|\cdot\|_1$ and let

$$X = \{u : u, u', u'' \in AC[0, 1] \text{ and } u''' \in L^1[0, 1]\}$$

be equipped with the norm $\|u\| = \max\{\|u\|_0, \|u'\|_0, \|u''\|_0\}$. Define the mapping $L : \text{dom } L \subset X \to Z$, where

$$\text{dom } L = \{u \in X : u \text{ satisfies (2) and (3)}\},$$

by

$$Lu(t) = u'''(t), \quad t \in [0, 1].$$

Define $N : X \to Z$ by

$$Nu(t) = f(t, u(t)), \quad t \in [0, 1].$$

Lemma 2.1 *The mapping $L : dom\ L \subset X \to Z$ is a Fredholm mapping of index zero.*

Proof. Let $g \in Z$, and for $t \in [0, 1]$, let

$$u(t) = a\left(t^2/2 - \rho t\right) + \int_0^t \int_\rho^s \int_0^r g(\tau)\,\mathrm{d}\tau\,\mathrm{d}r\,\mathrm{d}s. \tag{5}$$

Then, $u'''(t) = g(t)$ for a.e. $t \in [0, 1]$, $u(0) = 0$, and $u'(\rho) = 0$. Furthermore, if

$$\int_0^1 g(s)\,\mathrm{d}s - \lambda\left[\int_0^t g(s)\,\mathrm{d}s\right] = 0,$$

then u satisfies the nonlocal boundary condition $u''(1) = \lambda[u'']$. If (5) is satisfied, then $u \in \mathrm{dom}\ L$, and so,

$$\left\{g \in Z : \int_0^1 g(s)\,\mathrm{d}s - \lambda\left[\int_0^t g(s)\,\mathrm{d}s\right] = 0\right\} \subseteq \mathrm{Im}\ L.$$

Now let $g \in \mathrm{Im}\ L$. Then there exists a $u \in \mathrm{dom}\ L$ such that $u'''(t) = g(t)$ for a.e. $t \in [0, 1]$. We have,

$$u''(t) = u''(0) + \int_0^t g(s)\,\mathrm{d}s. \tag{6}$$

Apply the functional $\lambda[v]$ to both sides of (6) to obtain,

$$\lambda[u''] = u''(0) + \lambda\left[\int_0^t g(s)\,\mathrm{d}s\right]. \tag{7}$$

Also, evaluate (6) at $t = 1$.

$$u''(1) = u''(0) + \int_0^1 g(s)\,\mathrm{d}s. \tag{8}$$

From the boundary condition $u''(1) = \lambda[u'']$, and (7) and (8), we see that g satisfies

$$\int_0^1 g(s)\,\mathrm{d}s = \lambda\left[\int_0^t g(s)\,\mathrm{d}s\right].$$

Hence

$$\mathrm{Im}\ L \subseteq \left\{g \in Z : \int_0^1 g(s)\,\mathrm{d}s - \lambda\left[\int_0^t g(s)\,\mathrm{d}s\right] = 0\right\}.$$

Since both inclusions hold, then

$$\mathrm{Im}\ L = \left\{g \in Z : \int_0^1 g(s)\,\mathrm{d}s - \lambda\left[\int_0^t g(s)\,\mathrm{d}s\right] = 0\right\}.$$

Let $\varphi \in C[0,1]$ be such that $\varphi(t) > 0$ for $t \in [0,1]$ and

$$C \equiv \int_0^1 \varphi(t)\,\mathrm{d}t - \lambda \left[\int_0^t \varphi(s)\,\mathrm{d}s\right] \neq 0.$$

Define $Q_1 : Z \to \mathbb{R}$ by

$$Q_1 g \equiv \int_0^1 g(s)\,\mathrm{d}s - \lambda \left[\int_0^t g(s)\,\mathrm{d}s\right].$$

Now define the mapping $Q : Z \to Z$ by

$$(Qg)(t) = \frac{1}{C}(Q_1 g)\varphi(t). \tag{9}$$

The mapping Q is a continuous linear mapping and

$$\begin{aligned}
(Q^2 g)(t) &= (Q(Qg))(t) \\
&= \frac{1}{C}\left(\frac{1}{C}Q_1 g\right)\left(\int_0^1 \varphi(t)\,\mathrm{d}t - \lambda\left[\int_0^t \varphi(s)\,\mathrm{d}s\right]\right)\varphi(t) \\
&= \left(\frac{1}{C}Q_1 g\right)\varphi(t) \;=\; (Qg)(t).
\end{aligned}$$

That is, $Q^2 g = Qg$ for all $g \in Z$. Furthermore, $\mathrm{Im}\ L = \ker Q$.

For $g \in Z$ we have $g - Qg \in \ker Q = \mathrm{Im}\ L$ and $Qg \in \mathrm{Im}\ Q$. Hence $Z = \mathrm{Im}\ L + \mathrm{Im}\ Q$. Let $g \in \mathrm{Im}\ L \cap \mathrm{Im}\ Q$. Since $g \in \mathrm{Im}\ Q$, then there exists an $\eta \in \mathbb{R}$ such that $g(t) = \eta\varphi(t), t \in [0,1]$. Since $g \in \mathrm{Im}\ L = \ker Q$, then

$$0 = Q_1 g(t) = \eta\left(\int_0^1 \varphi(t)\,\mathrm{d}t - \lambda\left[\int_0^t \varphi(s)\,\mathrm{d}s\right]\right) = \eta C.$$

Since $C \neq 0$, then $\eta = 0$ and so $g(t) \equiv 0, t \in [0,1]$. Consequently, $Z = \mathrm{Im}\ L \oplus \ker Q$. Note $\ker L = \{c\,(t^2/2 - \rho t) : c \in \mathbb{R}\} \cong \mathbb{R}$, and so $\dim\ker L = \mathrm{codim}\ \mathrm{Im}\ L = \dim \mathrm{Im}\ Q = 1$. Thus, L is a Fredholm mapping on index zero. The proof is complete.

We are now ready to give the other projector employed in the proof of our main result. Define $P : X \to X$ by

$$(Pu)(t) = u''(0)\left(t^2/2 - \rho t\right), \quad t \in [0,1], \tag{10}$$

and note that $\ker P = \{u \in X : u''(0) = 0\}$ and $\mathrm{Im}\ P = \ker L$. Since $(Pu)''(t) = u''(0)$, then $(P^2 u)(t) = (Pu)(t)$, $t \in [0,1]$. For all $u \in X$ we have

$$u(t) = u''(0)\left(t^2/2 - \rho t\right) + (u(t) - u''(0)(t^2/2 - \rho t)),$$

and so, $X = \ker L \oplus \ker P$.

Define the mapping $K_P : \text{Im } L \to \text{dom } L \cap \ker P$ by

$$K_P g(t) = \int_0^t \int_\rho^s \int_0^r g(\tau)\,\mathrm{d}\tau\,\mathrm{d}r\,\mathrm{d}s.$$

It follows that $\|K_P g\|_0 \leq \frac{1}{2}(1-\rho)^2\|g\|_1$, $\|(K_P g)'\|_0 \leq (1-\rho)\|g\|_1$, and $\|(K_P g)''\|_0 \leq \|g\|_1$. As such, we have

$$\|K_p g\| = \max\{\|K_P g\|_0, \|(K_P g)'\|_0, \|(K_P g)''\|_0\} \leq \|g\|_1. \tag{11}$$

If $g \in \text{Im } L$ then,

$$(LK_P)g(t) = \frac{\mathrm{d}^3}{\mathrm{d}t^3}\int_0^t \int_\rho^s \int_0^r g(\tau)\,\mathrm{d}\tau\,\mathrm{d}r\,\mathrm{d}s = g(t).$$

And if $u \in \text{dom } L \cap \ker P$ then,

$$\begin{aligned}(K_p L)u(t) &= \int_0^t \int_\rho^s \int_0^r u'''(\tau)\,\mathrm{d}\tau\,\mathrm{d}r\,\mathrm{d}s \\ &= u(t) - u(0) - tu'(\rho) - u''(0)\left(t^2/2 - \rho t\right) \\ &= u(t).\end{aligned}$$

Consequently, $K_P = (L|_{\text{dom } L\cap\ker P})^{-1}$. The generalized inverse of L is defined by

$$K_{P,Q}u(t) = \int_0^t \int_\rho^s \int_0^r u(\tau) - Qu(\tau)\,\mathrm{d}\tau\,\mathrm{d}r\,\mathrm{d}s.$$

Lastly in this section, we show that N is L-compact.

Lemma 2.2 *The mapping $N : X \to Z$ given by $Nu(t) = f(t, u(t))$ is L-completely continuous.*

Proof. Let $E \subset X$ be a bounded set and let r be such that $\|u\| \leq r$ for all $u \in E$. Since f satisfies Carathéodory conditions, there exists an $\alpha_r \in L^1[0,1]$ such that for a.e. $t \in [0,T]$ and for all z such that $|z| < r$ we have $|f(t,z)| \leq \alpha_r(t)$. Let $M = \int_0^1 \mathrm{d}\Lambda(t)$. Then,

$$\begin{aligned}\|QNu(t)\|_1 &\leq \frac{1}{C}\int_0^1\int_0^1 |f(s,u(s))|\,\mathrm{d}s + \lambda\left[\int_0^t |f(s,u(s))|\,\mathrm{d}s\right]\varphi(t)\,\mathrm{d}t \\ &\leq \frac{1}{C}\left(\int_0^1\int_0^1 \alpha_r(s)\,\mathrm{d}s\varphi(t)\,\mathrm{d}t + \int_0^1 \lambda\left[\int_0^t \alpha_r(s)\,\mathrm{d}s\right]\varphi(t)\,\mathrm{d}t\right) \\ &\leq \frac{1}{C}\|\varphi\|_1\left(\|\alpha_r\|_1 + \int_0^1\int_0^t \alpha_r(s)\,\mathrm{d}s\,\mathrm{d}\Lambda(t)\right) \\ &\leq \frac{1}{C}\|\varphi\|_1\|\alpha_r\|_1(1+M).\end{aligned}$$

Hence, $QN(E)$ is uniformly bounded.

It is clear that the functions $QN(u)$ are equicontinuous on E. By the Arzelà-Ascoli Theorem, $QN(E)$ is relatively compact.

It can be shown that $K_{P,Q}N(E)$ is relatively compact as well. As such, the mapping $N : X \to Z$ is L-completely continuous and the proof is complete.

3 Existence of Solutions

We will assume that the following conditions hold.

(H_1) There exists a constant $c_1 > 0$ such that for all $u \in \text{dom } L \setminus \ker L$ satisfying $|u''(t)| > c_1, t \in [0, T]$, we have

$$QNu(t) \neq 0, \text{ for all } t \in [0, 1].$$

(H_2) There exist $\beta, \gamma \in L^1[0, T]$, such that for all $u \in \mathbb{R}$ and for all $t \in [0, 1]$,

$$|f(t, u)| \leq \beta(t) + \gamma(t)|u|.$$

(H_3) There exists a $B > 0$ such that for all $c_2 \in \mathbb{R}$ with $|c_2| > B$, either

$$c_2 \left(\int_0^1 f\left(s, c_2(s^2/2 - \rho s)\right) \, \mathrm{d}s - \lambda \left[\int_0^t f\left(s, c_2(s^2/2 - \rho s)\right) \, \mathrm{d}s \right] \right) < 0$$

or

$$c_2 \left(\int_0^1 f\left(s, c_2(s^2/2 - \rho s)\right) \, \mathrm{d}s - \lambda \left[\int_0^t f\left(s, c_2(s^2/2 - \rho s)\right) \, \mathrm{d}s \right] \right) > 0.$$

Theorem 3.1 *Assume that conditions (H_1)-(H_3) hold and that*

$$1 - 2\|\gamma\|_1 > 0. \tag{12}$$

Then the nonlinear periodic problem (1)-(3) has at least one solution.

Proof. Let $Q : Z \to Z$ and $P : X \to X$ be defined as in (9) and (10), respectively. We begin by constructing a bounded open set Ω that satisfies Theorem 2.1. To this end, define the set Ω_1 as follows.

$$\Omega_1 = \{u \in \text{dom } L \setminus \ker L : Lu = \mu Nu \text{ for some } \mu \in (0, 1)\}.$$

Let $u \in \Omega_1$ and write u as $u = Pu + (I - P)u$. Then

$$\|u\| \leq \|Pu\| + \|(I - P)u\|. \tag{13}$$

Since $u \in \Omega_1$ then $(I-P)u \in \operatorname{dom} L \cap \ker P = \operatorname{Im} K_P$, $0 < \mu < 1$ and $Nu = \frac{1}{\mu} Lu \in \operatorname{Im} L$. From (11) we have,

$$\|(I-P)u\| = \|K_P L(I-P)u\| \leq \|L(I-P)u\| = \|Lu\| < \|Nu\|. \tag{14}$$

From (H_2) we see that $\|Nu\| \leq \|\beta\|_1 + \|\gamma\|_1\|u\|$, and so by (13) and (14), we obtain,

$$\|u\| < \|Pu\| + \|\beta\|_1 + \|\gamma\|_1\|u\|. \tag{15}$$

Now, $Pu(t) = u''(0)\,(t^2/2 - \rho t)$. Since $0 < \rho < 1$, then $\|Pu\| = |u''(0)|$. Also, since $u \in \Omega_1$ and $\ker Q = \operatorname{Im} L$, then $QNu(t) = 0$ for all $t \in [0,1]$. By (H_1), there exists $t_0 \in [0,1]$ such that $|u''(t_0)| \leq c_1$. Now,

$$u''(0) = u''(t_0) - \int_0^{t_0} u'''(s)\,\mathrm{d}s,$$

and so,

$$|u''(0)| \leq |u''(t_0)| + \int_0^{t_0} |u'''(s)|\,\mathrm{d}s \leq c_1 + \|Nu\|.$$

Consequently,

$$\|Pu\| = |u''(0)| \leq c_1 + \|\beta\|_1 + \|\gamma\|_1\|u\|. \tag{16}$$

By (15) and (16), we have for $u \in \Omega_1$,

$$\|u\| \leq c_1 + 2\|\beta\|_1 + 2\|\gamma\|_1\|u\|,$$

which, using (12), implies that

$$\|u\| \leq \frac{c_1 + 2\|\beta\|}{1 - 2\|\gamma\|_1}.$$

The set Ω_1 is bounded.

Next define the set Ω_2 by

$$\Omega_2 = \{u \in \ker L : Nu \in \operatorname{Im} L\}.$$

Let $u \in \Omega_2$. Since $u \in \ker L$, then $u(t) = c_2\,(t^2/2 - \rho t)$ for some $c_2 \in \mathbb{R}$. We also know that $Nu \in \operatorname{Im} L = \ker Q$ and so,

$$0 = Q_1 Nu = \int_0^1 f\left(s, c_2(s^2/2 - \rho s)\right)\,\mathrm{d}s - \lambda\left[\int_0^t f\left(s, c_2(s^2/2 - \rho s)\right)\,\mathrm{d}s\right].$$

It follows from (H_3) that $|c_2| < B$ and so Ω_2 is bounded.

Before we define the set Ω_3, we must state our isomorphism, $J : \operatorname{Im} Q \to \ker L$. Let

$$J(c\varphi(t)) = c(t^2/2 - \rho t).$$

Suppose that the first part of (H_3) is satisfied. Then define

$$\Omega_3 = \left\{ u \in \ker L := -\mu J^{-1}u + (1-\mu)QNu = 0, \mu \in [0,1] \right\}$$

and note that for each $u \in \Omega_3$,

$$\mu c\varphi(t) = (1-\mu)\frac{1}{C}\left(\int_0^1 f\left(s, c_2(s^2/2 - \rho s)\right)\,\mathrm{d}s - \lambda\left[\int_0^t f\left(s, c_2(s^2/2 - \rho s)\right)\,\mathrm{d}s \right] \right).$$

Suppose that $\mu = 1$, then $c = 0$. If $|c| > B$, then

$$\begin{aligned} \mu c^2\varphi(t) &= (1-\mu)\frac{c}{C}\left(\int_0^1 f\left(s, c_2(s^2/2 - \rho s)\right)\,\mathrm{d}s \right. \\ &\quad \left. - \lambda\left[\int_0^t f\left(s, c_2(s^2/2 - \rho s)\right)\,\mathrm{d}s \right] \right) < 0. \end{aligned}$$

In either case we get a contradiction and hence Ω_3 is bounded.

If the second part of (H_3) is satisfied then define Ω_3 by

$$\Omega_3 = \left\{ u \in \ker L := \mu J^{-1}u + (1-\mu)QNu = 0, \mu \in [0,1] \right\}.$$

A similar argument as above shows that Ω_3 is bounded.

Let Ω be an open and bounded set such that $\cup_{i=1}^3 \overline{\Omega}_i \subset \Omega$. Then the assumptions (i) and (ii) of Theorem 2.1 are satisfied. By Lemma 2.1, $L : \operatorname{dom} L \subset X \to Z$ is a Fredholm mapping of index zero. By Lemma 2.2, the mapping $N : X \to Z$ is L-completely continuous. We only need to verify that condition (iii) of Theorem 2.1 is satisfied.

We apply the invariance under a homotopy property of the Brower degree. Let

$$H(u,\mu) = \pm \mathrm{Id}\, u + (1-\mu)JQNu.$$

If $u \in \ker L \cap \partial\Omega$, then

$$\begin{aligned} \deg_B\left(JQN|_{\ker L\cap\partial\Omega}, \Omega\cap\ker L, 0\right) &= \deg_B\left(H(\cdot,0), \Omega\cap\ker L, 0\right) \\ &= \deg_B\left(H(\cdot,1), \Omega\cap\ker L, 0\right) \\ &= \deg_B\left(\pm\,\mathrm{Id}, \Omega\cap\ker L, 0\right) \\ &\neq 0. \end{aligned}$$

All the assumptions of Theorem 2.1 are fulfilled and the proof is complete.

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