# Initial boundary value problems for some damped nonlinear conservation laws

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**Abstract.** In this paper, we study the non-negative solutions of initial boundary value problems for some damped nonlinear conservation laws on the half line modelled by first order nonlinear hyperbolic PDEs. We consider the class of initial profile which are non-negative, bounded and compactly supported. Using the method of characteristics and Rankine–Hugoniot jump condition, an entropy solution is constructed subject to a top-hat initial profile. Then the large time behaviour of the constructed entropy solution is obtained. Finally, taking recourse to some comparison principles and the method of super and sub solutions the large time behaviour of entropy solutions subject to the general class of bounded and compactly supported initial profiles are established as the large time behaviour of the entropy solution subject to top-hat initial profiles.

**Keywords:** damped nonlinear conservation laws, Riemann problem, entropy solutions, method of characteristics, large time behaviour.

**2010 Mathematics Subject Classification:** 35B06, 35B40, 35C06, 35F31.

## 1   Introduction

Analysis of initial boundary value problems for partial differential equations on the semi-infinite line describing nonlinear wave propagation, dispersion and dissipation phenomena is paramount to large time asymptotic behaviour of such systems. We refer to [1–6] as some interesting references in this direction. In this paper we consider an initial boundary value problem (IBVP) on the semi-infinite line for an inviscid generalized Burgers equation in the form

$$u_t + u^\alpha u_x + g(x,t)u = 0, \quad (x,t) \in (0,\infty) \times (0,\infty), \tag{1.1}$$

$$u(x,0) = u_0(x), \quad x \in [0,\infty), \tag{1.2}$$

$$u(0,t) = u_0(0)f(t), \quad t \ge 0, \tag{1.3}$$

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where $\alpha \geq 1$. The inviscid generalized Burgers equation (1.1) model a nonlinear conservation law with variable linear damping. We choose

$$g(x,t) = j/(2(t+1)) \quad \text{or} \quad \lambda \tag{1.4}$$

to study the IBVP (1.1)–(1.3) for the well studied inviscid non-planar Burgers equation and the inviscid $\alpha - \lambda$ equation, respectively. Here $j \geq 0$, $\lambda \geq 0$. Further, we choose

$$f(t) = (t+1)^{-j/2} \quad \text{or} \quad \mathrm{e}^{-\lambda t} \tag{1.5}$$

according as $g(x,t) = j/(2(t+1))$ or $\lambda$. We may note that the choice of $f$ makes the initial and boundary data given in (1.2) and (1.3) compatible with each other. We study the IBVP (1.1)–(1.3) using the method of characteristics for a typical 'top hat' initial profile, $u_0(x)$ of the form

$$u_0(x) = \begin{cases} h, & 0 \leq x \leq l, \\ 0, & x > l, \end{cases} \tag{1.6}$$

where $h$ and $l$ are some positive constants. We refer [9, 10] and the references therein for an extensive discussion on the physical background and applications of the inviscid generalized Burgers equation.

The motivation for studying the IBVP (1.1)–(1.3) is due to the work of Murray [6]. He considered an initial boundary value problem on the semi-infinite line of the form

$$u_t + g(u)u_x + \lambda h(u) = 0, \quad x \in (0,\infty) \times (0,\infty), \tag{1.7}$$
$$u(x,0) = u_0(x), \quad x \in [0,\infty), \tag{1.8}$$
$$u(0,t) = 0, \quad t \geq 0, \tag{1.9}$$

where $\lambda \geq 0$ and $g(u)$ and $h(u)$ are non-negative monotonic increasing functions of $u$. The initial profile $u_0(x)$ was taken as

$$u_0(x) = \begin{cases} 0, & x < 0, \\ f(x), & 0 < x < X, \\ 0, & x > X, \end{cases} \tag{1.10}$$

where $0 \leq f(x) \leq 1$ and $X > 0$. Murray [6] studied the IBVP (1.7)–(1.10) via the method of characteristics. He discussed the existence of discontinuities in the solution of the IBVP (1.7)–(1.10) and their propagation speeds. Rao and Yadav [9] established large time asymptotic behaviour of Cauchy problem for the inviscid non-planar Burgers equation subject to bounded, non-negative and compactly supported initial data. Large time asymptotic behaviour of entropy solution to Cauchy problem for a nonlinearly damped conservation law is studied in [7, 8].

In the present work, we are interested in analyzing the contribution of the bounadry data (1.3) to the formation of discontinuities in the solution of the IBVP (1.1)–(1.3) and their propagation speeds.

Now, we summarize the main results of this study. Let $u_0 \in L^\infty(\mathbb{R})$ be non-negative and compactly supported in $\mathbb{R}$ and $\operatorname{supp} u_0 := [0,l]$, where $l > 0$. Then, the solution $u(x,t)$ of the IBVP (1.1)–(1.3) is also non-negative. The support function $s : [0,\infty) \to [l,\infty)$ of $u(x,t)$ is defined as

$$s(t) := \sup\{x > 0 : u(x,t) > 0 \text{ in } (0,x)\}.$$

We have the following two theorems. The first theorem is for the IBVP (1.1)–(1.3) with $g(x,t) = j/(2(t+1))$ and $f(t) = (t+1)^{-j/2}$, whereas the second theorem is for the IBVP (1.1)–(1.3) with $g(x,t) = \lambda$ and $f(t) = e^{-\lambda t}$.

**Theorem 1.1.**

(i) *Let $\alpha j > 2$. Then, there exists $x_0 > 0$ such that*

$$l \leq s(t) \leq l + x_0, \quad \text{for all } t \geq 0.$$

(ii) *Let $0 < \alpha j \leq 2$. Then there exist three constants $c_0$, $c_1$ and $c_2$ such that*

$$\lim_{t \to \infty} (\log t)^{-1}(s(t) - l) = c_0, \quad \text{if } \alpha j = 2,$$

*and for t large*

$$c_1 t^{(2-\alpha j)/2} \leq s(t) - l \leq c_2 t^{(2-\alpha j)/2}, \quad \text{if } 0 < \alpha j < 2.$$

*Further, there exists $c > 0$ such that*

$$t \left[ h^{2/j} - t\|u(.,t)\|_\infty^{2/j} \right] \to c, \quad \text{as } t \to \infty.$$

**Theorem 1.2.** *Let $\alpha > 0$, $\lambda > 0$ and $u_0 \in L^\infty(\mathbb{R})$ be compactly supported. Then there exists $x_0 > 0$ such that*

$$l \leq s(t) \leq l + x_0, \quad \text{for all } t \geq 0.$$

*Further, the large time behaviour of the solution to the IBVP (1.1)–(1.3) is given by*

$$\lim_{t \to \infty} e^{\lambda t} \|u(.,t)\|_\infty = h.$$

The organisation of this paper is as follows. In Section 2, we have constructed an entropy solution of IBVP (1.1)–(1.3) corresponding to the inviscid non-planar Burgers equation. Further, we have proved Theorem 1.1 concerning large time behaviours of support function and the constructed entropy solution. In Section 3, we have constructed an entropy solution of IBVP (1.1)–(1.3) corresponding to the inviscid $\alpha - \lambda$ equation. Theorem 1.2 concerns large time behaviour of support function and the constructed entropy solution is proved. Finally, Section 4 presents the conclusions of the study.

## 2   Inviscid non-planar Burgers equation

In this section, we study the IBVP (1.1)–(1.3) with $g(x,t) = j/(2(t+1))$, via the method of characteristics.

Let $x = x(t, x_0)$ denote the characteristic curve emanating from $(x_0, 0)$ and set

$$U = U(x_0, t) := u(x(t, x_0), t).$$

Then, for some fixed $x_0$ the characteristic equations are

$$\begin{aligned}
\frac{dx}{dt} &= U^\alpha, \quad x(0, x_0) = x_0 \\
\frac{dU}{dt} &= -\frac{jU}{2(t+1)}, \quad U(x_0, 0) = u_0(x_0).
\end{aligned} \tag{2.1}$$

Integrating the system (2.1), we get

$$U(x_0, t) = u_0(x_0)(t+1)^{-j/2}, \tag{2.2}$$

$$x(t, x_0) = \begin{cases} x_0 + u_0^{\alpha}(x_0)\log(t+1), & \text{if } \alpha j = 2, \\ x_0 + \frac{2u_0^{\alpha}(x_0)}{\alpha j - 2}\left[1 - (t+1)^{-(\alpha j - 2)/2}\right], & \text{if } \alpha j \neq 2. \end{cases} \tag{2.3}$$

Let us denote the characteristic curve issued at $(0,0)$ by $x = c(t)$. Then,

$$c(t) = \begin{cases} h^{\alpha}\log(t+1), & \text{if } \alpha j = 2, \\ \frac{2h^{\alpha}}{\alpha j - 2}\left[1 - (t+1)^{-(\alpha j - 2)/2}\right], & \text{if } \alpha j \neq 2 \end{cases} \tag{2.4}$$

Let us denote the shock issued at $(l, 0)$ by $x = s_0(t)$. Then, the Rankine–Hugoniot jump condition requires $s_0$ to satisfy

$$\frac{ds_0}{dt} = \frac{1}{\alpha + 1}u^{\alpha}(s_0(t) - 0, t), \qquad s_0(0) = l. \tag{2.5}$$

As long as $c$ does not intersect $s_0$

$$u(s_0(t) - 0, t) = U(0, t) = h(t+1)^{-j/2}. \tag{2.6}$$

Using (2.6) we integrate (2.5) to obtain

$$s_0(t) = \begin{cases} l + \frac{h^{\alpha}}{\alpha + 1}\log(t+1), & \text{if } \alpha j = 2, \\ l + \frac{2h^{\alpha}}{(\alpha + 1)(\alpha j - 2)}\left[1 - (t+1)^{-(\alpha j - 2)/2}\right], & \text{if } \alpha j \neq 2. \end{cases} \tag{2.7}$$

**Proposition 2.1.**

(i) *Let $\alpha j > 2$. Then the characteristic $c$ intersects the shock $s_0$ if and only if*

$$2\alpha h^{\alpha} > l(\alpha + 1)(\alpha j - 2). \tag{2.8}$$

(ii) *Let $0 < \alpha j \leq 2$. Then there is a unique intersection between $s_0$ and $c$.*

*Proof.* It is easy to see from (2.4) and (2.7) that any intersection between $c$ and $s_0$ has the abscissa

$$\bar{x} = \frac{l(\alpha + 1)}{\alpha}. \tag{2.9}$$

Hence an intersection exists if and only if for some $\bar{t} > 0$, $c(\bar{t}) = \bar{x} = s_0(\bar{t})$. Since $c(0) = 0$, $c$ is strictly increasing in $t$ and

$$c_{\infty} := \lim_{t \to \infty} c(t) = \begin{cases} \frac{2h^{\alpha}}{\alpha j - 2} & \text{if } \alpha j > 2, \\ \infty & \text{if } 0 < \alpha j \leq 2; \end{cases}$$

the conclusions follow from the fact that $c(\bar{t}) < c_{\infty}$. $\qquad\square$

In case of intersection between $c$ and $s_0$ the shock path changes from time $\bar{t}$ onwards and

$$\bar{t} = \begin{cases} e^{l(\alpha + 1)/(\alpha h^{\alpha})} - 1, & \text{if } \alpha j = 2, \\ \left[1 - \frac{l(\alpha + 1)(\alpha j - 2)}{2\alpha h^{\alpha}}\right]^{-2/(\alpha j - 2)} - 1, & \text{if } \alpha j \neq 2. \end{cases} \tag{2.10}$$

Let $s_1$ denote the new shock path then, once again by Rankine–Hugoniot jump condition we have

$$\frac{ds_1}{dt} = \frac{1}{\alpha+1} u^\alpha(s_1(t) - 0, t), \qquad s_1(\bar{t}) = \bar{x}, \ t > \bar{t}, \tag{2.11}$$

where $u(s_1(t) - 0, t)$ is given by the characteristic solution of the IBVP (1.1)–(1.3) in the region $0 \le x < c(t)$, $t \ge \bar{t}$.

Now we consider the characteristics emanating from the line $x = 0$. Let $X = X(t, t_0)$ denote the characteristic curve issued from some point $(0, t_0)$, $t_0 > 0$. Further, let

$$V = V(t, t_0) = u(X(t, t_0), t)$$

denote the characteristic solution of the IBVP (1.1)–(1.3) along the curve $X = X(t, t_0)$. Then the characteristic equations for some fixed $t_0 > 0$ are

$$\begin{aligned}
\frac{dX}{dt} &= V^\alpha, \quad X(t_0, t_0) = 0 \\
\frac{dV}{dt} &= -\frac{jV}{2(t+1)}, \quad V(t_0, t_0) = u_0(0)(t_0 + 1)^{-j/2}.
\end{aligned} \tag{2.12}$$

Integrating the system (2.12), we get

$$V(t, t_0) = V(t_0, t_0) \left(\frac{t+1}{t_0+1}\right)^{-j/2} = u_0(0)(t+1)^{-j/2}, \tag{2.13}$$

$$X(t, t_0) = \begin{cases} V^\alpha(t_0, t_0)(t_0 + 1) \log \frac{t+1}{t_0+1}, & \text{if } \alpha j = 2, \\[2mm] \frac{2V^\alpha(t_0, t_0)(t_0+1)^{\alpha j/2}}{\alpha j - 2} \left[(t_0+1)^{1-\alpha j/2} - (t+1)^{1-\alpha j/2}\right], & \text{if } \alpha j \ne 2. \end{cases} \tag{2.14}$$

On substituting $V(t_0, t_0) = u_0(0)(t_0 + 1)^{-j/2}$ in (2.14), we get

$$X(t, t_0) = \begin{cases} u_0^\alpha(0) \log \frac{t+1}{t_0+1}, & \text{if } \alpha j = 2, \\[2mm] \frac{2u_0^\alpha(0)}{\alpha j - 2} \left[(t_0+1)^{1-\alpha j/2} - (t+1)^{1-\alpha j/2}\right], & \text{if } \alpha j \ne 2. \end{cases} \tag{2.15}$$

We may note that the characteristic, $X = X(t, t_0)$, emanating from $(0, t_0)$ intersect the characteristics $x = x_0$, $x_0 > \bar{x}$. Therefore,

$$u(s_1(t) - 0, t) = V(t, t_0), \qquad t > \bar{t}. \tag{2.16}$$

Now using (2.16) and (2.13) in (2.11), we get

$$\frac{ds_1}{dt} = \frac{1}{\alpha+1} u_0^\alpha(0)(t+1)^{-\alpha j/2}, \qquad s_1(\bar{t}) = \bar{x}, \ t > \bar{t}. \tag{2.17}$$

On integrating (2.17), the new shock path is obtained as

$$s_1(t) = \begin{cases} \bar{x} + \frac{u_0^\alpha(0)}{\alpha+1} \log \frac{t+1}{\bar{t}+1}, & \text{if } \alpha j = 2, \\[2mm] \bar{x} + \frac{2u_0^\alpha(0)}{(\alpha+1)(\alpha j - 2)} \left[(\bar{t}+1)^{1-\alpha j/2} - (t+1)^{1-\alpha j/2}\right], & \text{if } \alpha j \ne 2, \end{cases} \qquad t > \bar{t}. \tag{2.18}$$

On simplifying (2.18), we get

$$s_1(t) = s_0(t), \qquad t > \bar{t}. \tag{2.19}$$

Now we define a function $s : [0, \infty) \to [l, \infty)$ as

$$s(t) := s_0(t), \qquad t \in [0, \infty). \tag{2.20}$$

In due course we see that $s$ is nothing but the support function of $u(x, t)$. Now we divide the quarter-plane $[0, \infty) \times [0, \infty)$ into the following disjoint regions

$$I_1 := \{(x, t) : t \geq 0, \ 0 \leq x < \min(c(t), s(t))\},$$
$$I_2 := \{(x, t) : t \geq 0, \ \min(c(t), s(t)) \leq x \leq s(t)\},$$
$$I_3 := \{(x, t) : t \geq 0, \ x > s(t)\},$$

and define $u : [0, \infty) \times [0, \infty) \to [0, \infty)$ as

$$u(x, t) := \begin{cases} h(t+1)^{-j/2}, & (x, t) \in I_1 \cup I_2 \\ 0, & (x, t) \in I_3. \end{cases} \tag{2.21}$$

The regions $I_2$ and $I_3$ are separated by the curve $x = s(t)$. Hence, $s(t)$ is the support function of $u(x, t)$ defined by (2.21).

## 2.1 Large time behaviour of support function and solution

In this section, we study the large time behaviour of the support function defined in (2.20) and prove Theorem 1.1.

**Lemma 2.2.** $s \in C^1(0, \infty)$.

*Proof.* It is easy to see that $s_0 \in C^1(0, \bar{t})$. Now if $\bar{t} = \infty$, then $s(t) = s_0(t)$ for all $t \in [0, \infty)$ and the lemma is proved. In case of $\bar{t} < \infty$, we have $s_1 \in C^1(\bar{t}, \infty)$ and $s$ is continuous at $\bar{t}$. It remains to show the continuity of $ds/dt$ at $\bar{t}$. Using the expression for $V(t, t_0)$ in (2.13) and the initial value problems (2.5), (2.11) satisfied by $s_0$ and $s_1$, respectively, we have

$$\frac{ds}{dt}(\bar{t} - 0) = \frac{h^\alpha}{\alpha + 1}(\bar{t} + 1)^{-\alpha j/2},$$
$$\frac{ds}{dt}(\bar{t} + 0) = \frac{1}{\alpha + 1} V^\alpha(\bar{t}, t_0)$$
$$= \frac{h^\alpha}{\alpha + 1}(\bar{t} + 1)^{-\alpha j/2},$$

since $u_0(0) = h$. Thus, $ds/dt$ is continuous at $\bar{t}$. Hence the lemma.  □

**Lemma 2.3.** *For any $t \geq 0$, $s(t) \leq s_0(t)$.*

*Proof.* It is easy to see that $s$ satisfies the initial value problem

$$\frac{ds}{dt} = \frac{1}{\alpha + 1} u^\alpha(s(t) - 0, t), \qquad s(0) = l.$$

By the solution (2.21),

$$\frac{ds}{dt} \leq \frac{h^\alpha}{\alpha + 1}(t+1)^{-\alpha j/2}, \qquad s(0) = l, \ t \geq 0.$$

Hence $s(t) \leq s_0(t)$ for all $t \geq 0$.  □

We may note that $\lim_{t\to\infty} s_0(t) := s_0^\infty$ exist when $\alpha j > 2$ and

$$s_0^\infty = l + \frac{2h^\alpha}{(\alpha+1)(\alpha j - 2)}. \tag{2.22}$$

**Proof of Theorem 1.1.**

(i) It is easy to see that $s(0) = l$ and $s$ is increasing on $[0, \infty)$. Hence $l \leq s(t)$ for all $t \geq 0$. Now by Lemma 2.3 and equation (2.22), whenever $\alpha j > 2$, there exist $x_0 := s_0^\infty - l > 0$ such that $s(t) \leq l + x_0$ for all $t \geq 0$.

(ii) By Proposition 2.1, whenever $0 < \alpha j \leq 2$, there is a unique intersection point $(\bar{x}, \bar{t})$ between $s_0$ and $c$. Thus, $s(t) = s_1(t)$ for $t \geq \bar{t}$ and equation (2.19) gives

$$\lim_{t\to\infty}(\log t)^{-1}(s(t) - l) = \frac{h^\alpha}{\alpha+1}, \quad \text{if } \alpha j = 2,$$

$$s(t) - l \leq \frac{2h^\alpha}{(\alpha+1)(2-\alpha j)}(t+1)^{(2-\alpha j)/2}, \quad \text{if } \alpha j < 2.$$

Let $c_2 := \frac{2h^\alpha}{(\alpha+1)(2-\alpha j)}$, then we may choose $c_1 < c_2$ such that for $t$ large

$$c_1 t^{(2-\alpha j)/2} \leq s(t) - l \leq c_2 t^{(2-\alpha j)/2}.$$

Further, by (2.21) for all $\alpha \geq 1$ and $j > 0$, we have

$$\|u(.,t)\|_\infty = h(t+1)^{-j/2}, \qquad t \geq 0. \tag{2.23}$$

From (2.23), it is easy to see that

$$\lim_{t\to\infty} t[h^{2/j} - t\|u(.,t)\|_\infty^{2/j}] = h^{2/j}. \tag{2.24}$$

This completes the proof of Theorem 1.1. $\qquad\qquad\square$

## 3 Inviscid $\alpha - \lambda$ equation

In this section, we study the IBVP (1.1)–(1.3) with $g(x,t) = \lambda$, via the method of characteristics.

Let $x = x(t, x_0)$ denote the characteristic curve emanating from $(x_0, 0)$ and set

$$U = U(x_0, t) := u(x(t, x_0), t).$$

Then, for some fixed $x_0$ the characteristic equations are

$$\begin{aligned} \frac{dx}{dt} &= U^\alpha, & x(0, x_0) &= x_0 \\ \frac{dU}{dt} &= -\lambda U, & U(x_0, 0) &= u_0(x_0). \end{aligned} \tag{3.1}$$

Integrating the system (3.1), we get

$$U(x_0, t) = u_0(x_0)e^{-\lambda t}, \tag{3.2}$$

$$x(t, x_0) = x_0 + \frac{u_0^\alpha(x_0)}{\alpha\lambda}(1 - e^{-\alpha\lambda t}). \tag{3.3}$$

Let us denote the characteristic curve issued at $(0,0)$ by $x = c(t)$. Then,

$$c(t) = \frac{h^\alpha}{\alpha\lambda}(1 - e^{-\alpha\lambda t}). \tag{3.4}$$

Let us denote the shock issued at $(l, 0)$ by $x = s_0(t)$. Then, the Rankine–Hugoniot jump condition requires $s_0$ to satisfy

$$\frac{ds_0}{dt} = \frac{1}{\alpha+1}u^\alpha(s_0(t) - 0, t), \qquad s_0(0) = l. \tag{3.5}$$

As long as $c$ does not intersect $s_0$

$$u(s_0(t) - 0, t) = U(0, t) = he^{-\lambda t}. \tag{3.6}$$

Using (3.6) we integrate (3.5) to obtain

$$s_0(t) = l + \frac{c(t)}{\alpha+1}. \tag{3.7}$$

**Proposition 3.1.** *The characteristic curve $c$ intersects the shock path $s_0$ if and only if*

$$h^\alpha > l\lambda(\alpha+1). \tag{3.8}$$

*Proof.* It is easy to see from (3.4) and (3.7) that any intersection between $c$ and $s_0$ has the abscissa

$$\bar{x} = \frac{l(\alpha+1)}{\alpha}. \tag{3.9}$$

Hence an intersection exists if and only if for some $\bar{t} > 0$, $c(\bar{t}) = \bar{x} = s_0(\bar{t})$. Since $c(0) = 0$, $c$ is strictly increasing in $t$ and

$$c_\infty := \frac{h^\alpha}{\alpha\lambda},$$

the conclusions follow from the fact that $c(\bar{t}) < c_\infty$.  □

In case of intersection between $c$ and $s_0$ the shock path changes from time $\bar{t}$ onwards and

$$\bar{t} = \frac{1}{\alpha\lambda}\log\frac{h^\alpha}{h^\alpha - l\lambda(\alpha+1)}. \tag{3.10}$$

Let $s_1$ denote the new shock path then, once again by Rankine–Hugoniot jump condition we have

$$\frac{ds_1}{dt} = \frac{1}{\alpha+1}u^\alpha(s_1(t) - 0, t), \qquad s_1(\bar{t}) = \bar{x}, \; t > \bar{t}, \tag{3.11}$$

where $u(s_1(t) - 0, t)$ is given by the characteristic solution of the IBVP (1.1)–(1.3) in the region $0 \le x < c(t)$, $t \ge \bar{t}$.

Now we consider the characteristics emanating from the line $x = 0$. Let $X = X(t, t_0)$ denote the characteristic curve issued from some point $(0, t_0)$, $t_0 > 0$. Further, let

$$V = V(t, t_0) = u(X(t, t_0), t)$$

denote the characteristic solution of the IBVP (1.1)–(1.3) along the curve $X = X(t, t_0)$. Then the characteristic equations for some fixed $t_0 > 0$ are

$$\begin{aligned} \frac{dX}{dt} &= V^\alpha, \quad X(t_0, t_0) = 0 \\ \frac{dV}{dt} &= -\lambda V, \quad V(t_0, t_0) = u_0(0)e^{-\lambda t_0}. \end{aligned} \tag{3.12}$$

Integrating the system (3.12), we get

$$V(t, t_0) = V(t_0, t_0)e^{-\lambda(t-t_0)} = u_0(0)e^{-\lambda t}, \tag{3.13}$$

$$X(t, t_0) = \frac{V^\alpha(t_0, t_0)}{\alpha\lambda}\left(1 - e^{-\alpha\lambda(t-t_0)}\right). \tag{3.14}$$

On substituting $V(t_0, t_0) = u_0(0)e^{-\lambda t_0}$ in (3.14), we get

$$X(t, t_0) = \frac{u_0^\alpha(0)}{\alpha\lambda}\left(e^{-\alpha\lambda t_0} - e^{-\alpha\lambda t}\right). \tag{3.15}$$

We may note that the characteristic, $X = X(t, t_0)$, emanating from $(0, t_0)$ intersect the characteristics $x = x_0$, $x_0 > \bar{x}$. Therefore,

$$u(s_1(t) - 0, t) = V(t, t_0), \qquad t > \bar{t}. \tag{3.16}$$

Now using (3.16) and (3.13) in (3.11), we get

$$\frac{ds_1}{dt} = \frac{1}{\alpha + 1}u_0^\alpha(0)e^{-\alpha\lambda t}, \qquad s_1(\bar{t}) = \bar{x}, \; t > \bar{t}. \tag{3.17}$$

On integrating (3.17), the new shock path is obtained as

$$s_1(t) = \bar{x} + \frac{u_0^\alpha(0)}{\alpha\lambda(\alpha + 1)}\left(e^{-\alpha\lambda\bar{t}} - e^{-\alpha\lambda t}\right), \qquad t > \bar{t}. \tag{3.18}$$

On simplifying (3.18), we get

$$s_1(t) = s_0(t), \qquad t > \bar{t}. \tag{3.19}$$

Now we define a function $s : [0, \infty) \to [l, \infty)$ as

$$s(t) := s_0(t), \qquad t \in [0, \infty). \tag{3.20}$$

In due course we see that $s$ is nothing but the support function of $u(x, t)$. Now we divide the quarter-plane $[0, \infty) \times [0, \infty)$ into the following disjoint regions

$$\begin{aligned}
I_1 &:= \{(x, t) : t \geq 0, \; 0 \leq x < \min(c(t), s(t))\}, \\
I_2 &:= \{(x, t) : t \geq 0, \; \min(c(t), s(t)) \leq x \leq s(t)\}, \\
I_3 &:= \{(x, t) : t \geq 0, \; x > s(t)\},
\end{aligned}$$

and define $u : [0, \infty) \times [0, \infty) \to [0, \infty)$ as

$$u(x, t) := \begin{cases} he^{-\lambda t}, & (x, t) \in I_1 \cup I_2 \\ 0, & (x, t) \in I_3. \end{cases} \tag{3.21}$$

The regions $I_2$ and $I_3$ are separated by the curve $x = s(t)$. Hence, $s(t)$ is the support function of $u(x, t)$ defined by (3.21).

### 3.1   Large time behaviour of support function and solution

In this section, we study the large time behaviour of the support function defined in (3.20) and the large time behaviour of the entropy solution (3.21) of the IBVP (1.1)–(1.3) and prove Theorem 1.2. Before we proceed further, we have the following lemmas on $s(t)$.

**Lemma 3.2.** $s \in C^1(0, \infty)$.

*Proof.* It is easy to see that $s_0 \in C^1(0, \bar{t})$. Now if $\bar{t} = \infty$, then $s(t) = s_0(t)$ for all $t \in [0, \infty)$ and the lemma is proved. In case of $\bar{t} < \infty$, we have $s_1 \in C^1(\bar{t}, \infty)$ and $s$ is continuous at $\bar{t}$. It remains to show the continuity of $ds/dt$ at $\bar{t}$. Using the expression for $V(t, t_0)$ in (3.13) and the initial value problems (3.5), (3.17) satisfied by $s_0$ and $s_1$, respectively, we have

$$\frac{ds}{dt}(\bar{t} - 0) = \frac{h^\alpha}{\alpha + 1} e^{-\alpha\lambda\bar{t}},$$

$$\frac{ds}{dt}(\bar{t} + 0) = \frac{1}{\alpha + 1} V^\alpha(\bar{t}, t_0)$$

$$= \frac{h^\alpha}{\alpha + 1} e^{-\alpha\lambda\bar{t}},$$

since $u_0(0) = h$. Thus, $ds/dt$ is continuous at $\bar{t}$. Hence the lemma.    □

**Lemma 3.3.** *For any $t \geq 0$, $s(t) \leq s_0(t)$.*

*Proof.* It is easy to see that $s$ satisfies the initial value problem

$$\frac{ds}{dt} = \frac{1}{\alpha + 1} u^\alpha(s(t) - 0, t), \qquad s(0) = l.$$

By the solution (3.21),

$$\frac{ds}{dt} \leq \frac{h^\alpha}{\alpha + 1} e^{-\alpha\lambda t}, \qquad s(0) = l, \ t \geq 0.$$

Hence $s(t) \leq s_0(t)$ for all $t \geq 0$.    □

We may note that $\lim_{t \to \infty} s_0(t) := s_0^\infty$ exist and

$$s_0^\infty = l + \frac{h^\alpha}{\alpha\lambda(\alpha + 1)}. \tag{3.22}$$

**Proof of Theorem 1.2.** It is easy to see that $s(0) = l$ and $s$ is increasing on $[0, \infty)$. Hence $l \leq s(t)$ for all $t \geq 0$. Now by Lemma 3.3 and equation (3.22), there exist $x_0 := s_0^\infty - l > 0$ such that $s(t) \leq l + x_0$ for all $t \geq 0$.

Further, by (3.21) for all $\alpha \geq 1$ and $\lambda > 0$, we have

$$\|u(\cdot, t)\|_\infty = h e^{-\lambda t}, \qquad t \geq 0. \tag{3.23}$$

From (3.23), it is easy to see that

$$\lim_{t \to \infty} e^{\lambda t} \|u(\cdot, t)\|_\infty = h. \tag{3.24}$$

This completes the proof of Therem 1.2.    □

## 4 Conclusions

Inspired by Murray's work [6], we have studied initial boundary value problems (1.1)–(1.3) for the inviscid non-planar Burgers equation and the $\alpha - \lambda$ equation on the semi-infinite line. The boundary data (1.3) is suitably chosen so that it is compatible with the initial data (1.2). Using the method of characteristics and Rankine–Hugoniot jump condition an entropy solution is constructed. The novelty of this work lies in using the information coming from the boundary $t = 0$ for constructing the entropy solution. Further, we have found the large time behaviour of the entropy solution and its support function.

## Acknowledgements

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