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A NOTE ON THE SECOND ORDER BOUNDARY VALUE PROBLEM ON A HALF-LINE

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Honoring the Career of John Graef on the Occasion of His Sixty-Seventh Birthday

Abstract

We consider the existence of a solution to the second order nonlinear differential equation

$$(p(t)u'(t))' = f(t, u(t), u'(t)),\ a.e. \text{ in } (0, \infty),$$

that satisfies the boundary conditions

$$u'(0) = 0, \quad \lim_{t\to\infty} u(t) = 0,$$

where $f : [0, \infty) \times \mathbb{R}^2 \to \mathbb{R}$ is Carathéodory with respect to $L_r[0, \infty)$, $r > 1$. The main technique used in this note is the Leray-Schauder Continuation Principle.

Key words and phrases: A priori estimate, Carathéodory, Leray-Schauder Continuation Principle.
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1 Introduction and Preliminaries

We study the second order nonlinear differential equation

$$(p(t)u'(t))' = f(t, u(t), u'(t)),\ a.e. \text{ in } (0, \infty), \tag{1}$$

satisfying

$$u'(0) = 0, \quad \lim_{t\to\infty} u(t) = 0. \tag{2}$$

We assume that the right side of (1) satisfies the Carathéodory conditions with respect to $L_r[0, \infty)$ with $r > 1$.

In Section 1, we discuss several recent results in the theory of boundary value problems on unbounded domains. In Section 2, we provide the definitions and techniques that will be used in the proof of the main result. In particular, we use the Leray-Schauder continuation principle based on a priori estimates which we also derive in Section 2. In Section 3 we state and prove the existence theorem.

The topological degree approach and the method of upper and lower solutions were used in [2] to obtain multiplicity results under the assumption of the boundedness of a solution. The Leray-Schauder continuation principle was applied to boundary value problems on unbounded domains in several recent papers [3, 4]. In [5], the authors studied the boundary value problem

$$x''(t) + f(t, x(t), x'(t)) = 0, \quad 0 < t < \infty,$$

$$x(0) = \alpha x(\eta), \quad \lim_{t\to\infty} x'(t) = 0,$$

where $\alpha \neq 1$ and $\eta > 0$. The authors showed the existence of at least one solution where $tf(t, \cdot, \cdot)$ is L_1-Carathéodory on $[0, \infty)$. In this work we introduce new a priori estimates on the positive half-line pertinent to the case of L_r-Carathéodory inhomogeneous term with $r > 1$. In [6], the boundary value problem (1) (2) is considered in the case of an L_1-Carathéodory inhomogeneous term. The existence result in our paper is complementary to that in [6].

2 Technical Results

The following definition gives Carathéodory's conditions imposed on a map with respect to the Lebesgue space $L_r[0, \infty)$, $r > 1$.

Definition 2.1 *We say that the map $f\colon [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$, $(t, z) \mapsto f(t, z)$ is L_r-Carathéodory, if the following conditions are satisfied:*

(i) for each $z \in \mathbb{R}^n$, the mapping $t \mapsto f(t, z)$ is Lebesgue measurable;

(ii) for a. e. $t \in [0, \infty)$, the mapping $z \mapsto f(t, z)$ is continuous on $\mathbb{R}^n$;

(iii) for each $R > 0$, there exists an $\alpha_R \in L_r[0, \infty)$ such that, for a. e. $t \in [0, \infty)$ and every z such that $|z| \leq R$, we that $|f(t, z)| \leq \alpha_R(t)$.

The assumptions on $p(t)$ are the following:

(P_1) $p \in C[0, \infty) \cap C^1(0, \infty)$ and $p(t) > 0$ for all $t \in [0, \infty)$;

(P_2) $\dfrac{1}{p(t)} = O(t^{-2})$.

Let $AC[0, \infty)$ denote the space of absolutely continuous functions on the interval $[0, \infty)$. Define the Sobolev space

$$W[0, \infty) = \{u\colon [0, \infty) \to \mathbb{R} : u, pu' \in AC[0, \infty),\ u \text{ satisfies (2) and } (pu')' \in L_r[0, \infty)\}.$$

The underlying Banach space is

$$X = \{u \in C^1[0,\infty) : u(t) \text{ and } u'(t) \text{ are bounded on } [0,\infty)\}$$

endowed the norm $\|u\| = \max\{\|u\|_\infty, \|u'\|_\infty\}$, where $\|u\|_\infty = \sup_{t\in[0,\infty)} |u(t)|$. We also need the Lebesgue space $Z = L_r[0,\infty)$ with the usual norm denoted by $\|\cdot\|_r$.

The Nemetski operator $N\colon X \to Z$ is defined by

$$Nu(t) = f(t, u(t), u'(t)), \quad t \in [0,\infty).$$

We state the Leray-Schauder continuation principle (see, e. g., [7]).

Theorem 2.1 *Let X be a Banach Space and $T : X \to X$ be a compact map. Suppose that there exists an $R > 0$ such that if $u = \lambda Tu$ for $\lambda \in (0,1)$, then $\|u\| \leq R$. Then T has a fixed point.*

In applying Theorem 2.1 we establish the compactness of a certain integral operator associated with (1), (2). This is done by means of the following compactness criterion [1].

Theorem 2.2 *Let X be the space of all bounded continuous vector-valued functions on $[0,\infty)$ and $S \subset X$. Then S is relatively compact in X if the following conditions hold:*

(i) S is bounded in X;

(ii) the functions from S are equicontinuous on any compact interval of $[0,\infty)$;

(iii) the functions from S are equiconvergent, that is, given $\epsilon > 0$, there exists a $T = T(\epsilon) > 0$ such that $\|v(t) - v(\infty)\|_{\mathbb{R}^n} < \epsilon$, for all $t > T$ and all $v \in S$.

Define the function

$$\phi(t) = \int_t^\infty \frac{d\tau}{p(\tau)}$$

and set $q = \dfrac{p}{p-1}$. Note that by assumptions (P_1) and (P_2) the expressions

$$A = \sup_{t\in[0,\infty)} \left(t^{\frac{1}{q}} \phi(t) + \left(\int_t^\infty \phi^q(s)\, ds \right)^{1/q} \right), \quad B = \sup_{t\in[0,\infty)} \frac{t^{\frac{1}{q}}}{p(t)} \tag{3}$$

are positive constants.

The first technical lemma provides the solution to the linear analogue of (1) satisfying (2).

Lemma 2.1 *Let $g \in L_r[0,\infty)$. Then the unique solution of the differential equation*

$$(p(t)u'(t))' = g(t), \quad a.\,e. \text{ in } (0,\infty),$$

subject to the boundary conditions (2) is

$$u(t) = -\int_t^\infty \frac{d\tau}{p(\tau)} \int_0^t g(s)\,ds - \int_t^\infty \left(\int_s^\infty \frac{d\tau}{p(\tau)}\right) g(s)\,ds. \tag{4}$$

The following are the a priori estimates for the solution (4) in terms of the constants (3).

Lemma 2.2 *Let $g \in L_r[0,\infty)$. Then the solution (4) satisfies*

$$\|u\|_\infty \le A\|g\|_r \quad \text{and} \quad \|u'\|_\infty \le B\|g\|_r.$$

Proof: From (4),

$$u(t) = -\int_t^\infty \frac{d\tau}{p(\tau)} \int_0^t g(s)\,ds - \int_t^\infty \left(\int_s^\infty \frac{d\tau}{p(\tau)}\right) g(s)\,ds$$

for $t \in [0,\infty)$. By Hölder's inequality,

$$\begin{aligned}
|u(t)| &\le \phi(t)\int_0^t |g(s)|\,ds + \int_t^\infty \phi(s)|g(s)|\,ds \\
&\le \phi(t)\left(\int_0^t 1^q\,ds\right)^{\frac{1}{q}} \left(\int_0^t |g(s)|^r ds\right)^{\frac{1}{r}} + \left(\int_t^\infty \phi^q(s)\,ds\right)^{\frac{1}{q}} \left(\int_t^\infty |g(s)|^r\,ds\right)^{\frac{1}{r}} \\
&\le \left(t^{\frac{1}{q}}\phi(t) + \left(\int_t^\infty \phi^q(s)\,ds\right)^{1/q}\right)\|g\|_r
\end{aligned}$$

Then

$$\|u\|_\infty \le A\|g\|_r.$$

For all $t \in [0,\infty)$ we have

$$\begin{aligned}
|u'(t)| &\le \frac{1}{p(t)}\int_0^t |g(s)|\,ds \\
&\le \frac{1}{p(t)}\left(\int_0^t 1^q ds\right)^{\frac{1}{q}} \left(\int_0^t |g(s)|^r\,ds\right)^{\frac{1}{r}} \\
&= \frac{t^{\frac{1}{q}}}{p(t)}\|g\|_r,
\end{aligned}$$

so that

$$\|u'\|_\infty \le B\|g\|_r.$$

□

We introduce the integral mapping $T : X \to X$ defined by

$$\begin{aligned} Tu(t) &= -\int_t^\infty \frac{d\tau}{p(\tau)} \int_0^t f(s,u(s),u'(s))\,ds - \int_t^\infty \left(\int_s^\infty \frac{d\tau}{p(\tau)}\right) f(s,u(s),u'(s))ds \\ &= -\int_t^\infty \frac{d\tau}{p(\tau)} \int_0^t Nu(s)ds - \int_t^\infty \left(\int_s^\infty \frac{d\tau}{p(\tau)}\right) Nu(s)\,ds \qquad (5) \end{aligned}$$

for all $t \in [0,\infty)$.

The last technical result of this section establishes the desired properties of the operator (5).

Lemma 2.3 *The mapping $T : X \to X$ is compact.*

Proof: Employing the estimates identical to those in the proof of Lemma 2.2 one can easily show that T is well-defined. It is also clear that $T : X \to X$. The continuity of T follows readily from the dominated convergence theorem in view of f satisfying the Carathéodory conditions. It remains to show that the image under T of a bounded set in X is relatively compact in X.

Let $S \subset X$ be bounded, that is, there exists an $R > 0$ such that $R = \sup\{\|u\| : u \in S\}$. Since the function $f : [0,\infty) \times \mathbb{R}^2 \to \mathbb{R}$ is L_r-Carathéodory, there exists a function $\alpha_R \in L_r[0,\infty)$ such that, for all $u \in S$ and *a. e.* $s \in [0,\infty)$,

$$|Nu(s)| = |f(s,u(s),u'(s)| \le |\alpha_R(s)|.$$

Then for $u \in S$ we obtain

$$\|Tu\|_\infty \le A\|\alpha_R\|_r \quad \text{and} \quad \|(Tu)'\|_\infty \le B\|\alpha_R\|_r;$$

that is, the set $T(S)$ is bounded in X.

Let $L_1, L_2 \in [0,\infty)$, $[t_1,t_2] \subset [L_1,L_2]$ and $u \in S$. Then

$$\begin{aligned} |(Tu)'(t_2) - (Tu)'(t_1)| &= \left|\frac{1}{p(t_2)}\int_0^{t_2} Nu(s)\,ds - \frac{1}{p(t_1)}\int_0^{t_1} Nu(s)\,ds\right| \\ &\le \left|\frac{1}{p(t_2)} - \frac{1}{p(t_1)}\right| \int_0^{t_2} |Nu(s)|\,ds + \frac{1}{p(t_1)}\int_{t_1}^{t_2} |Nu(s)|\,ds \\ &\le \left|\frac{1}{p(t_2)} - \frac{1}{p(t_1)}\right| \int_0^{L_2} |Nu(s)|\,ds + \frac{1}{p(t_1)}\int_{t_1}^{t_2} |Nu(s)\,ds \\ &\le \left|\frac{1}{p(t_2)} - \frac{1}{p(t_1)}\right| L_2^{\frac{1}{q}}\|\alpha_R\|_r + \frac{(t_2-t_1)^{\frac{1}{q}}}{p(t_1)}\|\alpha_R\|_r. \end{aligned}$$

In addition, for some $\xi \in (t_1,t_2)$,

$$|(Tu)(t_2) - (Tu)(t_1)| \le |(Tu)'(\xi)||t_2 - t_1| \le B\|\alpha_R\|_r|t_2 - t_1|.$$

The last two inequalities show that the set $T(S)$ is equicontinuous on every compact subinterval of $[0,\infty)$.

For $u \in S$, it follows from (P_2) that

$$\lim_{t\to\infty} Tu(t) = 0 \quad \text{and} \quad \lim_{t\to\infty} (Tu)'(t) = 0.$$

To show that $T(S)$ is an equiconvergent set, note that

$$\begin{aligned}
|Tu(t) - \lim_{t\to\infty} Tu(t)| &= \left| -\int_t^\infty \frac{d\tau}{p(\tau)} \int_0^t Nu(s)\,ds - \int_t^\infty \left(\int_s^\infty \frac{d\tau}{p(\tau)} \right) Nu(s)\,ds \right| \\
&\le \int_t^\infty \frac{d\tau}{p(\tau)} \int_0^t |\alpha_R(s)|\,ds + \int_t^\infty \left(\int_s^\infty \frac{d\tau}{p(\tau)} \right) |\alpha_R(s)|\,ds \\
&\le \int_t^\infty \frac{d\tau}{p(\tau)} t^{\frac{1}{q}} \|\alpha_R\|_r + \int_t^\infty \phi(s)|\alpha_R(s)|\,ds
\end{aligned}$$

and

$$\begin{aligned}
|(Tu)'(t) - \lim_{t\to\infty} (Tu)'(t)| &\le \frac{1}{p(t)} \int_0^t |Nu(s)| ds \\
&\le \frac{t^{\frac{1}{q}}}{p(t)} \|\alpha_R\|_r.
\end{aligned}$$

In view of (P_2) and $\phi\alpha_R \in L_1[0,\infty)$, the expressions in the right sides of the above inequalities can be made arbitrarily small independently on $u \in S$. Hence the set $T(S)$ is equiconvergent. The set $T(S)$ is relatively compact in X by Theorem 2.2.

□

3 Existence of a Solution

Theorem 3.1 *Assume that $f : [0,\infty) \times \mathbb{R}^2 \to \mathbb{R}$ is L_r-Carathéodory. Suppose that there exist functions $\rho, \sigma, \gamma : [0,\infty) \to [0,\infty)$, $\alpha, \rho, \sigma \in L_r[0,\infty)$ such that*

$$|f(t, z_1, z_2)| \le \rho(t)|z_1| + \sigma(t)|z_2| + \gamma(t), \quad a.\,e. \text{ in } (0,\infty), \tag{6}$$

and

$$A\|\rho\|_r + B\|\sigma\|_r < 1, \tag{7}$$

where the constants A and B are given by (3).

Then the boundary value problem (1), (2) has at least one solution for every $\gamma \in L_r[0,\infty)$.

Proof: We consider for $\lambda \in (0,1)$,

$$(p(t)u'(t))' = \lambda f(t, u(t), u'(t)), \quad a.\,e. \text{ in } (0,\infty), \tag{8}$$

subject to the boundary conditions (2).

We show that the set of all possible solutions of (8), (2) is a priori bounded in X by a constant independent of $\lambda \in (0,1)$.

Using Lemma 2.2 and (3), we obtain from the condition (6), for $u \in W[0,\infty)$,

$$\begin{aligned}\|(pu')'\|_r &= \lambda \|f(t,u,u')\|_r \\ &\le \|\rho\|_r \|u\|_\infty + \|\sigma\|_r \|u'\|_\infty + \|\gamma\|_r \\ &\le A\,\|\rho\|_r \|(pu')'\|_r + B\,\|\sigma\|_r \|(pu')'\|_r + \|\gamma\|_r.\end{aligned}$$

Hence, by (7),

$$\|(pu')'\|_r \le \frac{\|\gamma\|_r}{1 - A\,\|\rho\|_r - B\,\|\sigma\|_r},$$

that is, the solution set of (8), (2) is a priori bounded on $L_r[0,\infty)$ by a constant independent of $\lambda \in (0,1)$. By Lemma 2.2 and the above inequality, the solution set is bounded in X by a constant independent of $\lambda \in (0,1)$ since

$$\|u\| \le \max\{A,B\} \|(pu')'\|_r \le \frac{\max\{A,B\}\|\gamma\|_r}{1 - A\,\|\rho\|_r - B\,\|\sigma\|_r}.$$

It can be easily shown that the boundary value problem (1), (2) has a solution if and only if it is a fixed point of the mapping (5). In view of (2.2), the mapping T is compact. By the above inequality, the a priori estimate condition for Theorem 2.1 is satisfied, the assertion follows from Theorem 2.1.

□

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